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Statistical Convergence of Matrix Sequences

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Abstract

This paper extends the statistical convergence of real or complex numbers to the real square matrices sequences . In this context, we investigate the relation between the statistical convergence, statistical Cauchy condition, strong Cesáro summability of matrices sequences. This leads us to an initial analysis of the Tauberian conditions for the statistical convergence of matrix sequences.

Keywords: *Cesáro Summability, matrix sequence, statistical convergence.*
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1. Introduction

The density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} card\{k \leq n : \quad k \in K\}, \tag{1.1}$$

whenever the limit exists. If (x_k) is a sequence , which satisfies a property P for all k except a set of density zero, then, we say that (x_k) satisfies P for "almost all k " , and we abbreviate this by "$a.a.k.$" Statistical convergence of sequences of real or complex numbers was introduced 1951 by Steinhaus in [16] and Fast in [5].
A sequence (x_k) of real or complex numbers is said to be statistically convergent to the number a, and denoted by $st-\lim x_k = a$, if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - a| \geq \varepsilon\}) = 0, \tag{1.2}$$

or equivalently there exists a subset $K \subset \mathbb{N}$ with $\delta(K) = 1$ and n_0 such that for any $k \in K$, $k > n_0$ we have $|x_k - a| < \varepsilon$ a.a.k. see e.g. [6, 10, 14]. It is known that any convergent sequence is statistically convergent, but not conversely. As an interpretation of this notion, we can say that a sequence is statistically convergent if however small $\varepsilon > 0$ we take, the frequency of elements of the sequence that differ from the statistical limit of more than $\varepsilon > 0$ can be made arbitrarily small provided we take a large enough sample. Similarly, a sequence (x_k) of real or complex numbers is said to be statistically Cauchy if for each $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ such that $\delta(\{k \in \mathbb{N} : |x_k - x_N| \geq \varepsilon\}) = 0$. Basic properties of the statistical convergence and of some related summability methods were established in [4, 15, 19, 20].
Over almost 70 years since its inception, the concept of statistical convergence has been studied in the context of numerous mathematical disciplines including: the summability theory [6, 4, 7, 8], trigonometric series theory [23], measure theory [10], optimization [13], and approximation theory [9], to give just a few prominent examples. The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [1]-[3], [11], [18]-[22]). Nuray and Rhoades [11] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [17] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Ulusu and Nuray [18] introduced the concept of Wijsman strongly lacunary summability for set sequences and discused its relation with Wijsman strongly Cesàro summability. Nuray et al.[12] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigated the relationship between them.
The paper is organised as follows:

(a) Section 2 introduces necessary concepts and facts related to the theory convergence of matrices sequences.

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(b) Section 3 provides definitions of major concepts related to statistical convergence and First Equivalence Theorem for the statistical convergence of matrices sequences.
(c) In Section 4, we define the strong p-Cesàro summability of matrix sequences and we discuss its relationship with the statistical convergence introduced in the previous section. This is followed by a Tauberian Theorem.

2. Preliminaries

Let $M_1, M_2, M_3, ...$ be a sequence of matrices belonging to $M_n(\mathbb{C})$, the set of all $n\times n$ matrices with complex elements, and let $m_{ij}^{(k)}$ be the i,jth element of $M_k, k=1,2,3,....$ The sequence $M_1, M_2, M_3, ...$ is said to converge to the matrix $M\in M_n(\mathbb{C})$ if there exit numbers m_{ij}(the elements of M) such that $m_{ij}^{(k)}\to m_{ij}$ as $k\to\infty$ for each pair of subscripts i,j. A sequence that does not convergence is said to diverge. Thus the convergence of sequences of matrices is defined as elementwise convergence. The definition includes the convergence of column and row vectors as special cases as well. As an example, let

$$M_k=\begin{pmatrix} (\frac{1}{5})^k & \frac{k}{2^{2k+1}} \\ 0 & (\frac{1}{5})^k \end{pmatrix}. \tag{2.1}$$

