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Constructing Uninorms Based on Closure and Interior Operators on Bounded Lattices

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Sınırlı Kafeslerde Kapanış ve İç Operatörlere Dayanan Uninormların İnşaası

Gül Deniz ÇAYLI*,

Karadeniz Teknik Üniversitesi, Fen Fakültesi, Matematik Bölümü, Trabzon, Türkiye

Abstract
Uninorms generalizing triangular norms and triangular conorms on bounded lattices have attracted considerable attention recently. In this article, two new approaches are suggested to generate uninorms with an identity element on a bounded lattice. These approaches exploit the existences of a triangular norm (triangular conorm) and a closure operator (interior operator) on a bounded lattice. Meanwhile, two structures of idempotent uninorms on bounded lattices are obtained. In addition, the relationship between the proposed approaches and the existing constructions is investigated.

***Keywords:** Bounded lattice; Uninorm; Closure operator; Interior operator*

Öz
Sınırlı kafesler üzerinde üçgensel normları ve üçgensel konormları genelleştiren uninormlar son zamanlarda oldukça ilgi çekmiştir. Bu makalede bir sınırlı kafes üzerinde bir birim elemanlı uninormları üreten iki yeni yaklaşım önerilmektedir. Bu yaklaşımlar, bir sınırlı kafes üzerinde bir üçgensel normun (üçgensel konormun) ve bir kapanış operatörün (iç operatörün) varlıklarından yararlanmaktadır. Bu esnada, sınırlı kafesler üzerinde idempotent uninormların iki yapısı elde edilmektedir. Ayrıca, önerilen yaklaşımlar ve mevcut inşaalar arasındaki ilişki araştırılmaktadır.

***Anahtar Kelimeler:** Sınırlı kafes; Uninorm; Kapanış operatörü; İç opertatör*

1. Introduction

Menger (1942) developed the notions of triangular norms, also known as t-norms, and triangular conorms, also known as t-conorms. Their thorough investigations were conducted on the unit interval by Schweizer and Sklar (1963,1983). Additionally, they applied t-norms and t-conorms to expand the familiar triangle inequality on metric spaces to probabilistic metric spaces. Numerous fields have demonstrated the significance of t-norms and t-conorms, including fuzzy systems modeling, decision-making, probabilistic metric spaces, information aggregation, fuzzy set theory, fuzzy logic (Beliakov et al. 2007, Dubios and Prade 1995, 2000, Klement et al. 2000, 2004a, 2004b).Yager and Rybalov (1996) proposed uninorms on the unit interval $[0, 1]$, which constitute substantial expansions of t-norms and t-conorms. Fodor et al. (1997) performed an extensive research on them. As opposed to point 1 (the circumstance that exists for t-norms) or point 0 (the circumstance that exists for t-conorms), uninorms permit the position of their identity anywhere on the unit interval. The composition of these operators is closely associated with that of t-norms and t-conorms. This characteristic has been invaluable in theoretical (De Baets 1999, De Baets et al. 2009, Drewniak and Drygaś 2002) and practical examinations of uninorms, especially neural networks (Benítez 1997), fuzzy system modeling (Takács 2008, Yager 1994, 2001), image processing (González-Hidalgo et al. 2015), decision-making (Yager 2003).

Recent works address uninorms as a component of fuzzy logic and fuzzy set theory, substituting bounded lattices for the unit interval. Bounded lattices were included in the description of uninorms on the unit interval by Karaçal and Mesiar (2015). They also determined that on a bounded lattice, there are always the greatest and smallest uninorms. Hitherto, a great deal of examination has concentrated on uninorms, specifically on how to generate uninorms, on bounded lattices with more intricate framework than the unit interval. Some generation techniques for uninorms exploting t-norms and t-conorms were provided on a bounded lattice by Bodjanova and Kalina (2018, 2019). Afterward, two types of approaches were suggested by Çaylı et al. (2019) to acquire uninorms being internal and locally internal that have an identity on a bounded lattice. Idempotent uninorms were examined on bounded lattices structurally by Çaylı (2019). In further research, Dan et al. (2019) demonstrated on bounded lattices the availability of different forms of uninorms composed of t-norms and t-conorms. Two methods for receiving uninorms by way of

*Sorumlu Yazar/Corresponding Author: Gül Deniz Çaylı e-posta/e-mail: guldeniz.cayli@ktu.edu.tr

only one of the t-conorm and the t-norm were enhanced on a bounded lattice by Dan and Hu (2020). Numerous studies have been conducted about uninorms on bounded lattices (Çaylı 2019, 2020, 2021, He and Wang 2021, Hua and Ji 2022, Sun and Liu 2022, Zhao and Wu 2021).

Ouyang and Zhang (2020) suggested two ways to build uninorms that possess the identity $i \in T \setminus \{0,1\}$ through interior and closure operators on a bounded lattice T. Their techniques generated by only one of the t-conorm Ψ on $[i,1]^2$ and the t-norm Φ on $[0,i]^2$ can be seen as expansions of the findings from (Karaçal and Mesiar 2015). We describe two new techniques in this study that present uninorms possessing an identity i, provided that certain necessary and sufficient requirements are fulfilled. When examining the closure and interior operators $\delta, \sigma: T \to T$, we identify two forms of the uninorms $N_{(\delta)}$ and $N_{(\sigma)}$ by means of a t-conorm Ψ on $[i,1]^2$ or a t-norm Φ on $[0,i]^2$, respectively. As an efficient result, two forms of idempotent uninorms are presented on bounded lattices. We explore the correspondence between the uninorms developed by our techniques and those outlined in (Çaylı 2019, 2021, Zhao and Wu 2021). Specifically, when we permit the interior operator σ on T to be $\sigma(l) = l$ for all $l \in T$, the uninorm $N_{(\sigma)}$ matches up to the uninorms found in (Çaylı 2021) with the infimum t-norm on $[0,i]^2$. Furthermore, if we take account of the closure operator δ on T defined by $\delta(l) = l$ for all $l \in T$, the uninorm $N_{(\delta)}$ matches up to the uninorms in (Çaylı 2021) on the basis of supremum t-conorm on $[i,1]^2$. To further demonstrate that our methods do not need to match up to the established ones in (Çaylı 2019, 2021, Zhao and Wu 2021), we also provide a few illustrated cases. Interior and closure operators on bounded lattices are important tools for generating new forms of uninorms. Thereupon, from a mathematical standpoint, such forms of uninorms are highly fascinating analyses on bounded lattices.

