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Research article

Metallic Structures on Product Manifolds and Chen-Ricci Inequalities

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Abstract

In this study, we discuss metallic structures on product manifolds and derive the Chen-Ricci inequalities for remarkable submanifolds determined by the behaviour of their tangent bundles with regard to the action of the metallic structure in a locally decomposable metallic Riemannian manifold whose components are spaces of constant curvature. Moreover, the equality cases are considered in order to characterize these submanifolds.

Keywords: *Metallic structure, metallic Riemannian manifold, Riemannian submanifold, real space form, Ricci curvature, k-Ricci curvature, Chen-Ricci inequality.*

AMS Subject Classification (2020): *53B25; 53C15; 53C25; 53C40.*

1. Introduction

The (p, q)-*metallic number* [10, 11, 12, 13], also called the *metallic ratio*, is a special number being the positive solution of the equation of degree 2

$$x^2 - px - q = 0,$$

where $p, q \in \mathbb{Z}^+$. It is denoted by $\sigma_{p,q}$, that is, $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$. By extending the (p, q)-metallic number $\sigma_{p,q}$ to tensor fields of type $(1, 1)$ on differentiable manifolds, the concept of a metallic structure [27] was defined by Crâşmăreanu and Hreţcanu in order to realize the idea of investigating its effect on differential geometry.

Recently, metallic structures on Riemannian manifolds have attracted great attention and been intensively analyzed. Gezer and Karaman examined metallic Riemannian manifolds by means of a particular operator in [16]. Some different types of submanifolds of metallic Riemannian manifolds, such as invariant, anti-invariant, slant, semi-slant, hemi-slant, bi-slant submanifolds were introduced and studied by Hreţcanu and Blaga in [3, 22, 23, 24]. In [5], the metallic structure on the product of two metallic Riemannian manifolds was characterized based on metallic maps and an equivalent condition was given for the warped product of two locally decomposable metallic Riemannian manifolds to be a locally decomposable metallic Riemannian manifold. In addition, a necessary and sufficient condition was obtained for the warped product of two metallic Riemannian manifolds to have the invariant Ricci tensor with respect to the metallic structure. The authors also investigated metallic conjugate connections with regard to the structural and virtual tensors of the metallic structure and their action on invariant distributions in [4]. Özgür and Özgür studied the full classification of metallic shaped hypersurfaces in real space forms [33] and Lorentzian space forms [34]. In [8], Choudhary and Blaga established two sharp inequalities including the normalized scalar curvature and the Casorati curvature for invariant, anti-invariant and slant submanifolds in metallic Riemannian real space forms and showed that the equality case holds in both inequalities if and only if these submanifolds are invariantly quasi-umbilical, that is, the equality case at every point is a characterization for such types of submanifolds to be invariantly quasi-umbilical. Additionally, golden manifolds being the best known subclass of metallic manifolds were explored in [2, 9, 15, 17, 18, 25, 26].

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On the other hand, inspired by Chen's inequality [7, Theorem 4] between the shape operator and the mean curvature of any isometric immersion in a real space form, Hong and Tripathi established a basic inequality, which is called the Chen-Ricci inequality, comprising the Ricci curvature and the squared mean curvature of any submanifold of a Riemannian manifold in [21]. The Chen-Ricci inequalities were created for some submanifolds of different kinds of ambient manifolds [14, 19, 20, 28, 29, 30, 31, 32, 35, 36, 38].

The essential target of this paper is to search the construction of a locally decomposable metallic Riemannian manifold and determine the Chen-Ricci inequalities for its invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds. Also, we show that the Chen-Ricci inequality is a useful tool to obtain some geometric characterizations for each of the above mentioned submanifolds satisfying its equality case.

The preparation of this paper is the following: Section 1 is devoted to introduction. In section 2, we review some basic facts from Riemannian geometry, in particular metallic Riemannian manifolds and their submanifolds. Section 3 is related to metallic structures on product manifolds. We give an equivalent condition for metallic Riemannian manifolds to be a locally decomposable metallic Riemannian manifold. We obtain a necessary and sufficient condition for both components of locally decomposable metallic Riemannian manifolds to be an Einstein manifold or a space of constant curvature. We also get some results regarding invariant and anti-invariant submanifolds in locally decomposable metallic Riemannian manifolds. In section 4, we establish the Chen-Ricci inequalities for well known submanifolds defined by the behaviour of their tangent bundles in terms of the metallic structure in a locally decomposable metallic Riemannian manifold under the assumption that the components of the ambient manifold are spaces of constant curvature. In addition, we analyze the equality cases.

2. Preliminaries

This section contains some fundamental definitions, concepts, formulas, notations and results regarding metallic Riemannian manifolds and their submanifolds which are needed for the paper.

A non-zero tensor field $\widehat{F}$ of type $(1,1)$ on a differentiable manifold $\widehat{M}$ is named a *metallic structure* if it yields the equation

$$\widehat{F}^2 = p\widehat{F} + qI \tag{2.1}$$

for $p, q \in \mathbb{Z}^+$, where I is the identity operator on $T\widehat{M}$. In this case, the pair $\left(\widehat{M}, \widehat{F}\right)$ is said to be a *metallic manifold*. If any Riemannian manifold $\left(\widehat{M}, \widehat{g}\right)$ admits a metallic structure $\widehat{F}$ such that

$$\widehat{g}\left(\widehat{F}X, Y\right) = \widehat{g}\left(X, \widehat{F}Y\right), \tag{2.2}$$

or equivalenty

$$\widehat{g}\left(\widehat{F}X, \widehat{F}Y\right) = p\widehat{g}\left(\widehat{F}X, Y\right) + q\widehat{g}\left(X, Y\right) \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$, then the pair $\left(\widehat{g}, \widehat{F}\right)$ and the triple $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ are called a *metallic Riemannian structure* and a *metallic Riemannian manifold*, respectively [27]. Particulary, the triple $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ is termed a *locally decomposable metallic Riemannian manifold* if $\widehat{\nabla}\,\widehat{F} = 0$, where $\widehat{\nabla}$ stands for the Riemannian connection on $\widehat{M}$ [22].