Then, the matrices sequence (M_k) is convergent to the matrix $M=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
If a matrix sequence (M_k) converges and its limit is M, then this limit is unique and we write $\lim_{k\to\infty} M_k = M$.
Let (M_k) be a matrix sequence. The ith row of (M_k) is said to be convergent to M if and only if

$$\lim_{k\to\infty} a_{ij}^{(k)} = a_{ij} \tag{2.2}$$

for all $j=1,2,...,n$.
The jth column of (M_k) is said to be convergent to M if and only if

$$\lim_{k\to\infty} a_{ij}^{(k)} = a_{ij} \tag{2.3}$$

for all $i=1,2,...,n$.
If the matrix sequences (M_k) and (N_k) are convergent to M and N, then the matrix sequences (M_k+N_k) and $(M_k\cdot N_k)$ are convergent to $M+N$ and $M\cdot N$ respectively.
The definition of convergence can also be given using the matrix norm:
A sequence of matrices (M_k) converges a limit M if for a matrix norm, we have

$$\lim_{k\to\infty} \|M_k - M\| = 0. \tag{2.4}$$

The definition of convergence does not depend on the chosen norm, since $M_n(\mathbb{C})$ is a vector space of finite dimension. If a vector space is finite dimension, then all norms are equivalent. Thus if a matrix sequence converges for one norm, it converges for all norms.
A norm $\|\cdot\|$ defined on $M_n(\mathbb{C})$ is a matrix norm if for all matrices $M,N\in M_n(\mathbb{C})$,

$$\|MN\|\le\|M\|\|N\|. \tag{2.5}$$

Let $\|\cdot\|$ be a vector norm on $\mathbb{K}^n$. It induces a matrix norm defined by

$$\|A\| = \sup_{x\in\mathbb{K}^n, x\neq 0}\frac{\|Ax\|}{\|x\|}, \tag{2.6}$$

which is said to be subordinate to this vector norm. The norm of a matrix is a measure of how large its elements are. It is a way of determining the size of a matrix that is not necessarily related to how many rows or columns the matrix has.
There are many matrix norms, but the following three are among the most commonly used.
The 1-norm

$$\|M\|_1 = \max_{1\le j\le n}\left(\sum_{i=1}^{n}|m_{ij}|\right) \tag{2.7}$$

(the maximum absolute column sum).
The infinity-norm

$$\|M\|_\infty = \max_{1\le i\le n}\left(\sum_{j=1}^{n}|m_{ij}|\right) \tag{2.8}$$

(the maximum absolute row sum).
The Euclidean norm

$$\|M\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(m_{ij})^2} \tag{2.9}$$

(the square root of the sum of all the squares).

3. Statistical Convergence

Let us start with the following definition. In all following definitions all the matrices will be $n \times n$ matrices, unless stated otherwise. Let $M_k = [m_{ij}^{(k)}]$, $k \in \mathbb{N}$.

Definition 3.1. *Let (M_k) be a matrix sequence, M be a matrix and $\|\cdot\|$ be a matrix norm.*

(a) We say that the matrix sequence (M_k) is statistically convergent to the matrix M and write $M_k \xrightarrow{st} M$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} card\{k \leq n : \quad \|M_k - M\| \geq \varepsilon\} = 0. \tag{3.1}$$

(b) A sequence (M_k) is called statistically Cauchy sequence if for each $\varepsilon > 0$, there is a positive integer $N = (N_\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n} card\{k \leq n : \quad \|M_k - M_N\| \geq \varepsilon\} = 0. \tag{3.2}$$

(c) A sequence (M_k) is called statistically bounded if for arbitrary matrix M, there is a real number $q > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} card\{k \leq n : \quad \|M_k - M\| \geq q\} = 0. \tag{3.3}$$

Example 3.2. *Define (M_k) and M by*

$$M_k = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \text{ is a square} \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

and

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$\frac{1}{n} card\{k \leq n : \quad \|M_k - M\| \geq \varepsilon\} \leq \frac{\sqrt{n}}{n}, \tag{3.4}$$

so $M_k \xrightarrow{st} M$. The matrix sequence (M_k) is not convergent.

A ball $B(M,r)$ is defined by $B(M,r) = \{P \in \mathbb{C}_{m\times n} : \|M - P\| \leq r\}$.
The following result will be used in the proof of the main result of this section.