This article is drawn up as follows for the remainder: Main characteristics and definition of uninorms on bounded lattices are given in Part 2. Part 3 indicates two innovative techniques for providing uninorms on a bounded lattice T, considering an additional criteria on the element $i \in T \setminus \{0,1\}$ that serves as an identity. These techniques utilize a t-conorm Ψ on $[i,1]^2$ and an interior operator on T, or a t-norm Φ on $[0,i]^2$ and a closure operator on T. Two forms of idempotent uninorms are acquired on bounded lattices instantaneously. Moreover, we discuss on how our approaches relate to the established ones in literature. To highlight the distinctions between our tools and the structures suggested in (Çaylı 2019, 2021, Zhao and Wu 2021), we correspondingly offer a few instances. The findings drawn from our discussion are reviewed in the concluding part.

2. Preliminaries

The basic ideas and findings of bounded lattices (for further detail, see, for example, (Birkhoff 1967)) and uninorms on them are reviewed in this part.A binary relation $\leq$ is an order relation if it is reflexive, antisymmetric and transitive. A nonempty set T with an order relation $\leq$ is said to be a poset that is written as $(T, \leq)$. For the elements $f, l \in T$, if $f \leq l$ and $f \neq l$, then the notation $f < l$ is used. If f and l are incomparable (that is neither $f \leq l$ nor $l < f$), the notation $f \parallel l$ is used. The set of all elements incomparable to f is denoted as H_f (that is $H_f = \{u \in T : f \parallel u\}$). For a subset A of T, the element $k \in A$ is said to be a gratest (resp. smallest) element of A when $u \leq k$ (resp. $u \leq k$) for all $u \in A$. If a poset $(T, \leq)$ has smallest (also known as bottom) and greatest (also known as top) elements, then it is said to be a bounded poset.

A poset $(T, \leq)$ is said to be a lattice if, for any two elements $f, l \in T$, they have a smallest upper bound (called join or supremum), written as $f \vee l$, and a greatest lower bound (called meet or infimum), written as $f \wedge l$. Unless otherwise indicated in this article, T represents a bounded lattice $(T, \leq, 0, 1)$ that possesses the bottom and the top elements, which are represented by 0 and 1, respectively.

Given the elements $f, l \in T$ satisfying that $f \leq l$, the subinterval $[f, l]$ of T is stated by $[f, l] = \{u \in T : f \leq u \leq l\}$.

Similarly, we can give the subintervals $[f, l[$, $]f, l]$, and $]f, l[$ of T. Notice that $([f, l], \leq)$ is a bounded lattice that possesses the top and bottom elements, represented by l and f, respectively.

Definition 2.1. (Çaylı et al. 2019, Karaçal and Mesiar 2015) A binary operation $N: T^2 \to T$ is called a uninorm if, for any $f, l, k \in T$, the requirements listed below are met:

(i) $N(l, f) = N(f, l)$ (commutativity);
(ii) If $l \leq f$, then $N(l, k) \leq N(f, k)$ (increasingness);
(iii) $N(l, N(f, k)) = N(N(l, f), k)$ (associativity);
(iv) An element $i \in T$ exists, called an identity, satisfying that $N(l, i) = l$ (identity).
In especial, a uninorm N becomes a t-conorm Ψ when $i = 0$ and a t-norm Φ when $i = 1$.

Example 2.1. (i) The greatest t-norm $\Phi^{\wedge}: [f, l]^2 \to [f, l]$ is delineated by $\Phi^{\wedge}(u, k) = u \wedge k$ for all $u, k \in [f, l]$. The fact remains that the smallest t-norm $\Phi^{W}: [f, l]^2 \to [f, l]$

gets the value of $u \wedge k$ if $l \in \{u, k\}$ and f otherwise. Accordingly, for any t-norm Φ on $[f, l]^2$, the inequality $\Phi^W \leq \Phi \leq \Phi^\wedge$ is obtained.
(ii) The smallest t-conorm $\Psi^\vee: [f, l]^2 \to [f, l]$ is delineated by $\Psi^\vee(u, k) = u \vee k$ for all $u, k \in [f, l]$. The fact remains that the greatest t-conorm $\Psi^W: [f, l]^2 \to [f, l]$ gets the value of $u \vee k$ if $f \in \{u, k\}$ and l otherwise. Accordingly, for any t-conorm Ψ on $[f, l]^2$, the inequality $\Psi^\vee \leq \Psi \leq \Psi^W$ is obtained.

Definition 2.2. (Drossos 1999, Drossos and Navara 1996, Everett 1994) A operation $\delta: T \to T$ is called a closure operator if, for any elements $f, l \in T$, the requirements listed below are met:
(i) $l \leq \delta(l)$ (expansion);
(ii) $\delta(l \vee f) = \delta(l) \vee \delta(f)$ (preservation of join);
(iii) $\delta(\delta(l)) = \delta(l)$ (idempotence).

