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An Extended UEHL Distribution: Properties and Applications

Murat GENÇ[1]*, Ömer ÖZBİLEN[2]

[1] Department of Management Information Systems, Faculty of Economics and Administrative Sciences, Tarsus University, Mersin, Türkiye
[2] Department of Mathematics Education, Faculty of Education, Mersin University, Mersin, Türkiye
* muratgenc@tarsus.edu.tr
* Orcid No: 0000-0002-6335-3044

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Abstract

This study introduces a new distribution, a Lehmann-type exponentiated distribution, which is built upon the unit exponentiated half-logistic distribution. The analytical characteristics of the proposed distribution, like moments, moment-generating function, quantiles, and stress-strength reliability, are explored in detail. The renowned maximum likelihood estimation method is employed for the statistical inference of the distribution's parameters. A computer experiment is run to explore the performance of the maximum likelihood estimates of the distribution parameters under diverse scenarios. Additionally, the practicality and efficacy of the distribution are illustrated through a numerical example using a real-world dataset.

Keywords: Exponentiated family, G-family, Maximum likelihood estimator, Reliability, UEHL distribution.

1. Introduction

Scientists have proposed many probability distributions in recent decades for data-modeling in such diverse fields as biological studies, engineering, economics and medical sciences. Recently, there has been strong interest in pursuing more flexible distributions. Researchers are constantly developing new families of probability distributions that not only expand upon existing ones but also offer greater flexibility in representing the characteristics of real-world data. In this context, constructing extended G-family distributions by applying a particular transformation to the baseline distribution is widely adopted in statistics [1, 2, 3].

The Weibull method, which builds upon the existing "exponential" method, became popular because it is better at describing many types of data across various fields. With the contribution of the extra parameters, the Weibull method can handle a wider range of data types than the exponential method. Also, the Weibull distribution is successful in modeling monotone hazard rates in reliability theory. However, the distribution has limitations in capturing scenarios with non-monotonic hazard rates, where the likelihood of failure can fluctuate over time [4, 5].

Over the years, many continuous distributions with bounded domains have been proposed and applied in various fields of application to model uncertainty in a bounded phenomenon. In particular, modeling approaches on the unit interval, have grown in popularity recently since they address specific difficulties such as the recovery rate, mortality rate, and so on. Many useful unit distributions have been proposed, such as Johnson S_B distribution [6], unit logistic distribution [7], logit slash [8], unit Johnson S_U [9] distribution, log-xgamma distribution [10], unit-Weibull distribution [11], unit Birnbaum-Saunders distribution [12], unit inverse Gaussian distribution [13], log-weighted exponential distribution [14] and Generalized exponentiated unit Gompertz distribution [15].

Dombi et al. [16] proposed the omega distribution as a generalization of the Weibull distribution with bounded support, which has several applications in reliability theory. Dombi et al. [16] demonstrated that as the parameter in the omega distribution approaches infinity, it behaves identically to the Weibull distribution. This is because the hazard rate of the omega distribution, under these conditions, becomes indistinguishable from that of the Weibull distribution. In this context, [17] proposed the unit exponentiated-half logistic (UEHL) distribution by assigning a specific value to the parameter of the omega distribution. The UEHL distribution is obtained by a simple transformation of the exponentiated half-logistic distribution, which has been widely utilized in reliability theory [18, 19, 20, 21].

This study extends the UEHL distribution by constructing an exponentiated G-family UEHL distribution using a specific member of the G-Family distributions. Thus, we aim to improve the performance of the UEHL distribution by utilising G-Family distributions. The characteristics of this new distribution are thoroughly examined, and a performance analysis is conducted.

The rest of the paper is structured as follows: Section 2 introduces the new distribution and explores its properties (moments, MGF, quantiles, etc.). Section 3 evaluates the distribution's performance through simulations and real-data analysis. Lastly, Section 4 concludes the paper.

