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Non-Newtonian Pell and Pell-Lucas Numbers

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Abstract — In the present paper, we introduce a new type of Pell and Pell-Lucas numbers in terms of non-Newtonian calculus, which we call non-Newtonian Pell and non-Newtonian Pell-Lucas numbers, respectively. In non-Newtonian calculus, we study some significant identities and formulas for classical Pell and Pell-Lucas numbers. Therefore, we derive some relations with non-Newtonian Pell and Pell-Lucas numbers. Furthermore, we investigate some properties of non-Newtonian Pell and Pell-Lucas numbers, including Catalan-like identities, Cassini-like identities, Binet-like formulas, and generating functions.

Subject Classification (2020): 11B83, 11U10

1. Introduction

Between 1967 and 1970, Grossman and Katz [1] introduced novel definitions for derivate and integral, which involved replacing addition and subtraction with multiplication and division. This development led to the establishment of a new family of calculi called non-Newtonian calculus [1, 2]. There exists an infinite number of non-Newtonian calculi. Examples of such calculi include geometric calculus, bigeometric calculus, anageometric calculus, harmonic calculus, and quadratic calculus [2–4]. The non-Newtonian calculi offer alternative approaches to the classical calculus of Newton and Leibniz. Every property in classical calculus has a corresponding property in non-Newtonian calculus, allowing for a different approach to problems that can be solved using appropriate calculus.

The non-Newtonian calculi provide diverse mathematical tools and are widely used in various fields, including science, mathematics, and engineering. The use of non-Newtonian calculus has numerous practical applications in multiple areas, such as quantum calculus, functional analysis, complex analysis, fractal geometry, differential equations, calculus of variations, image analysis, signal processing, and economics, for instance, see [5–21].

On the other hand, there has been significant attention in the literature towards examining integer sequences and their utilization in different scientific fields. Many researchers have been interested in integer sequences, particularly Fibonacci and Lucas sequences [22–24]. Following the Fibonacci and Lucas sequences, the Pell and Pell-Lucas sequences [25–27] have been extensively researched - several studies in the literature concern the Fibonacci, Lucas, Pell, and Pell-Lucas sequences. In many of

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the studies, it is observed that these sequences have been examined from different perspectives, for instance, see [28–41]. In particular, these sequences have been generalized by many authors in many ways as seen in [28–31, 33, 34, 37, 41]. Some of these generalizations have been done by changing the initial conditions, some by changing the recurrence relations, and others by changing both the initial conditions and recurrence relations. However, more recently, Değirmen and Duyar [42] have generalized the Fibonacci and Lucas numbers in a different way. The authors have combined the Fibonacci numbers, Lucas numbers and non-Newtonian real numbers, and introduced a new version of Fibonacci and Lucas numbers, called non-Newtonian Fibonacci and non-Newtonian Lucas numbers. Moreover, they have examined some properties of these numbers.

This paper extends the ideas explored in [42] to Pell and Pell-Lucas numbers. In a word, the main objective of this paper is to introduce and study non-Newtonian Pell and non-Newtonian Pell-Lucas numbers. In order to accomplish this, it is necessary to have some prior understanding of non-Newtonian calculus and Pell and Pell-Lucas numbers.

2. Preliminaries

This section provides some basic notions to be needed for the following section. Arithmetic refers to a system that meets all the axioms of an ordered field, with its domain being a subset of the real numbers. A generator α is a function that maps real numbers to a subset of real numbers in a one-to-one manner. $\mathbb{R}_\alpha$ symbolizes the range of generator α. Here, $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$ is a set of non-Newtonian real numbers. As a generator, if $\alpha = I$, where I is the identity function whose inverse is itself, then α generates the classical arithmetic. As a generator, if $\alpha = \exp$ (exponential function), where, for $\mathbb{R}_{\exp} = \mathbb{R}^+$,

$$\alpha : \mathbb{R} \to \mathbb{R}_{\exp}, \quad \alpha(x) = e^x = y$$

and

$$\alpha^{-1} : \mathbb{R}_{\exp} \to \mathbb{R}, \quad \alpha^{-1}(y) = \ln y = x$$

then α generates the geometric arithmetic [1].