Definition 2.3. (Drossos 1999, Drossos and Navara 1996, Everett 1994) A operation $\sigma: T \to T$ is called an interior operator if, for any elements $f, l \in T$, the requirements listed below are met:
(i) $\sigma(l) \leq l$ (contraction);
(ii) $\sigma(f \wedge l) = \sigma(f) \wedge \sigma(l)$ (preservation of meet);
(iii) $\sigma(\sigma(l)) = \sigma(l)$ (idempotence).

3. Construction techniques for uninorms

This part presents a novel technique for creating the uninorm $N_{(\delta)}$ on a bounded lattice T that possesses an identity i, as shown in Theorem 3.1. Notably, it makes use of both a closure operator $\delta: T \to T$ and a t-norm Φ: $[0, i]^2 \to [0, i]$. Additionally, we suggest an alternative approach in Theorem 3.10 for building uninorm $N_{(\sigma)}$ on a bounded lattice T that possesses an identity i. This approach uses the presences of an interior operator $\sigma: T \to T$ and a t-conorm Ψ: $[i, 1]^2 \to [i, 1]$.

Theorem 3.1. Assume that $i \in T \setminus \{0,1\}$ and Φ: $[0, i]^2 \to [0, i]$ is a t-norm. The undermentioned operation $N_{(\delta)}: T^2 \to T$ is a uninorm that possesses an identity i for every closure operator $\delta: T \to T$ iff $a > b$ and $d \vee a \in H_i \cup \{1\}$ for all $d, a \in H_i$ and $b \in [0, i[$.

$$N_{(\delta)}(f, l) = \begin{cases} \Phi(f, l) & \text{if } (f, l) \in [0, i]^2, \\ f \wedge l & \text{if } (f, l) \in [0, i[\times H_i \cup H_i \times [0, i[\\ & \quad \cup [0, i[\times [i, 1] \cup [i, 1] \times [0, i[, \\ f & \text{if } (f, l) \in (H_i \cup [i, 1]) \times \{i\}, \\ l & \text{if } (f, l) \in \{i\} \times (H_i \cup [i, 1]), \\ f \vee l & \text{if } (f, l) \in H_i \times H_i, \\ \delta(f) \vee \delta(l) & \text{otherwise.} \end{cases} \quad (1)$$

Proof: *Necessity:* Presume that the operation $N_{(\delta)}$ is a uninorm on T that possesses an identity i. We describe that $a > b$ for all $a \in H_i, b \in [0, i[$. Letting that there are some elements $a \in H_i$, $b \in]0, i[$ with $a \parallel b$, we receive that

$$\begin{aligned} N_{(\delta)}(b, N_{(\delta)}(a, 1)) &= N_{(\delta)}(b, \delta(a) \vee \delta(1)) \\ &= N_{(\delta)}(b, 1) \\ &= b \wedge 1 = b, \end{aligned} \quad (2)$$

and

$$\begin{aligned} N_{(\delta)}(N_{(\delta)}(b, a), 1) &= N_{(\delta)}(b \wedge a, 1) \\ &= b \wedge a \wedge 1 = b \wedge a. \end{aligned} \quad (3)$$

Since $a \parallel b$, the associativity feature of $N_{(\delta)}$ is contradicted. Therefore, $a > b$ for all $a \in H_i, b \in]0, i[$.
Now, we demonstrate $d \vee a \in H_i \cup \{1\}$ for all $d, a \in H_i$. Presume that there are some elements $d, a \in H_i$ with $i < d \vee a < 1$. In this case, for the closure operator $\delta: T \to T$ presented by $\delta(l) = 1$ for all $l \in T$, we get that

$$\begin{aligned} N_{(\delta)}(d, N_{(\delta)}(a, a)) &= N_{(\delta)}(d, a \vee a) \\ &= N_{(\delta)}(d, a) = d \vee a, \end{aligned} \quad (4)$$

and

$$\begin{aligned} N_{(\delta)}(N_{(\delta)}(d, a), a) &= N_{(\delta)}(d \vee a, a) \\ &= \delta(d \vee a) \vee \delta(a) \\ &= \delta(d \vee a) = 1. \end{aligned} \quad (5)$$

Then the associativity feature of $N_{(\delta)}$ is contradicted. Therefore, $d \vee a \in H_i \cup \{1\}$ for all $d, a \in H_i$.
Sufficiency: Presume that $a > b$ and $d \vee a \in H_i \cup \{1\}$ for all $d, a \in H_i$ and $b \in [0, i[$. We bring out that the operation $N_{(\delta)}$ is a uninorm on T that possesses an identity i. Clearly, $N_{(\delta)}$ is commutative and the element i is an identity of $N_{(\delta)}$. Hence, it remains to verify that $N_{(\delta)}$ is associative and increasing.
(i) Increasingness: We prove that, for all $f, l, k \in T$, $N_{(\delta)}(f, k) \leq N_{(\delta)}(l, k)$ if $f \leq l$. If $k = i$, then

$$\begin{aligned} N_{(\delta)}(f, k) &= N_{(\delta)}(f, i) = f \\ &\leq l = N_{(\delta)}(l, i) = N_{(\delta)}(l, k). \end{aligned} \quad (6)$$

If $(f, l) \in [0, i[^2 \cup \{i\}^2 \cup]i, 1]^2 \cup {H_i}^2$, the increasingness is obtained. Thence, we deal with all remaining possible cases.
(i-1) Let $f \in [0, i[$.

