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Research Article

The Hermite-Hadamard type inequalities for the functions whose derivative is logarithmic p-convex function

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Abstract

By means of an integral identity, several Hermite-Hadamard type inequalities are presented in this study for a function whose derivative's absolute value is the log-p-convex function. With the use of these findings, we are able to determine the boundaries in terms of elementary functions for certain specific functions, such as the imaginary error function, the exponential integral, the hyperbolic sine and cosine functions. Additionally, a relationship between beta function, the hyperbolic sine and cosine functions is stated. Through the obtained results, a bound for numerical integration of such type functions is provided.

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1. Introduction

The generalizations of convexity using various mathematical techniques has been the focus of a large number of authors recently [1, 6, 8, 10, 19–21]. The p-convex functions are among the most recent of these generalizations, whose origin extends back to p-normed spaces [19]. Based on this new generalized definition of convexity, log-p-convexity is defined [21]. Logarithmic convexity is one of the most studied types of convexity [14, 15]. The reason for this is the multitude of application areas of logarithmically convex functions. Studies in application areas such as geometric programming in optimization, structural stability issues in thermoelasticity theory, continuum mechanics, statistics, growth theory, and modeling of some inference paired multi-user systems in information theory have taken their places in the literature [2, 11, 12, 16, 17].

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The well-known Hermite-Hadamard inequality is one of the distinctive characteristics of convexity. Let us recall the Hermite-Hadamard inequality. Let f be real valued function defined on a real interval $[a, b]$. If f is convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

It provides a useful bound on the average value of a convex function over an interval, based on the values of the function at the endpoints, which can also be used to evaluate the accuracy of numerical approximations. For generalizations and applications of the Hermite-Hadamard inequality for some generalized convex functions, see [5, 7, 9, 13, 22, 23] and references therein.

In this paper, some Hermite-Hadamard type inequalities for the function whose derivative's absolute value is log-p-convex function via an integral identity are stated. Using these results, we obtain the bounds with respect to elementary functions for some special functions including imaginary error function, exponential integral, sinus and cosinus hyperbolic functions. Moreover, a relation between sinus and cosinus hyperbolic function and beta function are stated. A bound for numerical integration of such type functions are given via obtained results.

Let us give some essential definitions and results which will be used in the paper.

Definition 1.1 ([19])**.** Let $0 < p \leq 1$ and K be a subset of $\mathbb{R}^n$. K is said to be a p-convex set in $\mathbb{R}^n$ if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \in [0, 1]$ such that $\alpha^p + \beta^p = 1$.

Throughout the paper, unless otherwise stated, $K \subseteq \mathbb{R}^n$ is a p-convex set, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_{++} := (0, +\infty)$.

In the set of real numbers, any interval containing zero as interior or boundary point is a p-convex set [19].

Definition 1.2 ([19])**.** A function $f : K \to \mathbb{R}$ is called p-convex function if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x, y \in K$ and $\alpha, \beta \in [0, 1]$ such that $\alpha^p + \beta^p = 1$.

Definition 1.3 ([21])**.** The function $f : K \to \mathbb{R}_{++}$ is said to be logarithmic p-convex function if the function $log f$ is p-convex.

The logarithmic p-convex functions are denoted by *log*-p-convex functions for short.

The following property is clear from the definition, which will be used in obtaining results.

Theorem 1.4 ([21])**.** *The function $f : K \to \mathbb{R}_{++}$ is a log-p-convex function if and only if*

$$f(\alpha x + \beta y) \leq [f(x)]^\alpha [f(y)]^\beta$$

holds for all $x, y \in K$, $\alpha, \beta \in [0, 1]$ such that $\alpha^p + \beta^p = 1$.

The Bernoulli inequality is given below.

Theorem 1.5. *Let $x > -1$ and $t \in (0, 1]$. Then, $(1 + x)^t \leq 1 + tx$.*

The following auxiliary statements help us in elaboration of results. Although some of them are shown in proofs of the theorems [18] ambiguously, we need to reveal them explicitly and separately.