Lemma 3.3. *Assume that $\{B_m\}$ is a sequence of nonempty subsets such that $B_{m+1} \subset B_m$ for every $m \in \mathbb{N}$, and such that $diam(B_m) \to 0$ as $m \to \infty$. Then, there exists M_0 with*

$$\bigcap_{m\in\mathbb{N}} B_m = \{M_0\}. \tag{3.5}$$

Proof. Denote $B = \bigcap_{m\in\mathbb{N}} B_m$. For every m select an element $M_m \in B_m$. Note that $\{M_m\}$ is Cauchy because $diam(B_m) \to 0$. By the completeness there exist M_0 such that $M_m \to M_0$. We need to show that $M_0 \in B$. Assume to the contrary that $M_0 \notin B$. There exists then M_0 with $M_0 \notin B_{m_0}$, which means that M_0 belongs to an open set $\mathbb{C} \setminus B_{m_0}$ and hence by convergence definition there exist $r > 0$ such that the ball $B(M_0,r) \cap B_{m_0} = \emptyset$. Observe that for any $m \geq m_0$ we have $B_m \subset B_{m_0}$, which implies that $B(M_0,r) \cap B_m = \emptyset$. On the other hand, $M_m \to M_0$ and hence $M_m \in B(M_0,r)$ for all $m \geq m_1 \geq m_0$. Contradiction.
To show the uniqueness, let us assume that there exists another element $M_0' \in B$. Hence, $M_0, M_0' \in B_m$ for every m, which yields that $\|M_0 - M_{0'}\| \leq diam(B_m) \to 0$, implying immediately that $M_0 = M_0'$. □

We are now ready to prove a matrix sequence version of the equivalence theorem.

Theorem 3.4 (First Equivalence Theorem)**.** *Let (M_k) be a matrix sequence. The following statements are equivalent:*

(i) (M_k) is statistically convergent;
(ii) (M_k) is statistically Cauchy;
(iii) There is a convergent matrix sequence (P_k) such that $M_k = P_k$ a.a. k.

Proof. To prove that (i) implies (ii) we need to show that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n} card\{k \le n : \|M_k - M_N\| > \varepsilon\} = 0. \tag{3.6}$$

By (i) there exists $M_0 \in \mathbb{C}_{n\times n}$ such that $M_k \xrightarrow{st} M_0$. Let us fix arbitrarily $\varepsilon > 0$ and note that $\|M_k - M_0\| \le \frac{\varepsilon}{2}$ $a.a.k.$ If $N \in \mathbb{N}$ is chosen such that

$$\|M_N - M_0\| \le \frac{\varepsilon}{2}. \tag{3.7}$$

Then we get,

$$\|M_k - M_N\| \le \|M_k - M_0\| + \|M_N - M_0\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad a.a.k. \tag{3.8}$$

which means that (M_k) is statistically Cauchy.

Let us assume now (ii), that is that (M_k) is statistically Cauchy. Then, there exists a ball B_1 with radius equal to 1 and center at some M_{N_1} such that B_1 contains M_k $a.a.$ k. Similarly, we can choose a ball $B_2^{'}$ with radius equal to $\frac{1}{2}$ and center at some M_{N_2} such that $B_2^{'}$ contains M_k $a.a.$ k. Define $B_2 = B_1 \cap B_2^{'}$ and observe that B_2 is a nonempty subset of B_1 with $diam(B_2) \le \frac{1}{2}$. Observe that B_2 contains M_k $a.a.$ k because for every $n \in \mathbb{N}$

$$\{k \le n : M_k \notin B_1 \cap B_2^{'}\} = \{k \le n : M_k \notin B_1\} \cup \{k \le n : M_k \notin B_2^{'}\},$$

and hence

$$\delta(\{k : M_k \notin B_1 \cap B_2^{'}\}) \le \delta(\{k : M_k \notin B_1\}) + \delta(\{k : M_k \notin B_2^{'}\}) = 0.$$

By induction we can obtain a sequence $\{B_n\}$ of nonempty sets such that $B_{n+1} \subset B_n$ and $diam(B_n) \to 0$ and each B_n contains M_k $a.a.$ k. By Lemma 3.3 there exist M_0 such that $\bigcap_{m\in\mathbb{N}} B_m = \{M_0\}$. Because for every m, $M_k \in B_m$ $a.a.$ k, we can choose an increasing sequence $\{p_m\}$ of natural numbers such that for every $n > p_m$

$$\frac{1}{n} card\{k \le n : M_k \notin B_m\} \le \frac{1}{m}. \tag{3.9}$$

Define the sequence $\{R_k\}$ as a subsequence of $\{M_k\}_{k>p_1}$ such that if $p_m < k < p_{m+1}$ then, $M_k \notin B_m$. Define now $\{P_k\}$ by

