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E-EXACT SEQUENCE AND SOME RESULTS

E-TAM DİZİ VE BAZI SONUÇLAR

Abuzer GÜNDÜZ[1]

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Corresponding Author (Sorumlu Yazar)	*Received (Geliş Tarihi)*	*Revised (Revizyon Tarihi)*	*Accepted (Kabul Tarihi)*
abuzergunduz@sakarya.edu.tr	*09.02.2024*	*15.06.2024*	*26.06.2024*

Abstract

Let R be a commutative ring with identity, M be a $R-module$ and N be a submodule of M. N is called to be essential (large) in M if $N \cap Rm \neq 0$ for any nonzero element $m \in M$ and we showed by $N \leq_e M$. A sequence of $R-modules$ and $R-morphisms$

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \ldots$$

is called exact at M_i if $Im(f_{i-1}) = Ker\,(f_i)$. Also this sequence is called $e-exact$ at M_i if $Im(f_{i-1}) \leq_e Ker(f_i)$ and it is called $e-exact$ if it is $e-exact$ at each M_i. In this note, we present the concept of the characterization of $E-homotopy$ and $E-resolution$ with some results such as chain map for $e-exact$ sequence and comparing theorem for $e-exact$ sequence.

Keywords: E-injective modules, e-exact sequences, contravariant functor, homological algebra.

Öz

R birimli ve değişmeli bir halka, M bir R modül ve N, M 'nin bir alt modülü olsun. Eğer sıfırdan farklı bir $m \in M$ elemanı için $N \cap Rm \neq 0$ gerçekleniyorsa N'ye M 'nin bir büyük alt modülü denir ve $N \leq_e M$ ile gösterilir. Bir $R-modül$ dizisi için

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \ldots$$

her M_i için $Im(f_{i-1}) = Ker\,(f_i)$ oluyorsa bu diziye tam (exact) dizi denir. Ayrıca her M_i için $Im(f_{i-1}) \leq_e Ker(f_i)$ oluyorsa bu diziye e-exact dizi denir. Bu çalışmada $tam\ (exact)$ diziler teorisinin bir genişlemesi olan $E-exact$ diziler teorisi için $E-homotopy$ and $E-resolution$ tanımlanmış ve zincir map ve karşılaştırma teoremi gibi ilgili bir kısım sonuçlar verilmiştir.

Anahtar Kelimeler: E-injektif modüller, e-tam diziler, contravariant functor, homolojik cebir.

[1] Sakarya University, Faculty of Science, Department of Mathematics, Sakarya, Turkey.
abuzergunduz@sakarya.edu.tr, Orcid.org/ 0000-0003-3351-2443.

1. INTRODUCTION

Let R be a commutative ring with identity and M, A_i be an $R-module$, for $i = 1,2$. Consider

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

an exact sequence of R −modules. Hence we have $Im(f_1) = Ker\ (f_2)\ (=\ {f_2}^{-1}(\{0\}))$. We can think a natural question: if we change a submodule U of R, what does happen for the trivial submodule $\{0\}$ in the above definition? This sequence is called $U_3-exact$ at A_3 if $Im(f_1) = {f_2}^{-1}(U_3)$, where U_3 is a submodule of A_3. Firstly, In (Davaz, & Parnian-Garameleky, 1999), Davaz and Parnian-Garameleky answered this question. Also, In (Davvaz, 2002), Davaz and Shabani-Solt obtained a generaliation of some notations in homological algebra and new basic properties of U −homological algebra for $U-exact$ sequence theory. Besides, in (Anvariyeh, & Davvaz, 2002), Anvariyeh and Davvaz studied over $U-split$ sequences. In (Anvariyeh, & Davvaz, 2005), Anvariyeh and Davvaz proved further results about quasi-exact sequences such as an analogue of Schanuel's Lemma for quasi-exact sequences. On the other hand, a submodule N of M is said to be essential (large) in M if the intersection of N with each nonzero submodule of M is nonzero, namely, $N \cap Rm \neq 0$ for any nonzero element $m \in M$ and we showed by $N \leq_e M$. A sequence of $R-modules$

$$\ldots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \ldots$$

is called exact at M_i if $Im(f_{i-1}) = Ker\ (f_i)$. In (Akray & Zebari, 2020), Akray and Zebari introduced the $e-exact$ sequences as a generalization of exact sequences, like $U-exact$ theory. The previous sequence is called $e-exact$ at M_i if $Im(f_{i-1}) \leq_e Ker(f_i)$ and it is called $e-exact$ if it is $e-exact$ at each M_i. Particularly, they defined the sequence