Lemma 1.6. *Let $a, b \in \mathbb{R}$ with $a < b$ and $p \in (0, 1)$. Suppose $g : [0, 1] \longrightarrow \mathbb{R}$ defined as follows*

$$g(x) = x^{\frac{1}{p}-1}b - (1 - x)^{\frac{1}{p}-1}a.$$

In case $a \cdot b < 0$, $|g|$ attains maximum values at $x_* = \left(1 + \left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$ for $p > \frac{1}{2}$. In other cases, $|g|$ attains maximum values at the points $x_* = 0$ or $x_* = 1$.

Proof. Because of the parameters a, b, p in $g(x)$ we investigate the function with respect to them case by case. From the first derivatives of g, it is shown below that g is either monotonic function or unimodal function, that is, a function which has only one extreme point x_* such that it is increasing on $[0, x_*]$ and decreasing on $[x_*, 1]$ or vice versa. (Note that you can easily visualize and try the cases in this proof via [3]).

Derivative of g is found as below

$$g'(x) = \left(\frac{1}{p} - 1\right)\left(x^{\frac{1}{p}-2}b + (1-x)^{\frac{1}{p}-2}a\right).$$

Case 1: $a < 0 < b$

It is trivial that $g(x) > 0$ for $x \in [0,1]$. g becomes unimodal function such that for $p < \frac{1}{2}$ and $\frac{1}{2} < p$, g has one minimum point and one maximum point, respectively. For $p = \frac{1}{2}$, g is an affine function.

Let us examine these three cases for p:

Let $p > \frac{1}{2}$. Then g has maximum value at $x_* = \left(1 + \left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$. So, $\max\{|g(x)|\} = g(x_*)$.

Let $p < \frac{1}{2}$. If $g'(x) = 0$, then $x = \left(1 + \left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$. Then g has minimum value at $x_* = \left(1 + \left(\frac{-b}{a}\right)^{\frac{p}{1-2p}}\right)^{-1}$ and $|g|$ takes maximum values at $x = 0$ or $x = 1$.

For $p = \frac{1}{2}$, $g'(x)$ is a constant, $|g|$ attains extremum values at $x = 0$ or $x = 1$.

Case 2: $a < b < 0$

$g'(x) < 0$. Thus g is decreasing function for all $p \in (0,1)$, so $|g|$ takes maximum values at $x = 0$ or $x = 1$, which equals $|g(0)| = -a$ or $|g(1)| = -b$.

Case 3: $0 < a < b$

Since $g'(x) > 0$, g is increasing function for all $p \in (0,1)$. So $|g|$ takes maximum values at $x = 1$, which equals $|g(1)| = b$.

Let $a = 0$. Then $b > 0$ and $g(x) = x^{\frac{1}{p}-1}b$. It takes maximum value at $x_* = 1$, thus making $|g(1)| = b$.

Let $b = 0$. From $a < 0$, $g(x) = -(1-x)^{\frac{1}{p}-1}a > 0$ so $g'(x) < 0$. Thus g takes maximum value at $x_* = 0$, thus making $|g(0)| = -a$.

In conclusion, we can say that $|g|$ takes maximum values at 0, 1 or $\left(\left(-\frac{b}{a}\right)^{\frac{p}{1-2p}} + 1\right)^{-1}$. □

Note that one can conclude from Lemma 1.6 that $|g|$ attains maximum values at the points $x_* = 0$, $x_* = 1$ or $x_* = \left(\left(-\frac{b}{a}\right)^{\frac{p}{1-2p}} + 1\right)^{-1}$, i.e. $|g|$ takes maximum values at one of these three points independent of the choice of a, b and p.