- $l = i$ and $k \in [0, i[$,

$$N_{(\delta)}(f, k) = \Phi(f, k) \leq k = N_{(\delta)}(i, k) = N_{(\delta)}(l, k). \quad (7)$$

- $l = i$ and $k \in]i, 1] \cup H_i$,

$$N_{(\delta)}(f, k) = f \wedge k \leq k = N_{(\delta)}(i, k) = N_{(\delta)}(l, k). \quad (8)$$

- $l \in]i, 1] \cup H_i$ and $k \in [0, i[$,

$$N_{(\delta)}(f, k) = \Phi(f, k) \leq l \wedge k = N_{(\delta)}(l, k). \quad (9)$$

- ($l \in]i, 1] \cup H_i$ and $k \in]i, 1]$) or ($l \in]i, 1]$ and $k \in H_i$),

$$N_{(\delta)}(f, k) = f \wedge k \leq \delta(l) \vee \delta(k) = N_{(\delta)}(l, k). \quad (10)$$

- $l, k \in H_i$,

$$N_{(\delta)}(f,k) = f \wedge k \leq l \vee k = N_{(\delta)}(l,k). \tag{11}$$

(i-2) Let $f = i$ and $l \in]i,1]$.

- $k \in [0,i[$,

$$N_{(\delta)}(f,k) = N_{(\delta)}(i,k) = k = l \wedge k = N_{(\delta)}(l,k). \tag{12}$$

- $k \in]i,1] \cup H_i$,

$$N(\delta)(f,k) = k \leq \delta(l) \vee \delta(k) = N_{(\delta)}(l,k). \tag{13}$$

(i-3) Let $f \in H_i$ and $l \in]i,1]$.

- $k \in [0,i[$,

$$N_{(\delta)}(f,k) = f \wedge k \leq l \wedge k = N_{(\delta)}(l,k). \tag{14}$$

- $k \in]i,1]$,

$$N_{(\delta)}(f,k) = \delta(f) \vee \delta(k) \leq \delta(l) \vee \delta(k) = N_{(\delta)}(l,k). \tag{15}$$

- $k \in H_i$,

$$N_{(\delta)}(f,k) = f \vee k \leq \delta(l) \vee \delta(k) = N_{(\delta)}(l,k). \tag{16}$$

(ii) Associativity: We prove that for all $f, l, k \in T$, $N_{(\delta)}(f, N_{(\delta)}(l,k)) = N_{(\delta)}(N_{(\delta)}(f,l),k)$.
The associativity holds if $i \in \{f,l,k\}$. Thence, we deal with all remaining possible cases.

(ii-1) Let $f \in [0,i[$.

- $l, k \in [0,i[$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, \Phi(l,k)) \\ &= \Phi\big(f, \Phi(l,k)\big) \\ &= N_{(\delta)}(\Phi(f,l),k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{17}$$

- $l \in [0,i[$ and $k \in]i,1] \cup H_i$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, l \wedge k) = N_{(\delta)}(f,l) \\ &= \Phi(f,l) = \Phi(f,l) \wedge k \\ &= N_{(\delta)}(\Phi(f,l),k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{18}$$

- $l \in]i,1] \cup H_i$ and $k \in [0,i[$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, l \wedge k) = N_{(\delta)}(f,k) \\ &= N_{(\delta)}(f \wedge l, k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{19}$$

- ($l \in]i,1] \cup H_i$ and $k \in]i,1]$) or ($l \in]i,1]$ and $k \in H_i$),

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, \delta(l) \vee \delta(k)) \\ &= f \wedge \big(\delta(l) \vee \delta(k)\big) = f \\ &= N_{(\delta)}(f,k) \\ &= N_{(\delta)}\big(N_{(\delta)}(f,l),k\big). \end{aligned} \tag{20}$$

- $l, k \in H_i$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, l \vee k) = f \wedge (l \vee k) \\ &= f = N_{(\delta)}(f,k) \\ &= N_{(\delta)}(f \wedge l, k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{21}$$

(ii-2) Let $f \in]i,1] \cup H_i$.

- $l, k \in [0,i[$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, \Phi(l,k)) \\ &= f \wedge \Phi(l,k) = \Phi(l,k) \\ &= N_{(\delta)}(l,k) = N_{(\delta)}(f \wedge l, k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{22}$$

- $l \in [0,i[$ and $k \in]i,1] \cup H_i$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, l \wedge k) \\ &= N_{(\delta)}(f,l) = l \\ &= N_{(\delta)}(l,k) = N_{(\delta)}(f \wedge l, k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{23}$$

(ii-3) Let $f \in H_i$ and $l \in]i,1]$.

- $k \in [0,i[$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, l \wedge k) \\ &= N_{(\delta)}(f,k) = k \\ &= N_{(\delta)}(\delta(f) \vee \delta(l), k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{24}$$

- $k \in]i,1] \cup H_i$,

$$\begin{aligned} N_{(\delta)}(f, N_{(\delta)}(l,k)) &= N_{(\delta)}(f, \delta\,(l) \vee \delta(k)) \\ &= \delta(f) \vee \delta(l) \vee \delta(k) \\ &= N_{(\delta)}(\delta\,(f) \vee \delta\,(l), k) \\ &= N_{(\delta)}(N_{(\delta)}(f,l),k). \end{aligned} \tag{25}$$

Similarly, for the cases $f \in]i,1]$ or $f, l \in H_i$, the associativity hold.

Thence, $N_{(\delta)}$ is a commutative, associative, and increasing operation on T that possesses an identity i. Consequently, $N_{(\delta)}$ is a uninorm on T.