Let α be a generator with range $A = \mathbb{R}_\alpha$, which is a subset of the real numbers. For any generator α, when we refer to α-arithmetic, we mean the arithmetic whose domain is A and whose operations are defined as follows:

$$\begin{array}{ll}
\alpha-\text{addition} & x\dot{+}y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\} \\
\alpha-\text{subtraction} & x\dot{-}y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\} \\
\alpha-\text{multiplication} & x\dot{\times}y = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(y)\right\} \\
\alpha-\text{division} & x\dot{/}y = \frac{x}{y}_\alpha = \alpha\left\{\dfrac{\alpha^{-1}(x)}{\alpha^{-1}(y)}\right\}, (y \neq \dot{0}) \\
\alpha-\text{order} & x\dot{<}y \iff \alpha^{-1}(x) < \alpha^{-1}(y)
\end{array}$$

where $x, y \in A = \mathbb{R}_\alpha$ [1]. For $\alpha = \exp$, α-arithmetic reduces to the geometric arithmetic:

$$\begin{array}{ll}
\text{geometric addition} & x\dot{+}y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\} = e^{\{\ln x + \ln y\}} = x.y \\
\text{geometric subtraction} & x\dot{-}y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\} = e^{\{\ln x - \ln y\}} = \dfrac{x}{y}, (y \neq 0)
\end{array}$$

geometric multiplication $\quad x\dot{\times}y = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(y)\right\} = e^{\{\ln x \times \ln y\}} = x^{\ln y} = y^{\ln x}$

geometric division $\quad x\dot{/}y = \frac{x}{y}_\alpha = \alpha\left\{\frac{\alpha^{-1}(x)}{\alpha^{-1}(y)}\right\} = e^{\left\{\frac{\ln x}{\ln y}\right\}} = x^{\frac{1}{\ln y}}, (y \neq 1)$

Let $x \in \mathbb{R}_\alpha$. If $x\dot{>}\dot{0}$, then x is a α-positive number, and if $x\dot{<}\dot{0}$, then x is a α-negative number. Moreover,

$$\alpha(-x) = \alpha(-\alpha^{-1}(\dot{x})) = \dot{-}\dot{x}, \quad \sqrt{x} = \alpha(\sqrt{\alpha^{-1}(x)})$$

and

$$x^{\dot{2}} = x\dot{\times}x = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(x)\right\} = \alpha((\alpha^{-1}(x))^2)$$

see [1]. In addition, the α-summation can be given as [16]:

$$_\alpha\sum_{k=1}^{n} x_k = \alpha\left(\sum_{k=1}^{n} \alpha^{-1}(x_k)\right)$$

and the non-Newtonian real series (or α-series) is given as [12]:

$$_\alpha\sum_{n=1}^{\infty} a_n = a_1\dot{+}a_2\dot{+}a_3\dot{+}\dots\dot{+}a_n\dot{+}\dots$$

In this paper, we will consider the Pell and Pell-Lucas numbers. The sequence of Pell numbers is defined recursively by the relation

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2 \tag{2.1}$$

with initial conditions $P_0 = 0$ and $P_1 = 1$ [25, 26]. The Pell numbers are

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \dots$$

Besides, the sequence of Pell-Lucas numbers is defined recursively by the relation

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2 \tag{2.2}$$

with initial conditions $Q_0 = Q_1 = 2$ [25]. The Pell-Lucas numbers are

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, \dots$$

The Binet formulas of the Pell and Pell-Lucas numbers are given by

$$P_n = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \tag{2.3}$$

and

$$Q_n = {r_1}^n + {r_2}^n \tag{2.4}$$

respectively, where $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$ [25, 26]. Furthermore, for Pell and Pell-Lucas numbers, the followings hold [25–27, 37]:

$$P_n + P_{n-1} = \frac{1}{2}Q_n \tag{2.5}$$

$$P_{n+1} + P_{n-1} = Q_n \tag{2.6}$$

$$Q_n + Q_{n-1} = 4P_n \tag{2.7}$$

$$Q_{n+1} + Q_{n-1} = 8P_n \tag{2.8}$$

$$P_nQ_n = P_{2n} \tag{2.9}$$

$$P_{n+1}^2 + P_n^2 = P_{2n+1} \tag{2.10}$$

$$P_{n+1}^2 - P_{n-1}^2 = 2P_{2n} \tag{2.11}$$

$$Q_{n+1}^2 + Q_n^2 = 8P_{2n+1} \tag{2.12}$$

$$8P_n^2 - Q_n^2 = 4(-1)^{n-1} \tag{2.13}$$

$$P_n^2 - P_{n+r}P_{n-r} = (-1)^{n-r}P_r^2 \tag{2.14}$$

$$Q_n^2 - Q_{n+r}Q_{n-r} = 8(-1)^{n-r+1}P_r^2 \tag{2.15}$$

$$P_{n-1}P_{n+1} + Q_{n-1}Q_{n+1} = 9P_n^2 + 3(-1)^{n-1} \tag{2.16}$$

$$P_{m-1}P_n + P_mP_{n+1} = P_{m+n} \tag{2.17}$$

$$P_{m-1}Q_n + P_mQ_{n+1} = Q_{m+n} \tag{2.18}$$

and summation formulas

$$\sum_{r=1}^{n} P_r = \frac{P_{n+1} + P_n - 1}{2} \tag{2.19}$$

$$\sum_{r=1}^{n} P_{2r} = \frac{P_{2n+1} - 1}{2} \tag{2.20}$$

$$\sum_{r=1}^{n} P_{2r-1} = \frac{P_{2n}}{2} \tag{2.21}$$

$$\sum_{r=1}^{n} Q_r = \frac{Q_{n+1} + Q_n - 4}{2} \tag{2.22}$$

$$\sum_{r=1}^{n} Q_{2r} = \frac{Q_{2n+1} - 2}{2} \tag{2.23}$$

$$\sum_{r=1}^{n} Q_{2r-1} = \frac{Q_{2n} - 2}{2} \tag{2.24}$$

3. Main Results

Let P_n and Q_n be the nth Pell and Pell-Lucas numbers, respectively. For $n \geq 0$, the nth non-Newtonian Pell and Pell-Lucas numbers are defined as

$$NP_n = \dot{P}_n = \alpha(P_n) \tag{3.1}$$

and

$$NQ_n = \dot{Q}_n = \alpha(Q_n) \tag{3.2}$$

respectively.

The non-Newtonian Pell numbers are

$$\dot{0}, \dot{1}, \dot{2}, \dot{5}, \dot{12}, \dot{29}, \dot{70}, \dot{169}, \dot{408}, \dot{985}, \dot{2378}, \ldots$$

and the non-Newtonian Pell-Lucas numbers are

$$\dot{2}, \dot{2}, \dot{6}, \dot{14}, \dot{34}, \dot{82}, \dot{198}, \dot{478}, \dot{1154}, \dot{2786}, \dot{6726}, \ldots$$

If we choose α-generator as I, the identity function defined by $\alpha(x) = x$ for all $x \in \mathbb{R}$, then we obtain Pell and Pell-Lucas numbers relating to classical arithmetic, respectively.

Moreover, if we choose α-generator as exp, the exponential function defined by $\alpha(x) = e^x$ for all $x \in \mathbb{R}$, then we obtain Pell and Pell-Lucas numbers relating to geometric arithmetic as

$$e^0, e^1, e^2, e^5, e^{12}, e^{29}, e^{70}, e^{169}, e^{408}, e^{985}, e^{2378}, \ldots$$

and

$$e^2, e^2, e^6, e^{14}, e^{34}, e^{82}, e^{198}, e^{478}, e^{1154}, e^{2786}, e^{6726}, \ldots$$

respectively.

Theorem 3.1. For $n > 0$, let NP_n and NQ_n be the nth non-Newtonian Pell and Pell-Lucas numbers, respectively. Then, the followings hold:

$$\dot{2}\dot{\times}NP_{n+1}\dot{+}NP_n = NP_{n+2} \tag{3.3}$$

$$\dot{2}\dot{\times}NQ_{n+1}\dot{+}NQ_n = NQ_{n+2} \tag{3.4}$$

$$NP_{n+1}\dot{+}NP_{n-1} = NQ_n \tag{3.5}$$

$$NQ_{n+1}\dot{+}NQ_{n-1} = \dot{8}\dot{\times}NP_n \tag{3.6}$$

$$NQ_n\dot{+}NQ_{n+1} = \dot{4}\dot{\times}NP_{n+1} \tag{3.7}$$

$$NP_n\dot{+}NP_{n-1} = \frac{\dot{1}}{\dot{2}}{}_\alpha\dot{\times}NQ_n \tag{3.8}$$

$$NP_n\dot{\times}NQ_n = NP_{2n} \tag{3.9}$$

Proof. (3.3): From the addition and multiplication properties of non-Newtonian real numbers and (2.1),

$$\begin{aligned}
\dot{2}\dot{\times}NP_{n+1}\dot{+}NP_n &= \alpha(2)\dot{\times}\alpha(P_{n+1})\dot{+}\alpha(P_n) \\
&= \alpha\left\{\alpha^{-1}\alpha\left\{\alpha^{-1}\alpha(2) \times \alpha^{-1}\alpha(P_{n+1})\right\} + \alpha^{-1}\alpha(P_n)\right\} \\
&= \alpha\left(2P_{n+1} + P_n\right) \\
&= \alpha(P_{n+2}) \\
&= NP_{n+2}
\end{aligned}$$

(3.8): From the addition, multiplication, and division properties of non-Newtonian real numbers and (2.5),

$$
\begin{aligned}
NP_n \dot{+} NP_{n-1} &= \alpha(P_n) \dot{+} \alpha(P_{n-1}) \\
&= \alpha\left\{\alpha^{-1}\alpha(P_n) + \alpha^{-1}\alpha(P_{n-1})\right\} \\
&= \alpha\left(P_n + P_{n-1}\right) \\
&= \alpha\left(\frac{1}{2}Q_n\right) \\
&= \alpha\left\{\alpha^{-1}\alpha\left(\frac{1}{2}\right) \times \alpha^{-1}\alpha(Q_n)\right\} \\
&= \alpha\left(\frac{1}{2}\right) \dot{\times} \alpha(Q_n) \\
&= \frac{\dot{1}}{\dot{2}}_\alpha \dot{\times} NQ_n
\end{aligned}
$$