Lemma 1.7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $p \in (0,1]$. Suppose $h : [0,1] \longrightarrow \mathbb{R}$ defined as follows*

$$h(x) = a + b - 2(x^{\frac{1}{p}}b + (1-x)^{\frac{1}{p}}a).$$

In case $a \neq 0$, $|h|$ attains maximum values at $x = 0$, $x = 1$ or $x_* = \left(1 + \left(\left|\frac{b}{a}\right|\right)^{\frac{p}{1-p}}\right)^{-1}$ where $p \neq 1$. In other cases, $|h|$ attains maximum values at the points $x_* = 0$ or $x_* = 1$.

Proof. Likewise g in Lemma 1.6 in terms of the values of a, b, p, the function h is either monotonic function or unimodal function (you can track the cases visually in [4]).

The derivative of h is obtained

$$h'(x) = -\frac{2}{p}\left(x^{\frac{1}{p}-1}b - (1-x)^{\frac{1}{p}-1}a\right).$$

Suppose $a \neq 0$. Let us examine the three cases according to order relation of a and b

Case 1: $a < b < 0$

Checking the derivative of h, we get that the only root of the equation $h'(x) = 0$ is $x_* = \left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$ where $p \neq 1$. $h'(x)$ is negative on $(0, x_*)$ and $h'(x)$ is positive on $(x_*, 1)$. So it has minimum at x_*. So the maximum values of $|h|$ can be attained at $x = 0$, $x = 1$ or $x = x_*$, thereby making possible maximum values $b - a$ and $|h(x_*)|$.

Case 2: $a < 0 < b$

It is clear that h' is negative. So it is decreasing function, so $|h|$ takes maximum values at $x = 0$ or $x = 1$, which equals $|h(0)| = b - a$ or $|h(1)| = |a - b| = b - a$.

Case 3: $0 < a < b$

h' is negative on $(0, x_*)$ and h' is positive on $(x_*, 1)$. So it has minimum at x_*. So $|h|$ takes extremum values at $x = 0$ or $x = 1$ or $x_* = \left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$ where $p \neq 1$.

Let $a = 0$, thereby, $b > 0$. Then the function h defined as $h(x) = b(1-2x^{\frac{1}{p}})$ is decreasing function. So the maximum values of $|h|$ is attained at $x = 0$, $x = 1$, thereby making it b.

In conclusion, $|h|$ takes extremum values at the points $x = 0$ or $x = 1$ or $x = \left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$, thus taking the possible maximum values as b, $b-a$ or $\left(1+\left(\frac{b}{a}\right)^{\frac{p}{1-p}}\right)^{-1}$. □

2. Main results

By making use of the lemma below, we can suggest an upper bound for the right side of Hermite-Hadamard type inequality for *log-p*-convex functions:

Lemma 2.1 ([18]). *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If f' is integrable on $\mathbb{R}$ then*

$$\begin{aligned}\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2p(a-b)}\int_0^1 &\left[a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right] \\ &\times f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\left[t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right]dt\end{aligned}$$

holds.

Theorem 2.2. *Let $a, b \in \mathbb{R}_+$ with $a < b$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable function such that $|f'|$ is integrable log-p-convex function on $\mathbb{R}_+$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{3}{2p(b-a)}\left(|a|+|b|\right)^2 |f'(a)|\,|f'(b)|$$

holds.

Proof. By means of Lemma 2.1, the *log-p*-convexity of $|f'|$, triangle inequality and the Bernoulli inequality, one has

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{1}{2p(b-a)}\int_0^1 \left|a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|\left|t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a\right| \\ &\qquad\times\left|f'(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)\right|dt\end{aligned}$$

$$\leq \frac{1}{2p(b-a)} \int_0^1 \left| a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a) \right| \left| t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a \right|$$
$$\times \left[|f'(b)| \right]^{t^{\frac{1}{p}}} \left[|f'(a)| \right]^{(1-t)^{\frac{1}{p}}} dt$$
$$\leq \frac{1}{2p(b-a)} \int_0^1 \left| a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a) \right| \left| t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a \right|$$
$$\times \left[1+t^{\frac{1}{p}}(|f'(b)|-1) \right] \left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1) \right] dt$$
$$\leq \frac{3}{2p(b-a)} (|a|+|b|)(|a|+|b|) \int_0^1 \left[1+t^{\frac{1}{p}}(|f'(b)|-1) \right]$$
$$\times \left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1) \right] dt.$$
$$\leq \frac{3}{2p(b-a)} (|a|+|b|)^2 \left[1+(|f'(a)|-1) \right] \left[1+(|f'(b)|-1) \right].$$

□

The following two theorems make the results independent of the choice of the positive a and b in Theorem 2.2.