If we delimitate the closure operator $\delta: T \to T$ by $\delta(l) = l$ for all $l \in T$, the structure that corresponds to the uninorm in Theorem 3.1 is as follows:

Corollary 3.2. Assume that $i \in T \setminus \{0,1\}$ and $\Phi: [0,i]^2 \to [0,i]$ is a t-norm. The undermentioned operation $N_{(\Phi)}: T^2 \to T$ is a uninorm that possesses an identity i iff $a > b$ for all $a \in H_i$ and $b \in [0,i[$.

$$N_{(\Phi)}(f,l) = \begin{cases} \Phi(f,l) & \text{if } (f,l) \in [0,i]^2, \\ f \wedge l & \text{if } (f,l) \in [0,i[\times H_i \cup H_i \times [0,i[\\ & \quad \cup [0,i[\times [i,1] \cup [i,1] \times [0,i[, \\ f & \text{if } (f,l) \in H_i \times \{i\}, \\ l & \text{if } (f,l) \in \{i\} \times H_i, \\ f \vee l & \text{otherwise}. \end{cases} \quad (26)$$

Take a note that the uninorm $N_{(\Phi)}: T^2 \to T$ in Corollary 3.2 is equivalent to the one introduced in Theorem 6 by Çaylı (2019). Hence, the uninorm $N_{(\delta)}: T^2 \to T$ in Theorem 3.1 encompasses, as a specific case, the uninorm constructed in Theorem 6 by Çaylı (2019). Moreover, when taking in Corollary 3.2 the t-norm $\Phi: [0,i]^2 \to [0,i]$ determined by $\Phi = \Phi^{\wedge}$, we get Corollary 3.3 that presents the appearance of an idempotent uninorm on bounded lattices.

Corollary 3.3. Assume that $i \in T \setminus \{0,1\}$. The undermentioned operation $N_{(\wedge)}: T^2 \to T$ is an idempotent uninorm that possesses an identity i iff $a > b$ for all $a \in H_i$ and $b \in [0,i[$.

$$N_{(\wedge)}(f,l) = \begin{cases} f \vee l & \text{if } (f,l) \in [i,1]^2 \cup H_i \times H_i \\ & \quad \cup]i,1] \times H_i \cup H_i \times]i,1], \\ f & \text{if } (f,l) \in H_i \times \{i\}, \\ l & \text{if } (f,l) \in \{i\} \times H_i, \\ f \wedge l & \text{otherwise}. \end{cases} \quad (27)$$

If we admit being an atom of the element $i \in T \setminus \{0,1\}$, the structure that corresponds to the uninorm in Corollary 3.2 is as follows:

Corollary 3.4 Assume that $i \in T \setminus \{0,1\}$ is an atom. The undermentioned operation $N_{(i,\wedge)}: T^2 \to T$ is an idempotent uninorm that possesses an identity i.

$$N_{(i,\wedge)}(f,l) = \begin{cases} f \vee l & \text{if } (f,l) \in [i,1]^2 \cup H_i \times H_i \\ & \quad \cup]i,1] \times H_i \cup H_i \times]i,1], \\ f & \text{if } (f,l) \in H_i \times \{i\}, \\ l & \text{if } (f,l) \in \{i\} \times H_i, \\ 0 & \text{otherwise}. \end{cases} \quad (28)$$

Remark 3.5. Assume that $i \in T \setminus \{0,1\}$, $\Psi: [i,1]^2 \to [i,1]$ is a t-conorm and $\delta: T \to T$ is a closure operator. We develop in Theorem 3.1 an innovative generation for uninorms on bounded lattices. Specifically,

(i) if we choose f and l from H_i, then our technique puts for $N_{(\delta)}(f,l)$ the output $f \vee l$ while the one described in Theorem 8 by Çaylı (2019) puts for $N(f,l)$ the output $\Psi(f \vee i, l \vee i)$. If $(f,l) \in]i,1]^2 \cup]i,1] \times H_i \cup H_i \times]i,1]$, in our technique $N_{(\delta)}(f,l) = \delta(f) \vee \delta(l)$ while in Theorem 8 determined by Çaylı (2019) $N(f,l) = \Psi(f \vee i, l \vee i)$. On the other hand, both techniques equal in the remainder domains;

(ii) if we choose f and l from H_i, then our technique puts for $N_{(\delta)}(f,l)$ the output $f \vee l$ while the one presented in Theorem 3.1 by Çaylı (2021) puts for $N(f,l)$ the output $\delta(f) \vee \delta(l)$. On the other hand, both techniques equal in the remainder domains;

(iii) if we choose f and l from H_i, then our technique puts for $N_{(\delta)}(f,l)$ the output $f \vee l$ while the one presented in Theorem 3.4 by Çaylı (2021) puts for $N(f,l)$ the output $\delta(f) \vee \delta(l)$. If $(f,l) \in]i,1]^2 \cup]i,1] \times H_i \cup H_i \times]i,1]$, in our technique $N_{(\delta)}(f,l) = \delta(f) \vee \delta(l)$ while in Theorem 3.4 stated by Çaylı (2021) $N(f,l) = \Psi(f,l)$ for $(f,l) \in]i,1]^2$ and $N(f,l) = f \vee l$ for $(f,l) \in]i,1] \times H_i \cup H_i \times]i,1]$. On the other hand, both techniques equal in the remainder domains;

(iv) if we choose f and l from H_i, then our technique puts for $N_{(\delta)}(f,l)$ the output $f \vee l$ while the one proposed in Proposition 3.5 by Zhao and Wu (2021) puts for $N(f,l)$ the output $\delta(f) \vee \delta(l)$. If $(f,l) \in]i,1]^2 \cup]i,1] \times H_i \cup H_i \times]i,1]$, in our technique $N_{(\delta)}(f,l) = \delta(f) \vee \delta(l)$ while in Proposition 3.5 represented by Zhao and Wu (2021) $N(f,l) = 1$. On the other hand, both techniques equal in the remainder domains.

Remark 3.6. Assume that $i \in T \setminus \{0,1\}$. If we specify the closure operator $\delta: T \to T$ by $\delta(l) = l$ for all $l \in T$, then the below-mentioned statements are obtained:

(i) $N_{(\delta)}$ fits the definition of the uninorm described in Theorem 3.1 by Çaylı (2021);
(ii) $N_{(\delta)}$ fits the definition of the uninorm presented in Theorem 3.4 by Çaylı (2021) if defining the t- conorm $\Psi: [i,1]^2 \to [i,1]$ such that $\Psi = \Psi^{\vee}$;
(iii) $N_{(\delta)}$ fits the definition of the uninorm proposed in Proposition 3.6 by Zhao and Wu (2021) if $l_1 \parallel l_2$ for all $l_1 \in [i,1[$, $l_2 \in H_i$ and the t-conorm $\Psi: [i,1]^2 \to [i,1]$ is defined by $\Psi = \Psi^{\vee}$.