Using the addition and multiplication properties of non-Newtonian real numbers, (2.2) and (2.6)-(2.9), the assertions given in (3.4)-(3.7), and (3.9) can be obtained in a similar manner. □

Theorem 3.2. For $n > 0$, let NP_n and NQ_n be the nth non-Newtonian Pell and Pell-Lucas numbers, respectively. Then, the followings hold:

$$NP_{n+1}^{\dot{2}} \dot{+} NP_n^{\dot{2}} = NP_{2n+1} \tag{3.10}$$

$$NP_{n+1}^{\dot{2}} \dot{-} NP_{n-1}^{\dot{2}} = \dot{2} \dot{\times} NP_{2n} \tag{3.11}$$

$$NQ_{n+1}^{\dot{2}} \dot{+} NQ_n^{\dot{2}} = \dot{8} \dot{\times} NP_{2n+1} \tag{3.12}$$

$$\dot{8} \dot{\times} NP_n^{\dot{2}} \dot{-} NQ_n^{\dot{2}} = \dot{4} \dot{\times} (\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{1}} \tag{3.13}$$

Proof. We prove (3.13). Considering (2.10)-(2.12), (3.10)-(3.12) can be similarly obtained. From the addition, subtraction, and multiplication properties of non-Newtonian real numbers and (2.13),

$$
\begin{aligned}
\dot{8} \dot{\times} NP_n^{\dot{2}} \dot{-} NQ_n^{\dot{2}} &= \alpha(8) \dot{\times} \alpha(P_n) \dot{\times} \alpha(P_n) \dot{-} \alpha(Q_n) \dot{\times} \alpha(Q_n) \\
&= \alpha\left\{\alpha^{-1}\alpha(8) \times \alpha^{-1}\alpha(P_n) \times \alpha^{-1}\alpha(P_n)\right\} \dot{-} \alpha\left\{\alpha^{-1}\alpha(Q_n) \times \alpha^{-1}\alpha(Q_n)\right\} \\
&= \alpha\left(8P_n^2 - Q_n^2\right) \\
&= \alpha\left(4(-1)^{n-1}\right) \\
&= \alpha(4) \dot{\times} \alpha\left((-1)^{n-1}\right) \\
&= \alpha(4) \dot{\times} \overbrace{\alpha(-1) \dot{\times} \alpha(-1) \dot{\times} \dots \dot{\times} \alpha(-1)}^{n-1 \quad \text{times}} \\
&= \dot{4} \dot{\times} \overbrace{(\dot{-}\dot{1}) \dot{\times} (\dot{-}\dot{1}) \dot{\times} \dots \dot{\times} (\dot{-}\dot{1})}^{n-1 \quad \text{times}} \\
&= \dot{4} \dot{\times} (\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{1}}
\end{aligned}
$$

□

Theorem 3.3. For $m, n > 0$, the followings are true.

$$NP_{m-1} \dot{\times} NP_n \dot{+} NP_m \dot{\times} NP_{n+1} = NP_{m+n} \tag{3.14}$$

$$NP_{m-1} \dot{\times} NQ_n \dot{+} NP_m \dot{\times} NQ_{n+1} = NQ_{m+n} \tag{3.15}$$

$$NP_{n-1}\dot{\times}NP_{n+1}\dot{+}NQ_{n-1}\dot{\times}NQ_{n+1}=\dot{9}\dot{\times}NP_n^{\dot{2}}\dot{+}\dot{3}\dot{\times}(\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{1}} \tag{3.16}$$

Proof. (3.14): From the addition and multiplication properties of non-Newtonian real numbers and (2.17),

$$\begin{aligned}
NP_{m-1}\dot{\times}NP_n\dot{+}NP_m\dot{\times}NP_{n+1} &= \alpha(P_{m-1})\dot{\times}\alpha(P_n)\dot{+}\alpha(P_m)\dot{\times}\alpha(P_{n+1})\\
&= \alpha\left\{\alpha^{-1}\alpha(P_{m-1})\times\alpha^{-1}\alpha(P_n)\right\}\dot{+}\alpha\left\{\alpha^{-1}\alpha(P_m)\times\alpha^{-1}\alpha(P_{n+1})\right\}\\
&= \alpha\left(P_{m-1}P_n+P_mP_{n+1}\right)\\
&= \alpha\left(P_{m+n}\right)\\
&= NP_{m+n}
\end{aligned}$$

From the addition and multiplication properties of non-Newtonian real numbers and (2.18), the assertion given in (3.15) can be easily obtained.