Theorem 2.3. *Let $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable function satisfying the condition that $|f'|$ is integrable log-p-convex function on $\mathbb{R}$. Then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \frac{1}{2p(b-a)} \max\{|a|, |b|, |g(t_1)|\} \max\{b-a, |b|, |h(t_2)|\} |f'(a)| |f'(b)|$$

holds, where

$$g(t) = t^{\frac{1}{p}-1}b-(1-t)^{\frac{1}{p}-1}a \quad \text{and} \quad h(t) = a+b-2(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)$$

and for $a \neq 0$,

$$t_1 = \left(1+\left(\left| \frac{b}{a} \right| \right)^{\frac{p}{1-2p}} \right)^{-1} \quad \text{when} \quad p \neq \frac{1}{2}$$

and

$$t_2 = \left(1+\left(\left| \frac{b}{a} \right| \right)^{\frac{p}{1-p}} \right)^{-1} \quad \text{when} \quad p \neq 1$$

for $a = 0$, t_1 and t_2 equal to 0 or 1.

Proof. Using Lemma 2.1 in a similar way to the proof of Theorem 2.2, one has

$$\left| \tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \tfrac{1}{2p(b-a)} \int_0^1 |h(t)g(t)| \left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1) \right] \left[1+t^{\frac{1}{p}}(|f'(b)|-1) \right] dt.$$

From Lemma 1.6 and Lemma 1.7

$$|g(t)| \leq \max\{|a|, |b|, |g(t_1)|\} \text{ and } |h(t)| \leq \max\{b-a, |b|, |h(t_2)|\}$$

is derived. The *log-p*-convexity of $|f'|$ and the Bernoulli inequality are used in a manner similar to how they were used in the proof of Theorem 2.2 to produce the desired result. □

Theorem 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function satisfying the condition that $|f'|$ is integrable log-p-convex function on $\mathbb{R}$. Then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|$$

$$
\begin{aligned}
&\le \frac{1}{2p^2(b-a)}\left(|a|+|b|\right)^2+\frac{1}{2(b-a)}\left(a^2+b^2\right)\\
&+\frac{1}{12(b-a)}\Big[3\left(|ab|+b^2\right)\left(|f'(b)|-1\right)+3\left(|ab|+a^2\right)\left(|f'(a)|-1\right)\\
&+4b^2(|f'(b)|-1)+4a^2(|f'(a)|-1)\Big]\\
&+\frac{1}{4p(b-a)}\beta(\frac{1}{p},\frac{1}{p})\left[(|ab|+a^2)(|f'(b)|-1)+\left(|ab|+b^2\right)\left(|f'(a)|-1\right)+4\,|ab|\right]\\
&+\frac{1}{6p(b-a)}\beta(\frac{2}{p},\frac{1}{p})\Big[2(3\,|ab|+a^2)(|f'(b)|-1)+2\left(3\,|ab|+b^2\right)\left(|f'(a)|-1\right)\\
&+\left(|a|+|b|\right)^2\left(|f'(a)|-1\right)(|f'(b)|-1)\Big]\\
&+\frac{1}{p(b-a)}\beta(\frac{2}{p},\frac{2}{p})\,|ab|\left(|f'(b)|-1\right)(|f'(a)|-1)\\
&+\frac{1}{4p(b-a)}\beta(\frac{3}{p},\frac{1}{p})\left(a^2+b^2\right)\left(|f'(b)|-1\right)(|f'(a)|-1)
\end{aligned}
$$

holds, where β denotes the beta function.