$$P_k = \begin{cases} M_0 & \text{if } M_k \text{ is a term of } (R_n) \\ M_k & \text{otherwise} \end{cases} \tag{3.10}$$

and observe that either $P_k = M_0$ or $P_k = M_k \in B_{m_k}$, where p_{m_k} is the largest term of the sequence $\{p_m\}$ for which $p_m < k$. Hence, either $|P_k - M_0| = 0$ or $|P_k - M_0| \le diam(B_{m_k}) \to 0$, which means that $\{P_k\}$ is convergent to M_0. It remains to be shown that $M_k = P_k$ $a.a.$ k. Observe that if $p_m < n \le p_{m+1}$ then,

$$\{k \le n : P_k \ne M_k\} \subset \{k \le n : M \notin B_m\}, \tag{3.11}$$

hence by (3.9)

$$\frac{1}{n} card\{k \le n : P_k \ne M_k\} \le \frac{1}{n} card\{k \le n : M_k \notin B_m\} \le \frac{1}{m}, \tag{3.12}$$

and finally

$$\lim_{n\to\infty} \frac{1}{n} card\{k \le n : P_k \ne M_k\} = 0, \tag{3.13}$$

which means that $M_k = P_k$ $a.a.$ k, as claimed.
To prove that (iii) implies (i), let us assume that there exists a matrix sequence (P_k) which converges to a matrix M_0 and that $M_k = P_k$ $a.a.$ k. We will prove that $M_k \xrightarrow{st} M_0$. Let us fix $\varepsilon > 0$ and take any $n \in \mathbb{N}$. Observe that for each pair of subscripts i,j,

$$\{k \le n : \|M_k - M_0\| \ge \varepsilon\} \subset \{k \le n : M_k \ne P_k\} \cup \{k \le n : \|P_k - M_0\| \ge \varepsilon\}. \tag{3.14}$$

Since $P_k \to M_0$, it follows that there exists $\ell = \ell(\varepsilon) \in \mathbb{N}$ such that

$$card\{k \le n : \|P_k - M_0\| \ge \varepsilon\} = \ell,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} card\{k \le n : \|M_k - M_0\| \ge \varepsilon\} \le \lim_{n\to\infty} \frac{1}{n} card\{k \le n : M_k \ne P_k\} + \lim_{n\to\infty} \frac{\ell}{n} = 0,$$

because $M_k = P_k$ $a.a.$ k, proving that $M_k \xrightarrow{st} M_0$, as claimed.

□

4. Summability and Tauberian Properties

In this section we introduce a notion of the strong p-Cesàro-summability of matrix sequences and we discuss its relationship with the statistical convergence introduced in the previous section. This will lead us to establishing a Tauberian type results.
First let us define formally bounded sequences matrices.

Definition 4.1. *We say that a matrix sequence (M_k) is bounded if* $\sup_k \|M_k\| < \infty$.

Definition 4.2. *Let p be a positive real number. We say that a matrix sequence (M_k) is strongly p-Cesàro-summable to matrix M if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|M_k - M\|^p = 0. \tag{4.1}$$

The following results, following patterns of [4], demonstrates a close connection between the statistical convergence of matrix sequences and the strong p-Cesàro-summability. Note that for bounded matrix sequences both notions are equivalent.

Theorem 4.3. *[Second Equivalence Theorem] Let $p \in \mathbb{R}, 0 < p < \infty$.*

(i) If a matrix sequence is strongly p-Cesàro-summable to M, then it is statistically convergent to M.
(ii) If a bounded matrix sequence is statistically convergent to M, then it is strongly p-Cesàro-summable to M.

Proof. For any sequence of matrices (M_k) and $\varepsilon > 0$ we have that

$$\begin{aligned}\sum_{k=1}^{n} \|M_k - M\|^p &= \sum_{\substack{k=1\\ \|M_k-M\|\geq\varepsilon}}^{n} \|M_k - M\|^p + \sum_{\substack{k=1\\ \|M_k-M\|<\varepsilon}}^{n} \|M_k - M\|^p \\ &\geq \sum_{\substack{k=1\\ \|M_k-M\|\geq\varepsilon}}^{n} \|M_k - M\|^p \\ &\geq card\{k\leq n: \quad \|M_k - M\| \geq \varepsilon\}\varepsilon^p.\end{aligned}$$