2. Material and Method

The proposal of novel statistical distributions based on the G-family method has gained popularity [22, 23]. The Weibull-G family of distributions was introduced by Bourguignon et al. [24], who also studied the distribution’s characteristics. Shukla et al. [25] further explored the reliability characteristics of the Weibull-G family using progressively Type-II censored data. Tahir et al. [26] proposed a new G-family type generator based on the Weibull random variable and studied the mathematical properties of the distribution. Korkmaz [27] introduced the extended Weibull-G distribution based on an extended form of the Weibull distribution and discussed the special members of the new family. Alizadeh et al. [28] proposed the Gompertz-G family of distribution with some special models based on the distribution. To address the deficiencies of the odd Fréchet-G family proposed by [29], Badr et al. [30] proposed the Transmuted Odd Fréchet-G family of distributions. Eghwerido et al. [31] proposed the transmuted alpha power-G class of models for modeling lifetime processes. Chakraborty [32] introduced the Kumaraswamy Poisson-G distribution by mixing the distribution of the minimum of a random number of identically and independent Kumaraswamy-G random variables and zero truncated Poisson random variable. Alnssyan et al. [33] introduced the weighted Lindley-G family of probabilistic models as a novel family based on Lindley distribution.

Recently, depending on the structure of the dataset, the use of distributions with bounded support sets has become widespread. Mazucheli et al. [34] proposed the unit-Weibull distribution and compared the model performance with that of the Kumaraswamy distribution [35]. Guerra et al. [36] proposed the unit extended Weibull families of distributions and used the distribution to model the literacy rate data. Chakraborty et al. [37] obtained a generalized log-Lindley distribution defined in the unit interval by extending the Log–Lindley distribution and, use the distribution to model outpatient health expenditure. Masood et al. [38] brought up the unit interval exponentiated exponential distribution and illustrated the performance of the distribution using COVID-19 data. Korkmaz and Korkmaz [39] proposed the unit log-log distribution and an alternative quantile regression with application to educational measurement data. Akata et al. [40] proposed the Kumaraswamy unit-Gompertz distribution and studied the applications of the distribution in lifetime datasets. Genç and Özbilen [41] proposed the DUS-UEHL distribution by implementing the transformation of DUS to the distribution of UEHL and investigated its characteristics in detail. Genç and Özbilen [42] proposed the EUEHL distribution, which is derived from the exponentiated transformation of the UEHL distribution and explored its characteristics using a computer experiment and an analysis of real data.

2.1. EG-UEHL distribution

The Lehmann-type exponentiated distributions are commonly used in statistics. In this context, Nadarajah [43] and, Tahir and Nadarajah [23] consider the cumulative distribution function (CDF)

$$F(x) = \left[1 - G(e^{-x})^{\lambda}\right] \tag{1}$$

based on the log transformation on the Lehmann type 2 distribution, where $G(x)$ is the baseline CDF and λ is a shape parameter. In this section, we introduce the two-parameter exponentiated UEHL distribution based on the G-family of distributions given by Equation (1). The CDF of the UEHL distribution is given by

$$F_{UEHL}(x;\theta,\lambda) = 1 - \left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}, 0 < x < 1. \tag{2}$$

By applying the distribution function in Equation (2) to the G-family of distributions in Equation (1), the CDF of the exponentiated G-family UEHL distribution is obtained as

$$F_{EG-UEHL}(x;\theta,\lambda) = \left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\lambda}, 0 < x < \infty \tag{3}$$

where $\theta > 0$ and $\lambda > 1$. In this study, we will use the notation $EG - UEHL(\theta,\lambda)$ to represent the exponentiated G-family UEHL distribution with the parameters θ and λ. As described in Equation (3), the probability density function (PDF) of the $EG - UEHL(\theta,\lambda)$ is given by

$$f_{EG-UEHL}(x;\theta,\lambda) = \frac{2\lambda\theta}{e^{\theta x} - e^{-\theta x}}\left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\lambda}, \quad 0 < x < \infty. \tag{4}$$

Figure 1 illustrates the PDF of the $EG-UEHL(\theta,\lambda)$ distribution for the different values of the parameters θ and λ. The distribution is skewed to the right and the skewness of the distribution decreases as the parameter λ increases. In addition, for large values of θ, the probabilities are large for small values of the random variable X, while the reverse is the case for small values of θ.