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

to be short $e-exact$ if $Ker\ (f_i) = 0$, $Im(f_1) \leq_e Ker(f_2)$ and $Im(f_2) \leq_e A_3$. Also from (Akray & Zebari, 2020), an $R-morphism$ $f_1: A_1 \to A_2$ is called epic if $Im(f_1) \leq_e A_2$ and essential monic if $Ker(f_1) = 0$. Obviously, the class of $e-exact$ sequences is larger than the class of $exact$ sequences. For instance, consider the short $e-exact$ sequence

$$0 \longrightarrow 16\mathbb{Z} \xrightarrow{f_1} \mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/16\mathbb{Z} \longrightarrow 0$$

where $f_1(16n) = 8n$ and $f_2(n) = 8n + 16\mathbb{Z}$. Since f_1 is $monic$, $Im(f_1) \leq_e Ker\ (f_2)$ and f_2 are $epic$, the sequence is $e-exact$. But the sequence is not $exact$, since f_2 is not an $epimorphism$.

In (Gunduz & Osama 2022), Gunduz and Osama defined a characterization of e-injective module in terms of contravariant functor $Hom(-, E)$.

We recall from (Tercan & Yücel 2016) some basic definitions. An element m of M is said to be torsion of M if the exists a regular element $r \in R$ such that $rm = 0$. The set of all torsion elements $T(M)$ is a submodule of M. Also, an $R - module\ M$ is called a $torsion$ if $T(M) = M$ and called $torsion - free$ when $T(M) = \{0\}$.

The following theorem says that the contravariant functor $Hom(-, M)$ is a left $e - exact$ functor when M is a $torsion - free\ R - module$.

Theorem 1. (Akray & Zebari, 2020) Suppose that the following sequence of $R - module$ and $R - morphism$

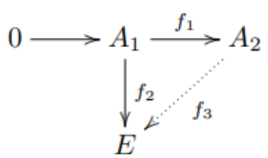
$$M_1 \overset{f_1}{\to} M_2 \overset{f_2}{\to} M_3 \longrightarrow 0$$

is $e - exact$. Then for all $torsion - free\ R - module\ M$, the sequence

$$0 \longrightarrow Hom(M_3, M) \xrightarrow{f_2^*} Hom(M_2, M) \xrightarrow{f_1^*} Hom(M_1, M)$$

is $e - exact$. The converse is true if $M_3/Im(f_2)$ and $M_2/Im(f_1)$ are $torsion - free$ $R - modules$.

Defınıtıon 1. (Gunduz & Osama, 2022) Let R be a ring and E an $R - module$. E is said to be $e - injective$ if the following condition is satisfied: For any monic map $f_1: A_1 \to A_2$ and any map $f_2: A_1 \to E$, there exist $0 \neq r \in R$ and $f_3: A_2 \to E$ such that $f_3 f_1 = r.f_2$.



Theorem 2. (Gunduz & Osama 2022) Let R be a ring and E an $R - module$. Then the following statements are equivalent:

i) E is an $e - injective\ R - module$.
ii) $Hom(-, E)$ is an $e - exact$ sequence.

Throughout section 2, all modules are assumed to be $torsion - free$. In this section, we introduce the definition of $e - homotopy$ and $e - resolution$ with some theorems such as chain map for $e - exact$ sequence comparing theorem for $e - exact$ sequence.

2. CHARACTERIZATION OF E-HOMOTOPY AND E-RESOLUTION

To define $e-homotopy$ and $e-resolution$, recall that some basic definitions. Let $\{K_n\}_{n\in\mathbb{Z}}$ be a family of $R-modules$ and $\{d_n: K_n \to K_{n-1}\}$ a family of $R-homomorphisms$. The family $\{K_n, d_n\}$ is called $chain\ complex$ if $d_n d_{n-1} = 0$ for each n.