Proof. Assume g and h as in Theorem 2.3. One has

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le \frac{1}{2p(b-a)}\int_0^1 |h(t)g(t)|\left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1)\right]\left[1+t^{\frac{1}{p}}(|f'(b)|-1)\right]dt.
\end{aligned}
$$

By making use of triangle inequality,

$$
\begin{aligned}
|h(t)g(t)| &= \Big|(ab+b^2)t^{\frac{1}{p}-1}-(a^2+ab)(1-t)^{\frac{1}{p}-1}-2b^2t^{\frac{2}{p}-1}\\
&+2ab(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}-t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}\Big|\\
&\le\ \left(|ab|+b^2\right)t^{\frac{1}{p}-1}+\left(a^2+|ab|\right)(1-t)^{\frac{1}{p}-1}+2b^2t^{\frac{2}{p}-1}\\
&+2\,|ab|\,(t^{\frac{1}{p}}(1-t)^{\frac{1}{p}-1}+t^{\frac{1}{p}-1}(1-t)^{\frac{1}{p}})+2a^2(1-t)^{\frac{2}{p}-1}. \qquad (2.1)
\end{aligned}
$$

Multiplying (2.1) by $\left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1)\right]\left[1+t^{\frac{1}{p}}(|f'(b)|-1)\right]$, then expanding and integrating on [0,1] with regard to t, one has

$$\begin{aligned}
&\int_0^1 |h(t)g(t)|\left[1+(1-t)^{\frac{1}{p}}(|f'(a)|-1)\right]\left[1+t^{\frac{1}{p}}(|f'(b)|-1)\right]dt\\
&\leq \frac{1}{p}\left(|a|+|b|\right)^2+p\left(a^2+b^2\right)\\
&+\frac{p}{6}\left[3\left(|ab|+b^2\right)(|f'(b)|-1)+3\left(|ab|+a^2\right)(|f'(a)|-1)\right.\\
&\left.+4b^2(|f'(b)|-1)+4a^2(|f'(a)|-1)\right]\\
&+\frac{1}{2}\beta(\frac{1}{p},\frac{1}{p})\left[(|ab|+a^2)(|f'(b)|-1)+\left(|ab|+b^2\right)(|f'(a)|-1)+4\,|ab|\right]\\
&+\frac{1}{3}\beta(\frac{2}{p},\frac{1}{p})\left[2(3\,|ab|+a^2)(|f'(b)|-1)+2\left(3\,|ab|+b^2\right)(|f'(a)|-1)\right.\\
&\left.+\left(|a|+|b|\right)^2(|f'(a)|-1)(|f'(b)|-1)\right]\\
&+2\beta(\frac{2}{p},\frac{2}{p})\,|ab|\,(|f'(b)|-1)(|f'(a)|-1)\\
&+\frac{1}{2}\beta(\frac{3}{p},\frac{1}{p})\left(a^2+b^2\right)(|f'(b)|-1)(|f'(a)|-1).
\end{aligned}$$

□

3. Applications

Proposition 3.1. *Let $0 \leq a < b$. Then*

$$\operatorname{erfi}(b)-\operatorname{erfi}(a) \leq \frac{1}{\sqrt{\pi}}\left[6(a+b)e^{a^2+b^2}+2\frac{(e^{b^2}-e^{a^2})}{a+b}\right]$$

where erfi *denotes the imaginary error function, i.e.*

$$\operatorname{erfi}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}dt.$$

Proof. Let $f(x)=\frac{\sqrt{\pi}}{2}\operatorname{erfi}(x)$ on $\mathbb{R}_+$ in Theorem 2.2, whose derivative is e^{x^2} and whose indefinite integral is $\frac{1}{2}\left(\sqrt{\pi}x\operatorname{erfi}(x)-e^{x^2}\right)$. We have

$$\begin{aligned}
&\left|\frac{\sqrt{\pi}}{2}\frac{\operatorname{erfi}(a)+\operatorname{erfi}(b)}{2}-\frac{1}{b-a}\left[\frac{1}{2}\left(\sqrt{\pi}b\operatorname{erfi}(b)-e^{b^2}-\sqrt{\pi}a\operatorname{erfi}(a)+e^{a^2}\right)\right]\right|\\
&\leq \frac{3}{2p(b-a)}\left(a+b\right)^2e^{a^2+b^2}.
\end{aligned}$$