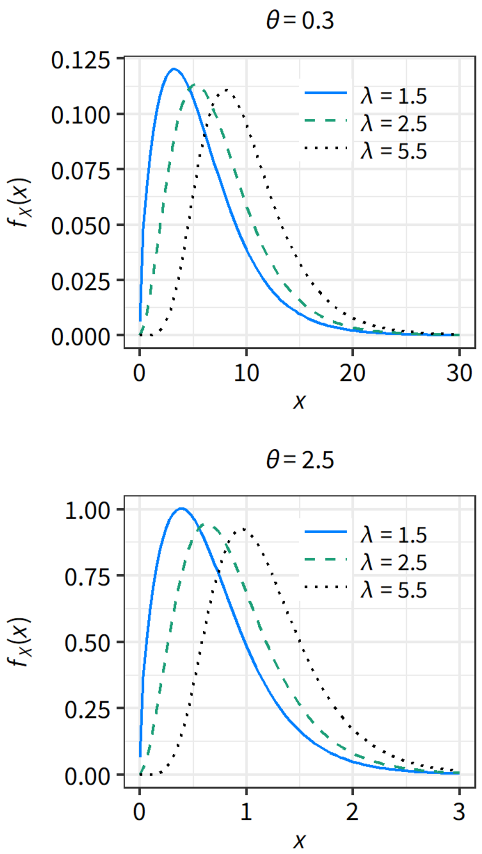

Figure 1. Plots of the PDF of the $EG-UEHL(\theta,\lambda)$ distribution for the selected θ and λ parameters.

The survival and the hazard rate functions of the $EG-UEHL(\theta,\lambda)$ distribution are provided, respectively, by

$$S_{EG-UEHL}(x;\theta,\lambda)=\frac{\left(1+e^{-\theta x}\right)^{\lambda}-\left(1-e^{-\theta x}\right)^{\lambda}}{(1+e^{-\theta x})^{\lambda}}$$

and

$$h_{EG-UEHL}(x;\theta,\lambda)=\frac{2\lambda\theta}{(e^{\theta x}-e^{-\theta x})\left(\left(\frac{1+e^{-\theta x}}{1-e^{-\theta x}}\right)^{\lambda}-1\right)}. \quad (5)$$

Figure 2 shows the hazard rate function for the chosen values of the parameters of the distribution. According to Figure 2, the hazard function of the $EG-UEHL(\theta,\lambda)$ distribution given by Equation (5) is an increasing function.

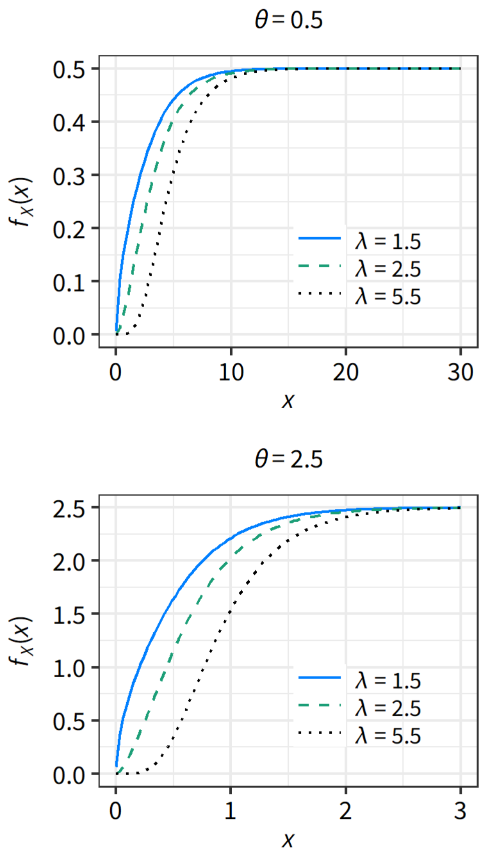

Figure 2. Plots of the hazard function of the $EG-UEHL(\theta,\lambda)$ distribution for the selected θ and λ parameters.