Observably, the uninorm depicted by the structure in Theorem 3.1 does not have to match those that are delineated in (Çaylı 2019, 2021, Zhao and Wu 2021). We show this assertion in the below-mentioned examples.

Example 3.7. Take into consideration the lattice T_1 described by Hasse diagram in Figure 1 and the t-norm $\Phi: [0,i]^2 \to [0,i]$ represented by $\Phi = \Phi^{\wedge}$. Identify the closure operator $\delta: T_1 \to T_1$ by $\delta(0) = 0$, $\delta(i) = i$, $\delta(s) = \delta(u) = u$, $\delta(t) = t$, $\delta(n) = \delta(m) = m$, $\delta(p) = \delta(q) = q$ and $\delta(1) = 1$. The uninorm $N_{(\delta)}^1: T_1 \times T_1 \to T_1$ is presented in Table 1 with the help of the framework

established in Theorem 3.1. In that case, we obtain the statements listed below:
(i) $N^1_{(\delta)}$ fulfills that $N^1_{(\delta)}(n,m)=m$ and $N^1_{(\delta)}(n,n)=n$;
(ii) the uninorm $N^1: T_1\times T_1\to T_1$ obtained by the generation mean in Theorem 8 in (Çaylı 2019) fulfills that $N^1(n,m)=1$;
(iii) the uninorms $N^2, N^3: T_1\times T_1\to T_1$, respectively, built by techniques in Proposition 3.6 in (Zhao and Wu 2021) and Theorem 3.4 in (Çaylı 2021) fulfill that $N^2(n,n)=N^3(n,n)=m$.
Hence, $N^1_{(\delta)}$ differs from the uninorms N^1, N^2 and N^3 on T_1.

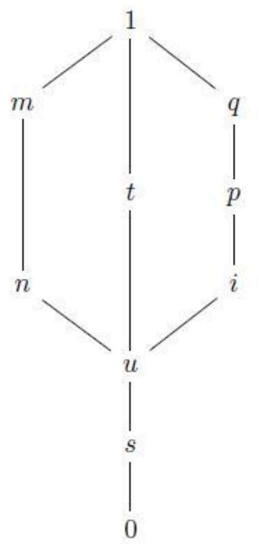

Figure 1. The lattice T_1

Table 1. Uninorm $N^1_{(\delta)}$ on T_1

$N^1_{(\delta)}$	0	s	u	i	t	n	m	p	q	1
0	0	0	0	0	0	0	0	0	0	0
s	0	s	s	s	s	s	s	s	s	s
u	0	s	u	u	u	u	u	u	u	u
i	0	s	u	i	t	n	m	p	q	1
t	0	s	u	t	t	1	1	1	1	1
n	0	s	u	n	1	n	m	1	1	1
m	0	s	u	m	1	m	m	1	1	1
p	0	s	u	p	1	1	1	q	q	1
q	0	s	u	q	1	1	1	q	q	1
1	0	s	u	1	1	1	1	1	1	1

Example 3.8. Take into consideration the lattice T_2 depicted by Hasse diagram in Figure 2. Determine the closure operator $\delta: T_2\to T_2$ by $\delta(0)=0$, $\delta(i)=\delta(m)=\delta(n)=n$, $\delta(p)=\delta(q)=\delta(s)=s$ and $\delta(1)=1$. The uninorm $N^2_{(\delta)}: T_2\times T_2\to T_2$ is presented in Table 2 with the help of the framework established in Theorem 3.1. In that case, we obtain the statements listed below:
(i) $N^2_{(\delta)}$ fulfills that $N^2_{(\delta)}(p,q)=q$;
(ii) the uninorms $N^4, N^5: T_2\times T_2\to T_2$, respectively, built by techniques in Theorem 3.4 in (Çaylı 2021) and Proposition 3.5 in (Zhao and Wu 2021) fulfill that $N^4(p,q)=N^5(p,q)=s$.
Hence, $N^2_{(\delta)}$ differs from the uninorms, N^4 and N^5 on T_2.

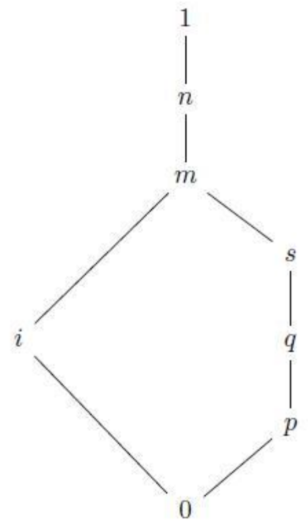

Figure 2. The lattice T_2

Table 2. Uninorm $N^2_{(\delta)}$ on T_2

$N^2_{(\delta)}$	0	i	p	q	s	m	n	1
0	0	0	0	0	0	0	0	0
i	0	i	p	q	s	m	n	1
p	0	p	p	q	s	n	n	1
q	0	q	q	q	s	n	n	1
s	0	s	s	s	s	n	n	1
m	0	m	n	n	n	n	n	1
n	0	n	n	n	n	n	n	1
1	0	1	1	1	1	1	1	1

Remark 3.9. The uninorm $N_{(\delta)}$ in Theorem 3.1 matches up to the t-conorm Ψ' on $[i,1]^2$ stated by

$$\Psi'(f,l)=\begin{cases}\delta(f)\vee\delta(l) & \text{if } (f,l)\in\,]i,1]^2,\\ f\vee l & \text{otherwise.}\end{cases} \tag{29}$$

Nevertheless, $N_{(\delta)}$ does not need to match up with any other t-conorm except Ψ' on $[i,1]^2$. To indicate this assertion, assume that the closure operator δ on the lattice T_2 in Figure 2 has the definition shown in Example 3.8 and the uninorm $N_{(\delta)}|[i,1]^2$ is the t-conorm $\Psi'': [i,1]^2\to[i,1]$ given in Table 3.