We take $\mathbb{K} = \{K_n\}$, $d = \{d_n\}$ and show a chain complexes as follows:

$$(\mathbb{K}, d): \qquad \dots \longrightarrow K_{n+1} \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \longrightarrow \dots$$

We also recall that $H_n(\mathbb{K}, d) = Z_n/B_n$, $n \in \mathbb{N}$ is called $n-th$ homology module of K, where $Z_n = Ker(d_n)$ and $B_n = Im(d_{n+1})$.

Let $(\mathbb{K}, d)$ and $(\mathbb{L}, d')$ be chain complexes. The sequence $f = \{f_n: K_n \longrightarrow L_n\}$ is called a chain map if the following diagram is commutative. In words for the diagram

$$\begin{array}{lccccccccc}
(\mathbb{K}, d): & \dots \longrightarrow & K_{n+1} & \xrightarrow{d_{n+1}} & K_n & \xrightarrow{d_n} & K_{n-1} & \longrightarrow \dots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & \\
(\mathbb{L}, d'): & \dots \longrightarrow & L_{n+1} & \xrightarrow{d'_{n+1}} & L_n & \xrightarrow{d'_n} & L_{n-1} & \longrightarrow \dots
\end{array}$$

that satisfies $f_{n-1} d_n = d'_n f_n$

For the theory of $e-exact$, we will define $f^* = H_n(f)$ from $H_n(K_n, d_n)$ to $H_n(L_n, d'_n)$ as follows:

Theorem 3. Let $(\mathbb{K}, d)$ and $(\mathbb{L}, d')$ be chain complexes. If $f = \{f_n\}$ is a chain map then it induces $R-module\ homomorphisms$ as follows

$H_n(f) = f^* = H_n(K_n, d_n) \to H_n(L_n, d'_n)$

such that $x + B_n \mapsto f_n(rx) + B'_n$, where $B_n = Im(d_{n+1})$, $B'_n = Im(d'_{n+1})$ and for some $0 \neq r \in R$.

Proof. To show f^* is well defined, suppose that $x + B_n = y + B_n$, then $x - y \in B_n$.

Let $x, y \in Ker(d_n)$ and implies that $x - y \in Ker(d_n)$. Since $Im(d_{n+1}) \leq_e Ker(d_n)$, we have $r(x - y) \in Im(d_{n+1})$ for some $0 \neq r \in R$. Hence $f_n(r(x - y)) = f_n(r(x) - r(y)) \in B'_n$ and so $f_n(rx) - f_n(ry) \in B'_n$. Therefore $f_n(rx) + B'_n = f_n(ry) + B'_n$ and we get $f_n^*(x) = f_n^*(y)$. Also, it can be seen that f^* is a homomorphism. Let $x + B_n$, $y + B_n \in H_n(K_n, d_n)$ then $f^*[(x + B_n), +(y + B_n)] = f^*[(x + y) + B_n] = f_n(r(x + y)) + B'_n = f_n(rx + ry) + B'_n = f_n(rx) + f_n(ry) + B'_n = (f_n(rx) + B'_n) + (f_n(ry) + B'_n) = f^*(x + B_n) + f^*(y + B_n)$, as desired.

Definıtıon 2. (E-homotopy). Let $(\mathbb{K}, d)$ and $(\mathbb{L}, d')$ be two chain complexes and $f = \{f, g: K \to L\}$ be two chain maps as 2.1. If there is a sequence $s = \{s_n\}$ such that $r[f_n - g_n] = d'_{n+1}s_n + r(s_{n-1}d_n)$ for all $n \in \mathbb{Z}$ and for some $0 \neq r \in R$, then f and g are chain $e - homotopic$ which is denoted by $f \simeq_e g$, where $s_n: K_n \to L_{n+1}$ is an $R - module\ homomorphism$ that is called a chain $e - homotopy$.

Lemma 1. The $e - homotopy$ relation "$f \simeq_e g$" is an equivalence relation.