2.2. Statistical Characteristics of the $\boldsymbol{EG}$-$\boldsymbol{UEHL(\theta,\lambda)}$ Distribution

This section explores several analytical properties of the $EG-UEHL(\theta,\lambda)$ distribution. These include the moments, quantile function, stress-strength reliability, and maximum likelihood estimation of the distribution parameters.

2.2.1. Moments

Moments serve as valuable tools for understanding various facets of a statistical distribution. This section delves into the moments of the $\boldsymbol{EG-UEHL(\theta,\lambda)}$ random variable. Let $\boldsymbol{X}$ follow a $\boldsymbol{EG-UEHL(\theta,\lambda)}$ distribution with the PDF given by Equation (4). Then the $\boldsymbol{r}$-th raw moment of $\boldsymbol{X}$ for $\boldsymbol{r=1,2,3,\ldots}$ is

$$E(X^r) = \int_0^\infty x^r f(x;\theta,\lambda)\,dx$$
$$= 2\lambda\theta \int_0^\infty \frac{x^r}{e^{\theta x} - e^{-\theta x}} \left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^\lambda dx$$

By the transformation $u = e^{-\theta x}$, the expectation is written as

$$E(X^r) = \frac{2\lambda}{\theta^r}\int_0^1 (-\log u)^r (1-u)^{\lambda-1}(1+u)^{-\lambda-1}\,du.$$

By using the Binomial series $(1-t)^z = \sum_{j=0}^\infty (-1)^j \binom{z}{j} t^j$, we obtain

$$E(X^r) = \frac{2\lambda}{\theta^r}\sum_{j=0}^\infty (-1)^j \binom{\lambda-1}{j}$$
$$\times \int_0^1 (-\log u)^r u^j (1+u)^{-\lambda-1} du.$$

Applying the Equation 4.272.14 provided in [44], we obtain the r-th raw moment of the $EG-UEHL(\theta,\lambda)$ random variable as

$$E(X^r) = \frac{2\lambda}{\theta^r} r! \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{(-1)^{j+k}}{(1+j+k)^{1+r}} \binom{\lambda-1}{j}\binom{\lambda+k}{k}.$$

2.2.2. Moment Generating Function

The moment generating function (MGF) offers an alternative approach for analytical characteristics instead of directly working with PDFs. Let X be an $EG-UEHL(\theta,\lambda)$ random variable with the PDF given by Equation (4). Then, the MGF of X

$$M_X(t) = \int_0^\infty e^{tx} f(x;\theta,\lambda)\,dx$$
$$= \int_0^\infty 2\lambda\theta \frac{e^{tx}}{e^{\theta x}-e^{-\theta x}}\left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^\lambda dx.$$

Using the transformation $u = e^{-\theta x}$, we write

$$M_X(t) = \int_0^1 \frac{2\lambda\theta u^{-\frac{t}{\theta}}}{1-u^2}\left(\frac{1-u}{1+u}\right)^\lambda du$$
$$= 2\lambda\theta \int_0^1 u^{-\frac{t}{\theta}} (1-u)^{\lambda-1}(1+u)^{-\lambda-1} du.$$

By applying Equation 3.197.3 provided in [44], we obtain the MGF as

$$M_X(t) = B\left(1-\frac{t}{\theta},\lambda\right)$$
$$\times {}_2F_1\left(\lambda+1,1-\frac{t}{\theta};1+\lambda-\frac{t}{\theta};-1\right)$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

is the beta function and

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n}\cdot\frac{z^n}{n!}$$

is the Gauss hypergeometric function and $(a)_n = a(a+1)\cdots(a+n+1)$.

