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Dyadic Maximal Function Maps the Weighted Hardy Space $H^1(w)$ to the Weighted $L^1(w)$ Space

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Abstract: Let $f:\mathbb{R}\to\mathbb{R}$ be a locally integrable function, and define the dyadic maximal function

$$Tf(x)=\sup_j \frac{1}{2^j}\left|\int_0^{2^j} f(x-t)dt\right|.$$

Let $1<p<\infty$, and $w\in A_p$, i.e.,

$$\sup_I\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}dx\right)<\infty,$$

where the supremum is taken over all intervals I in $\mathbb{R}$. In this research we prove that Tf maps the weighted Hardy space $H^1(w)$ to the weighted $L^1(w)$ space. More precisely, we show that there exists a positive constant α such that

$$\|Tf\|_{L^1(w)}\le\alpha\|f\|_{H^1(w)}$$

for all $f\in H^1(w)$.

İkili Maksimal Fonksiyon Ağırlıklı $H^1(w)$ Hardy Uzayını Ağırlıklı $L^1(w)$ Uzayına Dönüştürür

Anahtar Kelimeler
Ağırlıklı Hardy uzayı,
Muckenhoupt ağırlığı,
A_p ağırlığı,
Hardy uzayı,
İkili maksimal fonksiyon

Öz: Yerel olarak integrallenebilen bir $f:\mathbb{R}\to\mathbb{R}$ fonksiyonu için Tf ikili maksimal fonksiyonunu

$$Tf(x)=\sup_j \frac{1}{2^j}\left|\int_0^{2^j} f(x-t)dt\right|$$

olarak tanımlayalım. $1<p<\infty$ ve $w\in A_p$ olsun. Yani supremum $\mathbb{R}$ reel sayılar kümesindeki bütün I aralıkları üzerinden alınmak üzere

$$\sup_I\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}dx\right)<\infty$$

olsun. Bu araştırmada Tf fonksiyonunun ağırlıklı Hardy uzayı $H^1(w)$ yı ağırlıklı $L^1(w)$ uzayına dönüştürdüğü gösterilmiştir. Yani her $f\in H^1(w)$ için

$$\|Tf\|_{L^1(w)}\le\alpha\|f\|_{H^1(w)}$$

olacak şekilde bir pozitif α sabitinin varlığı gösterilmiştir.

1. INTRODUCTION

We say that a positive function $w \in L^1_{loc}(\mathbb{R})$ is a Muckenhoupt's A_p weight for some $1 < p < \infty$ if the following condition is satisfied:

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} dx \right) < \infty,$$

where the supremum is taken over all intervals I in $\mathbb{R}$.
We say that $w \in A_1$ if given any interval I in $\mathbb{R}$ there exists a constant $C > 0$ such that

$$\frac{1}{|I|} \int_I w(y)dy \le Cw(x)$$

for a.e. $x \in I$.
We say that $w \in A_\infty$ if there exist $\delta > 0$ and $\varepsilon > 0$ such that given an interval I in $\mathbb{R}$, for any measurable set $E \subset I$,

$$|E| < \delta \cdot |I| \Rightarrow w(E)(1-\varepsilon) \cdot w(I)$$

where

$$w(E) = \int_E w. \quad \text{(iv)}$$

One can find an extensive study of weighted Hardy spaces $H^p(w)$ in Garcia-Cuerva, J. (1979), where w is a Muckenhoupt's A_p weight. The atomic characterization of $H^p(w)$ has also been given in Garcia-Cuerva, J. (1979). Given a weight function w on $\mathbb{R}$, as usual we denote by $L^p(w)$ the space of all functions satisfying

$$\|f\|^p_{L^p(w)} = \int_{\mathbb{R}} |f(x)|^p w(x)dx < \infty.$$

When $p = \infty$, $L^\infty(w)$ is equal to the space L^∞ and

$$\|f\|_{L^\infty(w)} = \|f\|_{L^\infty}.$$

Let ϕ be a function in $S(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying

$$\int_{\mathbb{R}} \phi(x)dx = 1.$$

Define

$$\phi_t(x) = t^{-n}\phi(x/t), \qquad t > 0,\ x \in \mathbb{R},$$

and the maximal function f^* by

$$f^*(x) = \sup_{t>0} |f * \phi_t(x)|.$$

Then $H^p(w)$ consists of those tempered distributions $S'(\mathbb{R}^n)$ for which $f^* \in L^p(w)$ with

$$\|f\|_{H^p(w)} = \|f^*\|_{L^p(w)}.$$

These weighted Hardy spaces $H^p(w)$ can also be characterized in terms of these atoms in

the following way:

Definition 1. Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$ such that $w \in A_q$ with critical index q_w. Set $[\cdot]$ the integer function. For $s \in \mathbb{Z}$ satisfying $s \ge [n(q_w/p - 1)]$, a real-valued function a defined on $\mathbb{R}$ is called a (p,q,s)-atom with respect to w if

(i) $a \in L^p(w)$ and is supported on an interval I,

(ii) $\|a\|_{L^q(w)} \le w(I)^{1/q-1/p}$,

(iii) $\int_{\mathbb{R}} a(x)x^\alpha dx = 0$ for every multi-index α with $|\alpha| \le s$.

The real-valued atom defined above is called a (p,q,s)-atom centered at x_0 with respect to w ($w-(p,q,s)$-atom centered at x_0), where x_0 is the center of the interval I.

Remark. Let a be any real-valued $w-(p,q,s)$-atom supported in an interval I. Then we have

$$\int_I |a(x)|^p w(x)dx \le 1.$$

Proof. Let a be any B-valued $w-(p,q,s)$-atom. It is clear that $a \in L^p_B(w)$ and $\|a\|_{L^p_B(w)} \le 1$, since by Hölder's inequality

$$\int_I |a(x)|^p w(x)dx \le \|a^p\|_{L^r(w)} \left(\int_I w(x)dx \right)^{1/r'}$$
$$= \|a\|^p_{L^q(w)} \cdot w(I)^{1-p/q}$$
$$\le 1,$$

where $r = q/p$ and $1/r' = 1 - 1/r = 1 - p/q$.

Note that analog to the classical case any function in $H^p(w)$ admits a decomposition

$f=\sum \lambda_i a_i$, where a_i's are $w-(p,q,s)$-atoms and $\sum|\lambda_i|^p<\infty$. For a fixed weight function w and $f\in H^p(w)$ it is well known (see Garcia-Cuerva, J. (1979)) that

$$\|f\|_{H^p(w)}=\inf\left(\sum_i|\lambda_i|^p\right)^{1/p}.$$

2. RESULTS

Let $f:\mathbb{R}\to\mathbb{R}$ be a locally integrable function, and define the dyadic maximal function

$$Tf(x)=\sup_k \frac{1}{2^k}\left|\int_0^{2^k} f(x-t)dt\right|.$$

It is clear that $Tf(x)=\sup_k|K*f(x)|$, where

$$K(x)=\frac{1}{2^k}\chi_{[o,2^k]}(x).$$

Our first result is the following lemma that will be used when proving our main result:

Lemma 1. There exists a positive constant C independent of $y\in\mathbb{R}$ such that

$$\int_{|x|>2|y|}\sup_k|K(x-y)-K(x)|dx\le C.$$

Proof. Let

$$\Phi_k(x,y)=\frac{1}{2^k}\chi_{[0,2^k]}(x-y)-\frac{1}{2^k}\chi_{[0,2^k]}(x)$$
$$=\frac{1}{2^k}\chi_{[y,y+2^k]}(x)-\frac{1}{2^k}\chi_{[0,2^k]}(x).$$

First consider the case $x\ge0$, $y\ge0$. Since $x>2y$, we obviously have $\Phi_k(x,y)=\frac{1}{2^k}\chi_{[y,y+2^k]}(x)$. If for some $k\in\mathbb{Z}^+$ we have $x>y+2^k$, it is then clear that $\Phi_k(x,y)=0$. So we only need to consider the case $\Phi_k(x,y)=\frac{1}{2^k}\chi_{[y,y+2^k]}(x)$ when evaluating the integral.

Now assume that $x<0$, $y<0$. Since $|x|>2|y|$, we have $y>x$, and thus we obtain $\Phi_k(x,y)=0$. Also, same is true if $x\le0$, $y\ge0$ since this implies $x<y$. If $x\ge0$, $y\le0$, we have the same situation as in the first case.