(3.16): From the addition and multiplication properties of non-Newtonian real numbers and (2.16),

$$\begin{aligned}
NP_{n-1}\dot{\times}NP_{n+1}\dot{+}NQ_{n-1}\dot{\times}NQ_{n+1} &= \alpha(P_{n-1})\dot{\times}\alpha(P_{n+1})\dot{+}\alpha(Q_{n-1})\dot{\times}\alpha(Q_{n+1})\\
&= \alpha\left\{\alpha^{-1}\alpha(P_{n-1})\times\alpha^{-1}\alpha(P_{n+1})\right\}\\
&\quad\dot{+}\alpha\left\{\alpha^{-1}\alpha(Q_{n-1})\times\alpha^{-1}\alpha(Q_{n+1})\right\}\\
&= \alpha\left(P_{n-1}P_{n+1}+Q_{n-1}Q_{n+1}\right)\\
&= \alpha\left(9P_n^2+3(-1)^{n-1}\right)\\
&= \alpha(9)\dot{\times}\alpha(P_n)\dot{\times}\alpha(P_n)\dot{+}\alpha(3)\dot{\times}\alpha\left((-1)^{n-1}\right)\\
&= \dot{9}\dot{\times}NP_n^{\dot{2}}\dot{+}\dot{3}\dot{\times}(\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{1}}
\end{aligned}$$

□ The next result provides the Catalan-like identities for the non-Newtonian Pell and Pell-Lucas numbers.

Theorem 3.4. For non-negative integers n and r such that $r \leq n$, the followings are true.

$$NP_n^{\dot{2}}\dot{-}NP_{n+r}\dot{\times}NP_{n-r}=(\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{r}}\dot{\times}NP_r^{\dot{2}} \tag{3.17}$$

$$NQ_n^{\dot{2}}\dot{-}NQ_{n+r}\dot{\times}NQ_{n-r}=\dot{8}\dot{\times}(\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{r}\dot{+}\dot{1}}\dot{\times}NP_r^{\dot{2}} \tag{3.18}$$

Proof. From the subtraction and multiplication properties of non-Newtonian real numbers and (2.14),

$$\begin{aligned}
NP_n^{\dot{2}}\dot{-}NP_{n+r}\dot{\times}NP_{n-r} &= \alpha(P_n)\dot{\times}\alpha(P_n)\dot{-}\alpha(P_{n+r})\dot{\times}\alpha(P_{n-r})\\
&= \alpha\left\{\alpha^{-1}\alpha(P_n)\times\alpha^{-1}\alpha(P_n)\right\}\dot{-}\alpha\left\{\alpha^{-1}\alpha(P_{n+r})\times\alpha^{-1}\alpha(P_{n-r})\right\}\\
&= \alpha\left(P_n^2-P_{n+r}P_{n-r}\right)\\
&= \alpha\left((-1)^{n-r}P_r^2\right)\\
&= (\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{r}}\dot{\times}NP_r^{\dot{2}}
\end{aligned}$$

which completes the proof of the first assertion. From the subtraction and multiplication properties of non-Newtonian real numbers and (2.15),

$$\begin{aligned}
NQ_n^{\dot{2}} \dot{-} NQ_{n+r} \dot{\times} NQ_{n-r} &= \alpha(Q_n) \dot{\times} \alpha(Q_n) \dot{-} \alpha(Q_{n+r}) \dot{\times} \alpha(Q_{n-r}) \\
&= \alpha \left\{ \alpha^{-1}\alpha(Q_n) \times \alpha^{-1}\alpha(Q_n) \right\} \dot{-} \alpha \left\{ \alpha^{-1}\alpha(Q_{n+r}) \times \alpha^{-1}\alpha(Q_{n-r}) \right\} \\
&= \alpha \left(Q_n^2 - Q_{n+r}Q_{n-r} \right) \\
&= \alpha \left(8(-1)^{n-r+1} P_r^2 \right) \\
&= \dot{8} \dot{\times} (\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{r}\dot{+}\dot{1}} \dot{\times} NP_r^{\dot{2}}
\end{aligned}$$

□

The next corollary provides the Cassini-like identities for the non-Newtonian Pell and Pell-Lucas numbers.