2.2.3. Quantile Function

The quantile function of the $EG-UEHL(\theta,\lambda)$ distribution is given by

$$Q(u;\theta,\lambda) = \frac{1}{\theta}\log\left(\frac{1+u^{1/\lambda}}{1-u^{1/\lambda}}\right). \quad (6)$$

Equation (6) indicates that the median of the $EG-UEHL(\theta,\lambda)$ distribution is written as

$$Q(0.5;\theta,\lambda) = \frac{1}{\theta}\log\left(\frac{1+0.5^{1/\lambda}}{1-0.5^{1/\lambda}}\right). \quad (7)$$

Additionally, random numbers from the $EG-UEHL(\theta,\lambda)$ can be obtained by using the Equation (6) by the following algorithm:

S1. Generate a uniform random number from the interval $[0,1]$.
S2. Run the quantile function in Equation (6) on the uniform random number in S1.

2.2.3. Stress-Strength Reliability

Stress-strength reliability is denoted as R and is defined by the relation $R = P(Y < X)$, where Y and X represent the stress and strength random variables, respectively. The stress-strength reliability for the $EG-UEHL(\theta,\lambda)$ model is derived using Proposition 1.

Proposition 1. Given Y and X independent stress-strength random variables having $EG-UEHL(\theta,\lambda)$ distribution with parameters (θ,λ_1) and (θ,λ_2), respectively, the stress-strength reliability is obtained as

$$R = 2\lambda_2 \sum_{j=0}^\infty (-1)^j \left(\binom{\lambda_1+\lambda_2+j}{j} B(j+1,\lambda_1+\lambda_2)\right.$$
$$\left. - \binom{\lambda_2+j}{j} B(j+1,\lambda_2)\right)$$

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Proof:

By definition,

$$R = \int_0^\infty \left(1 - \left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\lambda_1}\right) \times \left\{\frac{2\lambda_2\theta}{e^{\theta x} - e^{-\theta x}}\left(\frac{1-e^{-\theta x}}{1+e^{-\theta x}}\right)^{\lambda_2}\right\} dx \tag{8}$$

By simplifications after substituting $u = e^{-\theta x}$ in Equation (8), we get

$$R = 2\lambda_2 \int_0^1 (1-u)^{\lambda_1+\lambda_2-1}(1+u)^{-\lambda_1-\lambda_2-1}\, du - 2\lambda_2 \int_0^1 (1-u)^{\lambda_2-1}(1+u)^{-\lambda_2-1}\, du \tag{9}$$

Utilizing the binomial expansion as outlined in Equation (9), the stress-strength reliability can be expressed as:

$$R = 2\lambda_2 \sum_{j=0}^{\infty} (-1)^j \left(\binom{\lambda_1+\lambda_2+j}{j} B(j+1, \lambda_1+\lambda_2) - \binom{\lambda_2+j}{j} B(j+1, \lambda_2) \right)$$

This concludes the proof.

2.2.3. Maximum Likelihood Estimation

Consider $X_1, X_2, \dots, X_n$ as a set of independent and identically distributed samples which are drawn from the $EG-UEHL(\theta,\lambda)$ distribution. The log-likelihood function of the parameters of the distribution is formulated as

$$\ell(\theta,\lambda) = n\log 2 + n\log\lambda + n\log\theta + \lambda \sum_{i=1}^{n} \log\left(\frac{1-e^{-\theta x_i}}{1+e^{-\theta x_i}}\right) - \sum_{i=1}^{n} \log\left(e^{\theta x_i} - e^{-\theta x_i}\right) \tag{10}$$

By taking the derivative of the log-likelihood function presented in Equation (10) concerning the parameters θ and λ, the log-likelihood equations are written, respectively, as

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \lambda \sum_{i=1}^{n} \left(\frac{2x_i e^{\theta x_i}}{e^{2\theta x_i} - 1}\right) - \sum_{i=1}^{n} \left(\frac{x_i\left(e^{2\theta x_i}+1\right)}{e^{2\theta x_i} - 1}\right) = 0, \tag{11}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(\frac{1-e^{-\theta x_i}}{1+e^{-\theta x_i}}\right) = 0. \tag{12}$$

Given that the likelihood equations (11) and (12) lack a closed-form solution, we must employ some iterative methods to obtain the maximum likelihood estimates (MLE) for the parameters of the $EG-UEHL(\theta,\lambda)$ distribution. In this study, the 'optim' function in R is utilized to compute these likelihood estimates.