We conclude that we only need to evaluate

$$\Phi_k(x,y)=\frac{1}{2^k}\chi_{[y,y+2^k]}(x),$$

and we have

$$\int_{|x|>2|y|}\sup_k|K(x-y)-K(x)|dx$$
$$=\int_{|x|>2|y|}\sup_k\frac{1}{2^k}\chi_{[y,y+2^k]}(x)dx$$
$$=\int_y^{y+2^k}\sup_k\frac{1}{2^k}dx$$
$$=1$$

and thus, our proof is complete.

Lemma 2. There exists a constant $C>0$ such that

$$\int_{\mathbb{R}}|Tf(x)|^p\,w(x)dx\le C\int_{\mathbb{R}}|f(x)|^p\,w(x)dx$$

for all $f\in L^p(w)$, $1<p<\infty$, where

$$L^p(w)=\left\{f:\mathbb{R}\to\mathbb{R}:\int_{\mathbb{R}}|f(x)|^p\,w(x)dx<\infty\right\}.$$

Proof. Recall the Hardy-Littlewood maximal function

$$Mf(x)=\sup_I\frac{1}{|I|}\int_I f(x-t)dt,$$

where the supremum is taken over all intervals I in $\mathbb{R}$. It is clear that for any $x\in\mathbb{R}$ we have $Tf(x)\le Mf(x)$ and it is also well known (see Muckenhoupt, B. (1972)) that there exists a constant $C>0$ such that

$$\int_{\mathbb{R}}|Mf(x)|^p\,w(x)dx\le C\int_{\mathbb{R}}|f(x)|^p\,w(x)dx$$

for all $f\in L^p(w)$, $1<p<\infty$. We thus obtain

$$\int_{\mathbb{R}}|Tf(x)|^p\,w(x)dx\le\int_{\mathbb{R}}|Mf(x)|^p\,w(x)dx$$
$$\le C\int_{\mathbb{R}}|f(x)|^p\,w(x)dx$$

and this completes our proof.

We can now state and prove our main result:

Theorem 1. Let $1<p<\infty$, and $w\in A_p$. Then there exists a constant $C>0$ such that

$$\|Tf\|_{L^1(w)}\le C\|f\|_{H^1(w)}$$

for all $f\in H^1(w)$.

Proof. Given an interval $I=I(x_0;R)$ in $\mathbb{R}$ with center x_0 and length 2R, and denoting by $\tilde{I}$ the double interval, $\tilde{I}=I(x_0;2R)$, we first claim that

$$\int_{\mathbb{R}-\tilde{I}} |T(f)|\, w(x)dx \le C\|f\|_{L^1(w)}$$

for every $f \in L^1(w)$ supported in I such that

$$\int f(x)dx = 0.$$

But for such a function f,

$$Tf(x) = \int_I \{K(x-y) - K(x-x_0)\} \cdot f(y)dy$$

$$(x \in \tilde{I})$$

and therefore

$$\int_{\mathbb{R}-\tilde{I}} |Tf(x)|\, w(x)dx$$

$$\le \int \int_{|x-x_0|\ge 2R > 2|y-x_0|} \sup_k |\{K(x-y) - K(x-x_0)\} \cdot f(y)|\, dy w(x)dx$$

$$\le C \int_{|y-x_0|<R} |f(y)|\, w(y)dy,$$

which proves our claim.

Let now $a(x)$ be an atom with supporting interval J, and let I be the smallest interval containing J, and $\tilde{J}$ as before. Then there exists a positive constant C_1 such that

$$\int_{\mathbb{R}-\tilde{I}} |Ta(x)|\, w(x)dx \le C_1.$$

On the other hand, since by Lemma 2

$$\int_{\mathbb{R}} |Ta(x)|^q\, w(x)dx \le C_2 \int_{\mathbb{R}} |a(x)|^q\, w(x)dx$$

we have by Hölder's inequality,

$$\int_{\tilde{I}} |Ta(x)|\, w(x)dx \le C_3 \|a(x)\|_{L^q(w)} (Cw(J))^{1/q'}$$

$$\le \text{Constant}.$$

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