Corollary 3.5. For non-negative integers n, the followings are true.

$$NP_n^{\dot{2}} \dot{-} NP_{n+1} \dot{\times} NP_{n-1} = (\dot{-}\dot{1})^{\dot{n}\dot{-}\dot{1}}$$

$$NQ_n^{\dot{2}} \dot{-} NQ_{n+1} \dot{\times} NQ_{n-1} = \dot{8} \dot{\times} (\dot{-}\dot{1})^{\dot{n}}$$

Theorem 3.6. For positive integer n, the followings hold:

$$ {}_{\alpha}\sum_{r=1}^{n} NP_r = \left(\frac{NP_{n+1} \dot{+} NP_n \dot{-} \dot{1}}{\dot{2}}\right)_{\alpha} \tag{3.19}$$

$$ {}_{\alpha}\sum_{r=1}^{n} NP_{2r} = \left(\frac{NP_{2n+1} \dot{-} \dot{1}}{\dot{2}}\right)_{\alpha} \tag{3.20}$$

$$ {}_{\alpha}\sum_{r=1}^{n} NP_{2r-1} = \left(\frac{NP_{2n}}{\dot{2}}\right)_{\alpha} \tag{3.21}$$

$$ {}_{\alpha}\sum_{r=1}^{n} NQ_r = \left(\frac{NQ_{n+1} \dot{+} NQ_n \dot{-} \dot{4}}{\dot{2}}\right)_{\alpha} \tag{3.22}$$

$$ {}_{\alpha}\sum_{r=1}^{n} NQ_{2r} = \left(\frac{NQ_{2n+1} \dot{-} \dot{2}}{\dot{2}}\right)_{\alpha} \tag{3.23}$$

$$ {}_{\alpha}\sum_{r=1}^{n} NQ_{2r-1} = \left(\frac{NQ_{2n} \dot{-} \dot{2}}{\dot{2}}\right)_{\alpha} \tag{3.24}$$

Proof. We only prove the first assertion given in (3.19). Using suitable identities provided in (2.20)-(2.24), assertions given in (3.20)-(3.24) can be obtained similarly. From the addition and division properties of non-Newtonian real numbers and (2.19),

$$\begin{aligned}
{}_{\alpha}\sum_{r=1}^{n} NP_r &= \alpha \left(\sum_{r=1}^{n} \alpha^{-1}\left(NP_r\right) \right) \\
&= \alpha \left(\sum_{r=1}^{n} \alpha^{-1}\alpha\left(P_r\right) \right) \\
&= \alpha \left(\sum_{r=1}^{n} P_r \right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha\left(\frac{P_{n+1}+P_n-1}{2}\right)\\
&= \left(\frac{\alpha(P_{n+1})\dot{+}\alpha(P_n)\dot{-}\alpha(1)}{\alpha(2)}\right)_{\alpha}\\
&= \left(\frac{NP_{n+1}\dot{+}NP_n\dot{-}\dot{1}}{\dot{2}}\right)_{\alpha}
\end{aligned}
$$

□

Theorem 3.7. Let NP_n and NQ_n be the nth non-Newtonian Pell and Pell-Lucas numbers, respectively. Then, for $n \geq 0$, the Binet-like formulas for these numbers are given by

$$NP_n = \left(\frac{\dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}}}{\dot{r_1}\dot{-}\dot{r_2}}\right)_{\alpha} \tag{3.25}$$

and

$$NQ_n = \dot{r_1}^{\dot{n}}\dot{+}\dot{r_2}^{\dot{n}} \tag{3.26}$$

respectively, where $\dot{r_1} = \dot{1}\dot{+}\sqrt{\dot{2}}$ and $\dot{r_2} = \dot{1}\dot{-}\sqrt{\dot{2}}$.

Proof. From the subtraction and division properties of non-Newtonian real numbers,

$$
\begin{aligned}
\left(\frac{\dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}}}{\dot{r_1}\dot{-}\dot{r_2}}\right)_{\alpha} &= \alpha\left(\frac{\alpha^{-1}(\dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}})}{\alpha^{-1}(\dot{r_1}\dot{-}\dot{r_2})}\right)\\
&= \alpha\left(\frac{\alpha^{-1}\left(\overbrace{(\dot{r_1}\dot{\times}\dot{r_1}\dot{\times}\ldots\dot{\times}\dot{r_1})}^{n \text{ times}}\dot{-}\overbrace{(\dot{r_2}\dot{\times}\dot{r_2}\dot{\times}\ldots\dot{\times}\dot{r_2})}^{n \text{ times}}\right)}{\alpha^{-1}(\dot{r_1}\dot{-}\dot{r_2})}\right)\\
&= \alpha\left(\frac{\alpha^{-1}\left(\alpha\left(\overbrace{\alpha^{-1}(\dot{r_1})\times\alpha^{-1}(\dot{r_1})\times\ldots\times\alpha^{-1}(\dot{r_1})}^{n \text{ times}}\right)\dot{-}\alpha\left(\overbrace{\alpha^{-1}(\dot{r_2})\times\alpha^{-1}(\dot{r_2})\times\ldots\times\alpha^{-1}(\dot{r_2})}^{n \text{ times}}\right)\right)}{\alpha^{-1}(\dot{r_1}\dot{-}\dot{r_2})}\right)\\
&= \alpha\left(\frac{\alpha^{-1}(\alpha(r_1{}^n)\dot{-}\alpha(r_2{}^n))}{\alpha^{-1}(\dot{r_1}\dot{-}\dot{r_2})}\right)\\
&= \alpha\left(\frac{r_1{}^n-r_2{}^n}{r_1-r_2}\right)
\end{aligned}
$$