Table 3. T-conorm Ψ'' on $[i,1]^2$

Ψ''	i	m	n	1
i	i	m	n	1
m	m	m	1	1
n	n	1	1	1
1	1	1	1	1

Utilizing the structure manner described in Theorem 3.1, then we conclude that

$$\begin{aligned} N_{(\delta)}\left(\left(N_{(\delta)}(m,m)\right),q\right) &= N_{(\delta)}(\Psi''(m,m),q)\\ &= N_{(\delta)}(m,q)\\ &= \delta(m)\vee\delta(q)=n\vee s=n, \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
N_{(\delta)}\left(m,\left(N_{(\delta)}(m,q)\right)\right) &= N_{(\delta)}\big(m,\delta(m)\vee\delta(q)\big)\\
&= N(\delta)(m,n)\\
&= \Psi''(m,n)=1.
\end{aligned} \tag{31}
$$

It contradicts the associativity property of $N_{(\delta)}$. Consequently, $N_{(\delta)}$ does not need to match up with any t-conorm excluding Ψ' on $[i,1]^2$.

We develop in the below-mentioned Theorem 3.10 a dual generation process of uninorms on bounded lattices. Accordingly, we delineate the form of uninorm $N_{(\sigma)}$ that possesses an identity i on T, exploiting a t-conorm $\Psi\colon [i,1]^2 \to [i,1]$ and an interior operator σ on T.

Theorem 3.10. Assume that $i \in T \setminus \{0,1\}$ and $\Psi\colon [i,1]^2 \to [i,1]$ is a t-conorm. The undermentioned operation $N_{(\sigma)}\colon T^2 \to T$ is a uninorm that possesses an identity i for every interior operator $\sigma\colon T \to T$ iff $a < c$ and $d \wedge a \in H_i \cup \{0\}$ for all $d, a \in H_i, c \in]i,1]$.

$$
N_{(\sigma)}(f,l)=
\begin{cases}
\Psi(f,l) & \text{if } (f,l)\in[i,1]^2,\\
f\vee l & \text{if } (f,l)\in]i,1]\times H_i\cup H_i\times]i,1]\\
& \quad \cup]i,1]\times[0,i]\cup[0,i]\times]i,1],\\
f & \text{if } (f,l)\in (H_i\cup[0,i])\times\{i\},\\
l & \text{if } (f,l)\in \{i\}\times(H_i\cup[0,i]),\\
f\wedge l & \text{if } (f,l)\in H_i\times H_i,\\
\sigma(f)\wedge\sigma(l) & \text{otherwise.}
\end{cases} \tag{32}
$$

Proof: It is proved using similar circumstances in that of Theorem 3.1.

If we delimitate in Theorem 3.10 the interior operator $\sigma\colon T \to T$ by $\sigma(l) = l$ for all $l \in T$, the structure that corresponds to the uninorm in Theorem 3.10 is as follows:

Corollary 3.11. Assume that $i \in T \setminus \{0,1\}$ and $\Psi\colon [i,1]^2 \to [i,1]$ is a t-conorm. The undermentioned operation $N_{(\Psi)}\colon T^2 \to T$ is a uninorm that possesses an identity i iff $a < c$ for all $a \in H_i, c \in]i,1]$.

$$
N_{(\Psi)}(f,l)=
\begin{cases}
\Psi(f,l) & \text{if } (f,l)\in[i,1]^2,\\
f\vee l & \text{if } (f,l)\in]i,1]\times H_i\cup H_i\times]i,1]\\
& \quad \cup]i,1]\times[0,i]\cup[0,i]\times]i,1],\\
f & \text{if } (f,l)\in H_i\times\{i\},\\
l & \text{if } (f,l)\in \{i\}\times H_i,\\
f\wedge l & \text{otherwise.}
\end{cases} \tag{33}
$$

Take a note that the uninorm $N_{(\Psi)}\colon T^2 \to T$ in Corollary 3.11 is equivalent to the one introduced in Theorem 9 by Çaylı (2019). Hence, the uninorm $N_{(\sigma)}\colon T^2 \to T$ in Theorem 3.10 encompasses, as a special case, the uninorm constructed in Theorem 9 by Çaylı (2019) Furthermore, when taking in Corollary 3.11 the t-conorm $\Psi\colon [i,1]^2 \to [i,1]$ stated by $\Psi = \Psi^{\vee}$, we get Corollary 3.12 that presents the appearance of an idempotent uninorm on bounded lattices.

Corollary 3.12. Assume that $i \in T \setminus \{0,1\}$. The undermentioned operation $N_{(\vee)}\colon T^2 \to T$ is an idempotent uninorm that possesses an identity i iff $a < c$ for all $a \in H_i, c \in]i,1]$.

$$
N_{(\vee)}(f,l)=
\begin{cases}
f\wedge l & \text{if } (f,l)\in[0,i]^2\cup H_i\times H_i\\
& \quad \cup H_i\times[0,i[\,\cup[0,i[\times H_i,\\
f & \text{if } (f,l)\in H_i\times\{i\},\\
l & \text{if } (f,l)\in \{i\}\times H_i,\\
f\vee l & \text{otherwise.}
\end{cases} \tag{34}
$$

If we admit being a coatom of the element $i \in T \setminus \{0,1\}$, the structure that corresponds to the uninorm in Corollary 3.12 is as follows:

Corollary 3.13. Assume that $i \in T \setminus \{0,1\}$ is a coatom. The undermentioned operation $N_{(i,\vee)}\colon T^2 \to T$ is an idempotent uninorm that possesses an identity i.

$$
N_{(i,\vee)}(f,l)=
\begin{cases}
f\wedge l & \text{if } (f,l)\in[0,i]^2\cup H_i\times H_i\\
& \quad \cup H_i\times[0,i[\,\cup[0,i[\times H_i,\\
f & \text{if } (f,l)\in H_i\times\{i\},\\
l & \text{if } (f,l)\in \{i\}\times H_i,\\
1 & \text{otherwise.}
\end{cases} \tag{35}
$$

Remark 3.14. Assume that $i \in T \setminus \{0,1\}$, $\Phi\colon [0,i]^2 \to [0,i]$ is a t-norm and $\sigma\colon T \to T$ is an interior operator. For uninorms on bounded lattices, we formulate a novel generation procedure in Theorem. 3.10. Specifically,

(i) if we choose f and l from H_i, then our technique puts for $N_{(\sigma)}(f,l)$ the output $f \wedge l$ while the one described in Theorem 11 by Çaylı (2019) puts for $N(f,l)$ the output $\Phi(f \wedge i, l \wedge i)$. If $(f,l) \in [0,i[^2 \cup [0,i[\times H_i \cup H_i \times [0,i[$, in our technique $N_{(\sigma)}(f,l) = \sigma(f) \wedge \sigma(l)$ while in Theorem 11 provided by Çaylı (2019) $N(f,l) = \Phi(f \wedge i, l \wedge i)$. On the other hand, both techniques match up in the remainder domains;

(ii) if we choose f and l from H_i, then our technique puts for $N_{(\sigma)}(f,l)$ the output $f \wedge l$ while the one proposed in Theorem 3.10 by Çaylı (2021) puts for $N(f,l)$ the output $\sigma(f) \wedge \sigma(l)$. On the other hand, both techniques equal in the remainder domains;

(iii) if we choose f and l from H_i, then our technique puts for $N_{(\sigma)}(f,l)$ the output $f \wedge l$ while the one in Theorem 3.12 by Çaylı (2021) puts for $N(f,l)$ the output $\sigma(f) \wedge \sigma(l)$. If $(f,l) \in [0,i[^2 \cup [0,i[\times H_i \cup H_i \times [0,i[$, in our

technique $N(\sigma)(f,l) = \sigma(f) \wedge \sigma(l)$ while in Theorem 3.12 presented by Çaylı (2021) $N(f,l) = \Phi\,(f,l)$ for $(f,l) \in [0,i[^2$, and $N(f,l) = f \wedge l$ for $(f,l) \in [0,i[\times H_i \cup H_i \times [0,i[$. On the other hand, both techniques match up in the remainder domains;
(iv) if we choose f and l from H_i, then our technique puts for $N_{(\sigma)}(f,l)$ the output $f \wedge l$ while the one in Corollary 4.2 introduced by Zhao and Wu (2021) puts for $N(f,l)$ the output $\sigma(f) \wedge \sigma(l)$. If $(f,l) \in [0,i[^2 \cup [0,i[\times H_i \cup H_i \times [0,i[$, in our technique $N(\sigma)(f,l) = \sigma(f) \wedge \sigma(l)$ while in Corollary 4.2 introduced by Zhao and Wu (2021) $N(f,l) = 0$. On the other hand, both techniques equal in the remainder domains.

Remark 3.15. Let $i \in T \setminus \{0,1\}$. If we specify the interior operator $\sigma: T \to T$ by $\sigma(l) = l$ for all $l \in T$, we obtain the statements listed below:
(i) the uninorm $N_{(\sigma)}$ in Theorem 3.10 matches up to the uninorm presented in Theorem 3.10 by Çaylı (2021);
(ii) the uninorm $N_{(\sigma)}$ in Theorem 3.10 matches up to the uninorm provided in Theorem 3.12 by Çaylı (2021) when defining the t-norm $\Phi: [0,i]^2 \to [0,i]$ such that $\Phi = \Phi^{\wedge}$;
(iii) the uninorm $N_{(\sigma)}$ in Theorem 3.10 matches up to the uninorm stated in Corollary 4.4 by Zhao and Wu (2021) if $f_1 \parallel f_2$ for all $f_1 \in]0,i]$, $f_2 \in H_i$ and the t-norm $\Phi: [0,i]^2 \to [0,i]$ is defined by $\Phi = \Phi^{\wedge}$.

Analogously to Examples 3.7 and 3.8, we can illustrate that the uninorm established by Theorem 3.10 does not need to match up to those that are described in (Çaylı 2019, 2021, Zhao and Wu 2021).

4. Conclusions

Uninorms have been thoroughly explored on bounded lattices similar to the way their investigations on the unit interval. Constructing uninorms has emerged on bounded lattices as a fascinating field of research recently. In this article, a novel method was presented to create uninorms possessing an identity $i \in T \setminus \{0,1\}$ on a bounded lattice T that benefit from both a closure operator δ and a t-norm Φ on T. Subsequently, we developed a dual construction mean for uninorms on T with the underlying not only the interior operator σ but also the t-conorm Ψ on T. We acquired on bounded lattices two forms of idempotent uninorms as a consequence of these techniques. For a better comprehension of the newly developed uninorms, some illustrated examples were also presented. Moreover, we examined the relative advantages of our tools against various techniques previously outlined in (Çaylı 2019, 2021, Zhao and Wu 2021). We came to the conclusion that the approaches in this article do not have to match up to those found in the literature.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement
Yazar: Conceptualization, Methodology/Study design, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing–original draft, Writing–review and editing, Visualization, Supervision

Declaration of Competing Interest
The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement
All data generated or analyzed during this study are included in this published article.

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