It follows that if (M_k) is strongly p-Cesàro summable to M then, (M_k) is statistically convergent to M, proving (i).
Now suppose that matrix sequence (M_k) is bounded and statistically convergent to M. Since (M_k) is bounded, there exists $0 < q < \infty$ such that $\|M_k - M\| < q$. Let $\varepsilon > 0$ be given and select n_ε such that

$$\frac{1}{n} card\{k\leq n: \quad \|M_k - M\| > \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\} < \frac{\varepsilon}{2q^p}$$

for all $n > n_\varepsilon$ and set $S_n = \{k \leq n: \quad \|M_k - M\| > (\frac{\varepsilon}{2})^{\frac{1}{p}}\}$. Now, for $n > n_\varepsilon$ we have that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} \|M_k - M\|^p &= \frac{1}{n}\left(\sum_{k\in S_n} \|M_k - M\|^p + \sum_{\substack{k\notin S_n\\ k\leq n}} \|M_k - M\|^p\right) \\ &< \frac{1}{n}\left(\frac{n\varepsilon}{2q^p}\right)q^p + \frac{1}{n}n\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, the matrix sequence (M_k) is strongly p-Cesàro summable to M, as claimed in (ii). □

Our next result establishes a Tauberian condition for the strong p-Cesàro-summablity or the statistical convergence of matrix sequence. Similarly to other Tauberian theorems, Theorem 4.4 demonstrates that under some order type constraints imposed on the forward differences $\Delta M_k = M_k - M_{k+1}$, the strong p-Cesàro-summablity and the statistical convergence are actually reduced to the convergence of matrix sequences.

Theorem 4.4. *[Tauberian Theorem] Let $0 < p < \infty$. If a sequence (M_k) is strongly p-Cesàro summable to M (or statistically convergent to M) and $\sup_{k\in\mathbb{N}} \|k\Delta M_k\| < \infty$ then (M_k) is convergent to M.*

Proof. In view of Theorem 4.3 it is enough to assume that (M_k) is statistically convergent to M. Using Theorem 3.4 Part (iii), choose a matrix sequence (P_k) such that $P_k \to M$ and $M_k = P_k$ *a.a.k*. Because $M_k = P_k$ *a.a.k*, there exists $k_0 \in \mathbb{N}$ such that the set $\{i \leq k: M_i = P_i\}$ is nonempty for any $k \geq k_0$. For each $k \geq k_0$ write $k = m_k + p_k$, where $m_k = \max\{i \leq k: M_i = P_i\}$. We will show that

$$\lim_{k\to\infty} \frac{p_k}{m_k} = 0. \tag{4.2}$$

Assume to the contrary that there exists $\gamma > 0$ such that $\frac{p_k}{m_k} > \gamma > 0$ for infinitely many $k \geq k_0$. Then for such k,

$$\begin{aligned}\frac{1}{k} card\{i\leq k: M_i \neq P_i\} &= \frac{1}{m_k + p_k} p_k \\ &> \frac{p_k}{\frac{p_k}{\gamma} + p_k} = \frac{\gamma}{1+\gamma}\end{aligned}$$

contradicting the fact that $M_k = P_k$ $a.a.k$. The contradiction proves (4.2). From (4.2) it follows that there exists $k_1 \geq k_0$ such that $p_k \leq m_k$ for any $k \geq k_1$. Since $\sup_{k\in\mathbb{N}} \|k\Delta M_k\| < \infty$, there is a finite constant $q > 0$ such that $\|\Delta M_k\| \leq \frac{q}{k}$ for any $k \in \mathbb{N}$. Therefore, for each pair of subscripts, we obtain that

$$
\begin{aligned}
\|P_{m_k} - M_k\| &= \|M_{m_k} - M_{m_k+p_k}\| \\
&= \|M_{m_k} - M_{m_k+1} + M_{m_k+1} - M_{m_k+2} + \ldots + M_{m_k+p_k-1} - M_{m_k+p_k}\| \\
&\leq \sum_{s=m_k}^{m_k+p_k-1} \|\Delta M_s\| \leq p_k \frac{q}{m_k}
\end{aligned}
$$

Finally,

$$
\lim_{k\to\infty} \|P_{m_k} - M_k\| = 0. \tag{4.3}
$$

Since $P_k \to M$, it follows from (4.3) that $M_k \to M$, as claimed. □

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