Since it is valid for all p, the maximum value of left term of inequality in terms of p is attained for $p=1$. Using that erfi and e^{x^2} is increasing function on $\mathbb{R}_+$ and making some easy algebraic manipulations, one has

$$\sqrt{\pi}\left(a+b\right)\left(\operatorname{erfi}(b)-\operatorname{erfi}(a)\right)\leq 6\left(a+b\right)^2e^{a^2+b^2}+2(e^{b^2}-e^{a^2}).$$

The required result is obtained from this inequality. □

Proposition 3.2. *Let $x \in (0,1]$. Then*

$$\left|\cosh x-\frac{1}{x}\sinh x\right| \leq \begin{cases} \dfrac{1}{2} & , \quad 0 < x \leq \frac{1}{2} \\[2ex] 2^{1-\frac{1}{x}} & , \quad \frac{1}{2} < x < 1. \end{cases}$$

Proof. In Theorem 2.3, consider $f(x) = e^x$ on $[a, b] = [-p, p]$ where $[-p, p]$ with $p \in (0, 1)$ is p-convex set and $|f'|$ is *log*-p-convex on this interval. Let $p < \frac{1}{2}$. Then $t_1 = \frac{1}{2}$ and $\max\{|a|, |b|, |g(t_1)|\} = \max\{p, 2^{2-\frac{1}{p}}p\} = p$. Also $\max\{b-a, |b|, |h(t_2)|\} = \max\{2p, p, 0\} = 2p$. Thus

$$\left|\cosh p - \frac{1}{p}\sinh p\right| \leq \frac{1}{2}.$$

Let $p > \frac{1}{2}$. Then $t_1 = \frac{1}{2}$ and $\max\{|a|, |b|, |g(t_1)|\} = \max\{p, 2^{2-\frac{1}{p}}p\} = 2^{2-\frac{1}{p}}p$. Also $\max\{b-a, |b|, |h(t_2)|\} = \max\{2p, p, 0\} = 2p$. Thus

$$\left|\cosh p - \frac{1}{p}\sinh p\right| \leq 2^{1-\frac{1}{p}}.$$

Making substitution $p = x$ yields to the required inequality. □

Proposition 3.3. *Let $x > 1$. Then*

$$\mathrm{Ei}(1) + \left(\frac{3}{2x} - e\right)e^x + e\left(e + \ln x - \frac{1}{2x}\right) \leq \mathrm{Ei}(x)$$

where

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}dt,$$

which is defined in terms of Cauchy principal value.

Proof. In Theorem 2.3, consider $f(x) = e^x$ on real numbers and $[a, b] = [1, t]$ with $t \in [1, \infty)$ and $|f'|$ is *log*-p-convex on real number. Let $p = \frac{1}{2}$. Then $\max\{1, t\} = t$. Also $\max\{t-1, t, \frac{1+t^2}{1+t}\} = t$. Thus

$$\left|\frac{e+e^t}{2} - \frac{e^t - e}{t-1}\right| \leq \frac{t^2 e^{t+1}}{t-1}.$$

So

$$\frac{e+e^t}{2}\frac{(t-1)}{t^2} - \frac{e^t}{t^2} + \frac{e}{t^2} \leq e^{t+1}.$$

Integrating both side on $[1, x]$ with respect to t, we have

$$\mathrm{Ei}(1) - \mathrm{Ei}(x) + \frac{3e^x}{2x} - e + \frac{e\ln x}{2} - \frac{e}{2x} \leq e^{x+1} - e^2.$$