Thus, from (2.3),

$$
\begin{aligned}
\left(\frac{\dot{r_1}^{\dot{n}}\dot{-}\dot{r_2}^{\dot{n}}}{\dot{r_1}\dot{-}\dot{r_2}}\right)_{\alpha} &= \alpha(P_n)\\
&= NP_n
\end{aligned}
$$

Moreover, from (2.4),

$$
\begin{aligned}
\dot{r_1}^{\dot{n}}\dot{+}\dot{r_2}^{\dot{n}} &= \overbrace{(\dot{r_1}\dot{\times}\dot{r_1}\dot{\times}\ldots\dot{\times}\dot{r_1})}^{n \text{ times}}\dot{+}\overbrace{(\dot{r_2}\dot{\times}\dot{r_2}\dot{\times}\ldots\dot{\times}\dot{r_2})}^{n \text{ times}}\\
&= \alpha(r_1{}^n)\dot{+}\alpha(r_2{}^n)\\
&= \alpha(r_1{}^n + r_2{}^n)\\
&= \alpha(Q_n)\\
&= NQ_n
\end{aligned}
$$

□

Theorem 3.8. Let $z \in \mathbb{R}_\alpha$. Then, the generating functions for the non-Newtonian Pell and Pell-Lucas numbers are given by

$$g(z) = {\frac{z}{\dot{1}\dot{-}\dot{2}\dot{\times}z\dot{-}z^{\dot{2}}}}_\alpha \tag{3.27}$$

and

$$h(z) = {\frac{\dot{2}\dot{-}\dot{2}\dot{\times}z}{\dot{1}\dot{-}\dot{2}\dot{\times}z\dot{-}z^{\dot{2}}}}_\alpha \tag{3.28}$$

respectively.

Proof. We prove (3.28). (3.27) can be obtained in a similar manner. Let $h(z)$ be the generating function of the non-Newtonian Pell-Lucas numbers. Then,

$$h(z) = {}_\alpha\sum_{n=0}^{\infty} NQ_n\dot{\times}z^{\dot{n}}$$

Thus, from (3.4),

$$\begin{aligned}
h(z) &= {}_\alpha\sum_{n=0}^{\infty} NQ_n\dot{\times}z^{\dot{n}} \\
&= \dot{2}\dot{+}\dot{2}\dot{\times}z\dot{+}{}_\alpha\sum_{n=2}^{\infty} NQ_n\dot{\times}z^{\dot{n}} \\
&= \dot{2}\dot{+}\dot{2}\dot{\times}z\dot{+}{}_\alpha\sum_{n=2}^{\infty} (\dot{2}\dot{\times}NQ_{n-1}\dot{+}NQ_{n-2})\dot{\times}z^{\dot{n}} \\
&= \dot{2}\dot{+}\dot{2}\dot{\times}z\dot{+}{}_\alpha\sum_{n=2}^{\infty} \dot{2}\dot{\times}NQ_{n-1}\dot{\times}z^{\dot{n}}\dot{+}{}_\alpha\sum_{n=2}^{\infty} NQ_{n-2}\dot{\times}z^{\dot{n}} \\
&= \dot{2}\dot{+}\dot{2}\dot{\times}z\dot{+}\dot{2}\dot{\times}z\dot{\times}{}_\alpha\sum_{n=2}^{\infty} NQ_{n-1}\dot{\times}z^{\dot{n}\dot{-}\dot{1}}\dot{+}z^{\dot{2}}\dot{\times}{}_\alpha\sum_{n=2}^{\infty} NQ_{n-2}\dot{\times}z^{\dot{n}\dot{-}\dot{2}} \\
&= \dot{2}\dot{-}\dot{2}\dot{\times}z\dot{+}\dot{2}\dot{\times}z\dot{\times}{}_\alpha\sum_{n=0}^{\infty} NQ_n\dot{\times}z^{\dot{n}}\dot{+}z^{\dot{2}}\dot{\times}{}_\alpha\sum_{n=0}^{\infty} NQ_n\dot{\times}z^{\dot{n}} \\
&= \dot{2}\dot{-}\dot{2}\dot{\times}z\dot{+}\dot{2}\dot{\times}z\dot{\times}h(z)\dot{+}z^{\dot{2}}\dot{\times}h(z)
\end{aligned}$$