3. Results

This section delves into the properties and efficacy of the proposed $EG-UEHL(\theta,\lambda)$ distribution, utilizing both a computer experiment and an analysis of real-world data.

3.1. Simulation Experiment

A simulation experiment was performed to examine the properties of the Maximum Likelihood Estimators (MLEs), which are discussed in detail in Section 2. Table 1 presents the sample size, n, along with different values of the distribution parameters θ and λ. It also includes the biases and Mean Squared Errors (MSE) of the parameter estimates, derived from 5000 repeated experiments. The data in Table 1 suggests that the MLEs display a positive bias. However, the MLEs of the parameters are asymptotically unbiased. Moreover, as anticipated, the MSEs of the MLEs of the parameters converge to zero as the sample size increases.

3.2. Real – World Dataset Application

In this section, we compare the performance of the $EG-UEHL(\theta,\lambda)$ distribution with the performance of some well-known distributions using a real-world dataset. The data for this study focuses on the maximum flood level of the Susquehanna River in Harrisburg, Pennsylvania. These levels are measured in millions of cubic feet per second (cfs) [45, 46], and it is called the flood data. The performance of the proposed distribution was compared with the Weibull, Beta, Kumaraswamy [35], $UEHL$ and $DUS-UEHL$ [41] distributions. The PDFs fot the examined distributions are presented below:

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Table 1. Bias and MSEs of MLEs for selected parameter values for the $EG-UEHL(\theta,\lambda)$ distribution.

			Bias		MSE	
θ	λ	n	θ	λ	θ	λ
2	2	50	0.05055	0.13872	0.07496	0.26700
		100	0.02098	0.05780	0.03504	0.10325
		200	0.01128	0.02667	0.01705	0.04482
		300	0.00687	0.01722	0.01108	0.02959
		500	0.00385	0.00889	0.00665	0.01745
0.5	2	50	0.01262	0.13861	0.00468	0.26697
		100	0.00525	0.05781	0.00219	0.10324
		200	0.00283	0.02674	0.00107	0.04481
		300	0.00169	0.01702	0.00069	0.02962
		500	0.00096	0.00891	0.00042	0.01746
2	1.5	50	0.05294	0.09102	0.08372	0.12490
		100	0.02197	0.03793	0.03916	0.04963
		200	0.01185	0.01738	0.01906	0.02184
		300	0.00715	0.01119	0.01237	0.01446
		500	0.00400	0.00567	0.00740	0.00854

Table 2. Parameter estimates, comparison criteria values and AD test results for the flood data.

	θ	λ	AIC	BIC	-2ℓ	AD (stat)	AD (p-value)
Weibull	3.5259	0.4689	−22.5280	−20.5365	−26.5280	0.8213	0.4643
Beta	6.7565	9.1110	−24.1245	−22.1330	−28.1245	0.7327	0.5302
Kumaraswamy	3.3632	11.7888	−21.7324	−19.7409	−25.7324	0.9321	0.3936
UEHL	3.5050	7.0242	−22.4045	−20.4131	−26.4045	0.8471	0.4467
DUS-UEHL	3.0336	6.6478	−21.7484	−19.7569	−25.7484	0.8648	0.4351
EG-UEHL	11.1194	30.2369	−28.2789	−26.2874	−32.2789	0.2941	0.9420

- Weibull distribution

$$f_{Weibull}(x;\theta,\lambda)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-(x/\lambda)^{\theta}},\quad \theta,\lambda>0$$