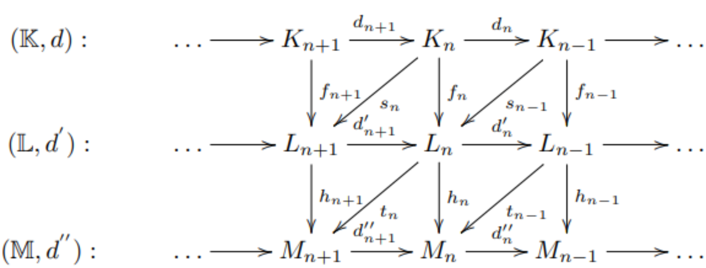
Proof. If we choose $s'_n = -s_n$ for all $n \in \mathbb{Z}$ and for some $0 \neq r \in R$, then $r[f_n - g_n] = d'_{n+1}(-s_n) + r((-s_{n-1})d_n)$ and implies that $r[g_n - f_n] = d'_{n+1}s_n + r(s_{n-1}d_n)$, namely "$g \simeq_e f$" and "$\simeq_e$" is symmetric.

If we choose $s_n = 0, \forall n \in \mathbb{Z}$ and for some $0 \neq r \in R$, then $r[f_n - g_n] = 0$ and implies that "$f \simeq_e f$" and "$\simeq_e$" is reflexive.

To check transtivity, let $f \simeq_e g$ and $g \simeq_e h$. Then for some $0 \neq r_i,\ r_j \in R$ and $i, j \in \mathbb{I}$ (an index set) there exist $s_n, t_n : K_n \to L_{n+1}$, $R - module$ homomorphisms such that we have $r_i[f_n - g_n] = d'_{n+1}s_n + r_i(s_{n-1}d_n)$ and $r_j[g_n - h_n] = d'_{n+1}t_n + r_j(t_{n-1}d_n)$. Define $x_n : K_n \to L_{n+1}$ homeomorphism such that $x_n = s_n + t_n$. This implies that $r[f_n - h_n] = r(f_n - g_n) + r(g_n - h_n) = d'_{n+1}s_n + d'_{n+1}t_n + r(s_{n-1}d_n) + r(t_{n-1}d_n) = d'_{n+1}(s_n + t_n) + r\big((s_{n-1} + t_{n-1})d_n\big) = d'_{n+1}x_n + r(x_{n-1}d_n)$, where for some $0 \neq r = r_ir_j \in R$. Namely "$\simeq_e$" is transitivity. Hence, "$\simeq_e$" is an equivalence relation.

Theorem 4. If "$f \simeq_e g$" and "$h \simeq_e k$", then "$hf \simeq_e kg$", where hf is equal $h \circ f$.

Proof. Let $f, g: (\mathbb{K}, d) \to (\mathbb{L}, d')$ be chain complexes. Then, there exist $s_n: K_n \to L_{n+1}$ and $t_n: L_n \to M_{n+1}$, $R - module$ homomorphisms such that $r_i[f_n - g_n] = d'_{n+1}s_n + r_i(s_{n-1}d_n)$ and $r_j[h_n - k_n] = d''_{n+1}t_n + r_j(t_{n-1}d'_n)$, some $0 \neq r_i, r_j \in R$, where each g_n is defined as $g_n: K_n \to L_n$.



Define $x_n: K_n \to M_{n+1}$, $\forall n \in \mathbb{Z}$ and some $0 \neq r = r_ir_j \in R$ such that $x_n = h_{n+1}s_n + t_ng_n$, then we get $r[h_nf_n - k_ng_n] = r[h_nf_n] - r[h_ng_n] + r[h_ng_n] - r[k_ng_n] = r(h_n[f_n - g_n]) + r\big([h_n - k_n]g_n = h_n(r[f_n - g_n]) + r[h_n - k_n]g_n = h_n(d'_{n+1}s_n + r(s_{n-1}d_n)\big) + \big(d''_{n+1}t_n + r(t_{n-1}d'_n)\big)g_n = h_nd'_{n+1}s_n + r(h_ns_{n-1}d_n + d''_{n+1}t_ng_n) + rt_{n-1}d'_ng_n = d''_{n+1}[h_{n+1}s_n + t_ng_n] + r(h_ns_{n-1} + t_{n-1}g_{n-1})d_n = d''_{n+1}x_n +$

$r(x_{n-1}d_n)$, as desired. Here $h_n d'_{n+1} = d''_{n+1}h_{n+1}$ and $d'_n g_n = g_{n-1}d_n$ are used by the above diagram. Hence "$hf \simeq_e kg$" and the proof is completed.