Thus it yields to required inequality. □

Proposition 3.4. *Let $x \geq 1$. Then*

$$\begin{aligned}&\left(1 - \frac{5}{6x}\right)\cosh\frac{1}{x} - x(1 + \sinh\frac{1}{x}) + \frac{1}{3x}\\ &\qquad\leq \frac{1}{4}(1 - \cosh\frac{1}{x})\left(2B(2x, 2x) + B(3x, x)\right) + \frac{1}{2}\cosh\frac{1}{x}B(x, x).\end{aligned}$$

Proof. In Theorem 2.4, consider $f(x) = e^x$ on $[-p, p]$ where $[-p, p]$ is p-convex set. We have

$$\begin{aligned}\frac{e^p + e^{-p}}{2} - \frac{e^p - e^{-p}}{2p} &\leq \frac{1}{p} + \frac{p}{2} + \frac{1}{24}10p\left(e^p + e^{-p} - 2\right)\\ &+ \frac{1}{4}B(\frac{1}{p}, \frac{1}{p})\left[e^p + e^{-p}\right] + \frac{1}{3}B(\frac{2}{p}, \frac{1}{p})\left[e^p + e^{-p} - 2\right]\\ &- \left[\frac{1}{2}B(\frac{2}{p}, \frac{2}{p}) + \frac{1}{4}B(\frac{3}{p}, \frac{1}{p})\right]\left[e^p + e^{-p} - 2\right].\end{aligned}$$

Using $\sinh p = \frac{e^p - e^{-p}}{2}$ and $\cosh p = \frac{e^p + e^{-p}}{2}$, then making the replacement $x = \frac{1}{p}$ and some easy algebraic manipulations, one can get the result. □

Using the composite trapezoid rule and some of the results mentioned above, we can also determine an upper bound on numerical integration error for functions whose absolute first derivative values are *log-p*-convex.

Assume that P is a partition of $[a,b]$ and f is $L^1[a,b]$, i.e. $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $\Delta x_{i+1} = x_{i+1} - x_i$. Then

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} \Delta x_{k+1} + E(f,P) \tag{3.1}$$

where $E(f,P)$ is the error of integral regard to P. For *log-p*-convex functions, we present an upper bound for $E(f,P)$.

Proposition 3.5. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying the condition that $|f'|$ is integrable log-p-convex function on $\mathbb{R}$. Consider that P is a partition of $[a,b]$ as denoted before. Then*

$$|E(f,P)| \leq \frac{3}{2p} \sum_{k=0}^{n-1} (|x_k| + |x_{k+1}|)^2 \left|f'(x_k)\right| \left|f'(x_{k+1})\right|.$$

Proof. Taking account of Theorem 2.2 on $[x_k, x_{k+1}]$, one can write

$$\begin{aligned} &\left| \frac{f(x_k) + f(x_{k+1})}{2} - \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(x)dx \right| \\ &\leq \frac{3}{2p(x_{k+1} - x_k)} (|x_k| + |x_{k+1}|)^2 \left|f'(x_k)\right|, \left|f'(x_{k+1})\right|. \end{aligned} \tag{3.2}$$

By (3.1),

$$\begin{aligned} |E(f,P)| &= \left| \sum_{k=0}^{n-1} \frac{f(x_k) + f(x_{k+1})}{2} \Delta x_{k+1} - \int_a^b f(x)dx \right| \\ &= \left| \sum_{k=0}^{n-1} \left(\frac{f(x_k) + f(x_{k+1})}{2} \Delta x_{k+1} - \int_{x_k}^{x_{k+1}} f(x)dx \right) \right| \\ &\leq \sum_{k=0}^{n-1} \left| \frac{f(x_k) + f(x_{k+1})}{2} \Delta x_{k+1} - \int_{x_k}^{x_{k+1}} f(x)dx \right| \\ &= \sum_{k=0}^{n-1} \Delta x_{k+1} \left| \frac{f(x_k) + f(x_{k+1})}{2} - \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(x)dx \right|. \end{aligned} \tag{3.3}$$

Thus using (3.2) in (3.3), we have the result. □

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