- Beta distribution

$$f_{Beta}(x;\theta,\lambda)=\frac{1}{B(\theta,\lambda)}x^{\theta-1}(1-x)^{\lambda-1},\quad \theta,\lambda>0$$

- Kumaraswamy distribution

$$f_{Kw}(x;\theta,\lambda)=\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda-1},\quad \theta,\lambda>0$$

- $DUS-UEHL$ distribution

$$f_{DUS-UEHL}(x)=\frac{1}{e-1}2\lambda\theta x^{\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{\left(1+x^{\theta}\right)^{\lambda+1}}e^{1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}},\quad \theta,\lambda>0.$$

We use the maximum likelihood method to estimate the parameters of all compared distributions. The goodness-of-fit for each model is evaluated using the Anderson-Darling test statistic (A-D (stat)) and its corresponding p-value (A-D (p-value)). To assess the models' relative performance, we consider the log-likelihood, Akaike information criterion (AIC), and Bayesian information criterion (BIC). The formulas for calculating AIC and BIC are presented below:

$$\text{AIC}=2m-2\ell(\theta,\lambda)\quad\text{and}\quad\text{BIC}=m\log n-2\ell(\theta,\lambda)$$

Here, m represents the number of parameters, n is the sample size of the dataset, and ℓ is the maximum value of the likelihood function for the respective distribution.

The maximum likelihood estimates, information criteria and goodness-of-fit results for all the models for flood data are given in Table 2. According to the A-D (stat) and A-D (p-value) in Table 2, the compared distributions offer a good fit for modeling the flood data. Table 2 shows that the $EG - UEHL(\theta, \lambda)$ model gives the smallest -2ℓ, AIC and BIC values. The $EG - UEHL(\theta, \lambda)$ is followed by the beta and Weibull distributions in terms of these criteria. Hence, the $EG - UEHL(\theta, \lambda)$ model stands out among its competitors due to its exceptional performance in modeling the flood data. Also, Figure 2 visually presents the empirical and fitted curves derived from the flood data.

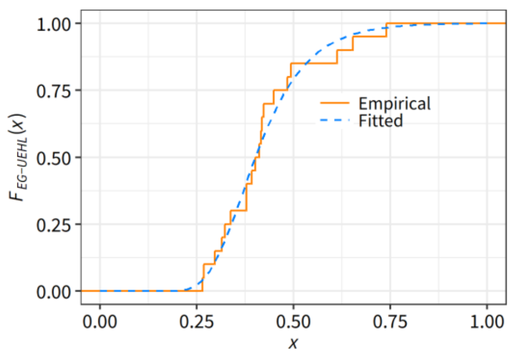

Figure 3. The distribution functions for the flood dataset (smooth: empirical, dashed: fitted)

4. Conclusion

This paper is based on the principle of enhancing the performance of a unit distribution through a G-family transformation. Particularly, the paper introduces a novel statistical distribution, the the $EG - UEHL(\theta, \lambda)$, built upon the G-family transformation. This new distribution exhibits enhanced functional capabilities and a strengthened mathematical structure. We delve into the distribution's key characteristics, including moments, the moment-generating function, the quantile function, and its relevance to stress-strength reliability analysis. These explorations offer a thorough understanding of the $EG - UEHL(\theta, \lambda)$ distribution, opening doors for both theoretical advancements and applications in various practical settings.

To investigate the behavior of the proposed distribution's parameter estimates, we conducted a computer simulation. Additionally, we compared the performance of the EG-UEHL model against established models using real-world data. The real-data analysis revealed that the EG-UEHL model outperforms the other models according to the AIC and BIC criteria.

Author's Contributions

Murat Genç presented the research idea, authored the manuscript, established the theoretical framework, and conducted the data analysis.

Ömer Özbilen analytically examined the structure, conducted a supporting role in data analysis, interpreted findings, and facilitated manuscript preparation.

Ethics

The authors have ensured that the manuscript adheres to all ethical considerations.

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