Theorem 5. If two chain maps $f, g: K \to L$ are $e - homotopic$, then $H_n(f) = H_n(g)$.

Proof. Suppose that $r[f_n - g_n] = d'_{n+1}s_n + r(s_{n-1}d_n)$ for all $0 \neq r \in R$. Let $x + B_n \in H_n(K)$ for $x \in \mathbb{Z}$. Since $d_n(x) = 0$ and $H_n(f) = f^*: H_n(K) \to H_n(L)$ such that $x + B_n \longmapsto f_n(rx) + B'_n$, then $r[f_n - g_n](x) = d'_{n+1}s_n(x) + r(s_{n-1}d_n)(x) = d'_{n+1}s_n(x)$. Since $x \in Ker(d_n)$, $d_n(x) = 0$ and $d'_{n+1}s_n(x) \in B'_n$, we get $r[f_n - g_n](x) = f_n(rx) - g_n(rx) \in B'_n$, which implies $f_n(rx) \in B'_n = g_n(rx) \in B'_n$. Hence $H_n(f)(x + B_n) = H_n(g)(x + B_n)$, and so $H_n(f) = H_n(g)$.

To give the following theorems, recall that Let $(\mathbb{X}, \varepsilon)$ be a left complex over a module A, where

$$\mathbb{X}: \ldots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \ldots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} 0$$

and $\varepsilon: X_0 \to A$ such that $\varepsilon \circ d_1 = 0$.

To get further results, we will give the following definitions.

Defınıtıon 3. If the above sequence is $e - exact$ then it is called $e - resolution$. Moreover if each X_n is an $e - projective$ module then it is called $e - projective\ resolution$.

Likewise, recall that let $(\mathbb{Y}, \delta)$ be the right complex over a module B, where

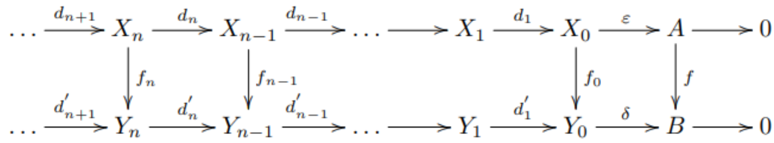
$$\mathbb{Y}: 0 \longrightarrow Y^0 \xrightarrow{d^0} Y^1 \xrightarrow{d^1} \ldots \qquad \xrightarrow{d^{n-1}} Y^n \xrightarrow{d^n} \ldots$$

and $\delta: B \to Y_0$ such that $d_0 \circ \delta = 0$.

Defınıtıon 4. If the above sequence is $e - exact$, then it is called $e - resolution$. Moreover, if each Y^n is an $e - injective\ module$ then it is called $e - injective\ resolution$.

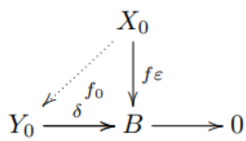
Under the above new definitions, the following theorem is characterized by comparing theorem for $e - exact$ theory that explains why the above definitions are important.

Theorem 6. Let $(\mathbb{X}, \varepsilon)$ be a left complex over $R - module\ A$, $(\mathbb{Y}, \delta)$ a left complex over $R - module\ B$ and $f: A \to B$ a homomorphism. If each X_n is $e - projective$ and $(\mathbb{Y}, \delta)$ is $e - resolution$, then

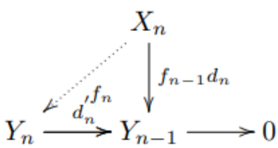


there exists a chain map f : $\{f\}$: $\mathbb{X} \to \mathbb{Y}$ such that the above diagram is commutative. Moreover, if f′ is another chain map that satisfies the same condition, then f $\simeq_e$f′ .