Then, it follows that

$$(\dot{1}\dot{-}\dot{2}\dot{\times}z\dot{-}z^{\dot{2}})\dot{\times}h(z) = \dot{2}\dot{-}\dot{2}\dot{\times}z$$

Hence,

$$h(z) = {\frac{\dot{2}\dot{-}\dot{2}\dot{\times}z}{\dot{1}\dot{-}\dot{2}\dot{\times}z\dot{-}z^{\dot{2}}}}_\alpha$$

□

Remark 3.9. As a generator, if $\alpha = I$ (identity function), the results given in this paper correspond to known results of classical Pell and Pell-Lucas numbers.

Remark 3.10. As a generator, if $\alpha = \exp$ (exponential function), the results given in this paper correspond to geometric identities. These corresponding geometric identities are listed below, respectively.

$$e^{2P_{n+1}+P_n} = e^{P_{n+2}}$$

$$e^{2Q_{n+1}+Q_n} = e^{Q_{n+2}}$$

$$e^{P_{n+1}+P_{n-1}} = e^{Q_n}$$

$$e^{Q_{n+1}+Q_{n-1}} = e^{8P_n}$$

$$e^{Q_n+Q_{n+1}} = e^{4P_{n+1}}$$

$$e^{P_n+P_{n-1}} = e^{\frac{Q_n}{2}}$$

$$e^{P_nQ_n} = e^{P_{2n}}$$

$$e^{P_{n+1}^2+P_n^2} = e^{P_{2n+1}}$$

$$e^{P_{n+1}^2-P_{n-1}^2} = e^{2P_{2n}}$$

$$e^{Q_{n+1}^2+Q_n^2} = e^{8P_{2n+1}}$$

$$e^{8P_n^2-Q_n^2} = e^{4(-1)^{n-1}}$$

$$e^{P_{m-1}P_n+P_mP_{n+1}} = e^{P_{m+n}}$$

$$e^{P_{m-1}Q_n+P_mQ_{n+1}} = e^{Q_{m+n}}$$

$$e^{P_{n-1}P_{n+1}+Q_{n-1}Q_{n+1}} = e^{9P_n^2+3(-1)^{n-1}}$$

$$e^{P_n^2-P_{n+r}P_{n-r}} = e^{(-1)^{n-r}P_r^2}$$

$$e^{Q_n^2-Q_{n+r}Q_{n-r}} = e^{8(-1)^{n-r+1}P_r^2}$$

$$e^{P_n^2-P_{n+1}P_{n-1}} = e^{(-1)^{n-1}}$$

$$e^{Q_n^2-Q_{n+1}Q_{n-1}} = e^{8(-1)^n}$$

$$e^{\sum_{r=1}^n P_r} = e^{\frac{P_{n+1}+P_{n}-1}{2}}$$

$$e^{\sum_{r=1}^n P_{2r}} = e^{\frac{P_{2n+1}-1}{2}}$$

$$e^{\sum_{r=1}^n P_{2r-1}} = e^{\frac{P_{2n}}{2}}$$

$$e^{\sum_{r=1}^n Q_r} = e^{\frac{Q_{n+1}+Q_{n}-4}{2}}$$

$$e^{\sum_{r=1}^n Q_{2r}} = e^{\frac{Q_{2n+1}-2}{2}}$$

$$e^{\sum_{r=1}^n Q_{2r-1}} = e^{\frac{Q_{2n}-2}{2}}$$

$$e^{P_n} = e^{\frac{{r_1}^n-{r_2}^n}{r_1-r_2}}$$

$$e^{Q_n} = e^{{r_1}^n + {r_2}^n}$$

$$G(z) = e^{\frac{\ln z}{1-2\ln z-(\ln z)^2}}$$

$$H(z) = e^{\frac{2-2\ln z}{1-2\ln z-(\ln z)^2}}$$

4. Conclusion

In this paper, we introduce and study non-Newtonian Pell and Pell-Lucas numbers. We obtain several identities and formulas for these numbers. By choosing I (Identity function) as a generator, the obtained results correspond to known results of classical Pell and Pell-Lucas numbers. Therefore, non-Newtonian Pell and non-Newtonian Pell-Lucas numbers can be considered generalizations of classical Pell and Pell-Lucas numbers, respectively. Moreover, just like the non-Newtonian Fibonacci and non-Newtonian Lucas numbers [42], non-Newtonian Pell and non-Newtonian Pell-Lucas numbers can be offer an alternative viewpoint in applications such as encryption theory. In future work, considering those that have not been studied before, the researchers can investigate different types of second-order and third-order integer sequences in terms of non-Newtonian calculus.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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