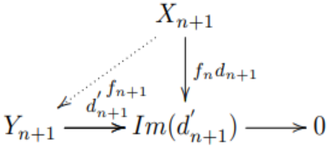
Proof. To proof this, we will use induction. Since δ is an epimorphism and X_0 is $e-projective$, there exists a homomorphism $f_0: X_0 \to Y_0$ such that $\delta f_0 = r[f\varepsilon]$, for some $0 \neq r \in R$. Thus, we have the following diagram



is hold. Now, suppose that $f_1, f_2, \ldots, f_n$ are homomorphisms. By hypothesis, we have the following diagram



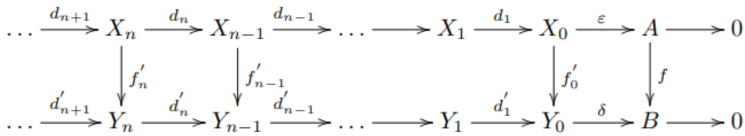
such that $d'f_n = r[f_{n-1}d_n]$ for some $0 \neq r \in R$. By the above diagram $d'f_nd_{n+1} = r[f_{n-1}d_nd_{n+1}] = 0$. Since $d_nd_{n+1} = 0$, it implies $f_nd_{n+1} \in Ker(d'_n)$. Also, since $Im(d'_{n+1}) \leq_e Ker(d'_n)$, then $r(f_nd_{n+1}) \in Im(d'_{n+1})$ for some $0 \neq r \in R$. This implies that there exists $f_{n+1}: X_{n+2} \to Y_{n+1}$ such that $d'_{n+1}(f_{n+1}) = rf_nd_{n+1}$. Thus we get the following diagram



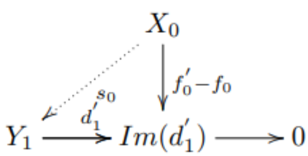
is hold.

Hence, we can say that there exists an $e-projective$ module X_{n+1} such that the above diagram is commutative.

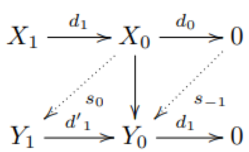
Now, let $\mathbf{f}' = f_n'\colon \mathbb{X} \to \mathbb{Y}$ be another chain map that make he following diagram commutative



To show f $\simeq_e$ f$'$, we will construct a homomorphism s_n. By induction, let $f_0' - f_0\colon X_0 \to Y_0$ be a homomorphism. Since $\delta(f_0' - f_0) = \delta f_0' - \delta f_0 = f\varepsilon - f\varepsilon = 0$, then $f_0' - f_0 \in Ker(\delta)$. Since, $Im(d_1') \leq_e Ker(\delta)$, that implies $r(f_0' - f_0) \in Im(d_1')$, for some $0 \neq r \in R$. So there exists an $s_0\colon X_0 \to Y_0$ with the commutative diagram

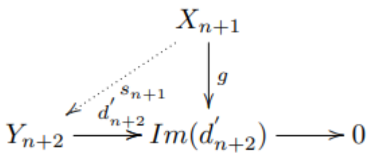


such that $d_1' s_0 = r(f_0' - f_0)$. Since $X_{-1} = 0$, we take $s_{-1} = 0$. So, we get $r[f_0' - f_0] = d_1' s_0 + r(s_{-1} d_0)$ for all $n \in \mathbb{Z}$ and for some $0 \neq r \in R$.
Now, suppose that there exist $s_0, s_1, \dots, s_n$, then the equality



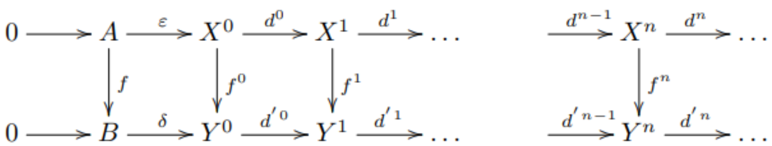
$r[f_n' - f_n] = d_{n+1}' s_n + r(s_{n-1} d_n)$ for all $n \in \mathbb{Z}$ and for some $0 \neq r \in R$ is satisfied. Now, we will show that there exists a homomorphism $s_{n+1}\colon X_{n+1} \to Y_{n+2}$ such that $r[f_{n+1}' - f_{n+1}] = d_{n+2}' s_{n+1} + r(s_n d_{n+1})$. Namely, it implies that $d_{n+2}' s_{n+1} = r[f_{n+1}' - f_{n+1} - s_n d_{n+1}]$.

Also, $d_{n+1}'(r[f_{n+1}' - f_{n+1} - s_n d_{n+1}]) = r d_{n+1}' f_{n+1}' - r d_{n+1}' f_{n+1} - r d_{n+1}' s_n d_{n+1} = r f_n' d_{n+1} - r f_n d_{n+1} - r d_{n+1}' s_n d_{n+1} = r[f_n' - f_n] d_{n+1} - r d_{n+1}' s_n d_{n+1} = r[d_{n+1}' s_n + r(s_{n-1} d_n)] d_{n+1} - r d_{n+1}' s_n d_{n+1} = r d_{n+1}' s_n d_{n+1} + r s_{n-1} d_n d_{n+1} - r d_{n+1}' s_n d_{n+1} = 0$, since $d_n d_{n+1} = 0$, where $d_{n+1}' f_{n+1}' = f_n' d_{n+1}$ from 2.4 and $d_{n+1}' f_{n+1} = f_n d_{n+1}$ from 2.3 If we take $g = f_{n+1}' - f_{n+1} - s_n d_{n+1}$ then we can see that $d_{n+1}'(g) = 0$. This implies $g \in Ker(d_{n+1}')$. Since $Im(d_{n+2}') \leq_e Ker(d_{n+1}')$, then $rg \in Im(d_{n+2}')$, for some $0 \neq r \in R$, it means that, there exists an $s_{n+1}\colon X_{n+1} \to Y_{n+2}$ with the following commutative diagram



is hold and such that $r[f'_{n+1} - f_{n+1}] = d'_{n+2}s_{n+1} + r(s_n d_{n+1})$. In conclusion that f $\simeq_e$ f′.

Theorem 7. Let $(\mathbb{X}, \varepsilon)$ be a right complex over $R - module\ A$, $(\mathbb{Y}, \delta)$ a right complex over $R - module\ B$ and $f: A \to B$ a homomorphism. If each Y^n is $e\ -\ injective$ and $(\mathbb{X}, \varepsilon)$ is $e\ -\ resolution$, then



there exists a chain map f : $\{f\}$: $\mathbb{X} \to \mathbb{Y}$ such that the above diagram is commutative. Moreover, if f′ is another chain map that satisfies the same condition, then f ≃e f′ .

Proof. The proof can be done as Theorem 6 in similar way.

3. RESULTS AND RECOMMENDATIONS

In this paper, we present some new definitions, theorems and results about e-exact sequences of theory, which is the generalization of exact sequence of module theory, like U-exact sequence theory. Similarly, many results of homological algebra can be obtained for e-exact sequences such as the Lambek lemma, Snake lemma, Connecting homomorphism and Exact triangle for this theory.

Statement of Research and Publication Ethics
Research and publication ethics were observed in the study.

REFERENCES

Akray, I., & Zebari, A. (2020). Essential exact sequences. *Communications of the Korean Mathematical Society,* 35(2), 469-480.

Alizade, R., & Pancar, A. (1999). *Homoloji cebire giris.* Ondokuz Mayıs Universitesi Yayınları, Samsun.

Anvariyeh, S. M., & Davvaz, B. (2005). On quasi-exact sequences. *Bull.Korean Math. Soc.*, 42(1), 149-155.

Anvariyeh, S. M., & Davvaz, B. (2002). U-Split-Exact Sequences. *Far East J. Math Sci. (FJMS),* 4 , 2, 2009-219.

Davaz, B. (2002). A generalization of homological algebra, *J. Korean math. Soc.* 39, no. 6, 881-898.

Davaz, B. & Parnian-Garameleky, Y. A. (1999). A note on Exact Sequences, *Bull. Malays. Math. Sci. Soc*. (2) 22, no. 1, 53-56.

Gunduz, A., & Osama, N. A. J. I. (2022). A Note On E-Injective Modules. *Sakarya University Journal of Science,* 26(6), 1262-1266.

Hungerford, T. W., & Hungerford, T. W. (1974). Categories. *Algebra,* 464-484.

Tercan, A., & Yücel, C. C. (2016). *Module theory. Extending Modules and Generalizations,* Bassel: Birkhäuser–Springer.