Any metallic structure $\widehat{F}$ on a differentiable manifold $\widehat{M}$ defines two almost product structures on the same manifold as follows [27]:

$$\widehat{\Phi_1} = \frac{2\widehat{F} - pI}{2\sigma_{p,q} - p} \text{ and } \widehat{\Phi_2} = -\frac{2\widehat{F} - pI}{2\sigma_{p,q} - p}.$$

Conversely, if there exists an almost product structure $\widehat{\Phi}$ on $\widehat{M}$, then it induces two metallic structures on $\widehat{M}$ given by the following rules [27]:

$$\widehat{F_1} = \frac{p}{2}I + \frac{2\sigma_{p,q} - p}{2}\widehat{\Phi} \text{ and } \widehat{F_2} = \frac{p}{2}I - \frac{2\sigma_{p,q} - p}{2}\widehat{\Phi}.$$

We consider any m-dimensional isometrically immersed submanifold M of an $\widehat{m}$-dimensional metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$. Then the tangent space $T_P\widehat{M}$ at a point $P \in M$ is decomposed as follows:

$$T_P\widehat{M} = T_P M \oplus T_P M^{\perp},$$

where $T_P M$ and $T_P M^{\perp}$ are its tangent and normal spaces at $P \in M$, respectively. Hereafter for simplicity, unless otherwise stated, we denote by the same notation $\widehat{g}$ the Riemannian metric induced on M.

For any $X \in \Gamma(TM)$, we write

$$\widehat{F}X = TX + NX, \tag{2.4}$$

where $TX \in \Gamma(TM)$ and $NX \in \Gamma\left(TM^{\perp}\right)$. Hence, T is an endomorphism on TM and N is a normal bundle-valued 1-form. Similarly, for any $U \in \Gamma\left(TM^{\perp}\right)$, we put

$$\widehat{F}U = tU + nU, \tag{2.5}$$

where $tU \in \Gamma(TM)$ and $nU \in \Gamma\left(TM^{\perp}\right)$. We also have [3]

$$\widehat{g}(TX, Y) = \widehat{g}(X, TY) \tag{2.6}$$

and

$$\widehat{g}(nU, V) = \widehat{g}(U, nV) \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma\left(TM^{\perp}\right)$, i.e., T and n are $\widehat{g}$-symmetric operators. It can be also shown that the following relations hold [24]:

$$pT + qI = T^2 + tN, \tag{2.8}$$

$$pN = NT + nN, \tag{2.9}$$

$$pt = Tt + tn \tag{2.10}$$

and

$$pn + qI = n^2 + Nt. \tag{2.11}$$

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulas of M in $\widehat{M}$ are given, respectively, by

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.12}$$

and

$$\widehat{\nabla}_X U = -A_U X + \nabla^{\perp}_X U \tag{2.13}$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, where h is the second fundamental form, A_U is the Weingarten map with respect to U and $\nabla^{\perp}$ is the normal connection on M. Also, there exists a relation between h and A such that

$$\widehat{g}(h(X, Y), U) = \widehat{g}(A_U X, Y) \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$. The *covariant derivative* of h is defined by

$$(\nabla_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.15}$$

for all $X, Y, Z \in \Gamma(TM)$, where $\nabla^{\perp}$ denotes the normal connection on M [1]. The *squared norm* of h is given by

$$\|h\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \widehat{g}\left(h(B_i, B_j), h(B_i, B_j)\right), \tag{2.16}$$

where $\{B_1, \ldots, B_m\}$ is a local orthonormal frame for TM. The *relative null space* of M at a point $P \in M$ is defined by

$$\mathcal{N}_P = \{X_P \in T_P M : h(X_P, Y_P) = 0 \text{ for all } Y_P \in T_P M\},$$

which is also called the *kernel of* h at $P \in M$ [6, 7]. The *mean curvature vector* H of M is given by

$$H = \frac{1}{m} trh, \tag{2.17}$$

where $trh = \sum_{i=1}^{m} h(B_i, B_i)$. If h is identically zero, then M is called a *totally geodesic manifold*. If $H = 0$, we say that then M is a *minimal submanifold*. Furthermore, M is named a *totally umbilical submanifold* if $h(X,Y) = \widehat{g}(X,Y)H$ for all $X, Y \in \Gamma(TM)$ [37].

Let us denote by $\widehat{R}$ and R the Riemannian curvature tensors of $\widehat{M}$ and M, respectively. Then the Gauss, Codazzi and Ricci equations are given, respectively, as follows:

$$\widehat{g}\left(\widehat{R}(X,Y)Z, W\right) = \widehat{g}(R(X,Y)Z, W) - \widehat{g}(h(X,W), h(Y,Z)) + \widehat{g}(h(Y,W), h(X,Z)), \tag{2.18}$$

$$\left(\widehat{R}(X,Y)Z\right)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \tag{2.19}$$

and

$$\widehat{g}\left(\widehat{R}(X,Y)U, V\right) = \widehat{g}\left(R^{\perp}(X,Y)U, V\right) + \widehat{g}([A_U, A_V]X, Y) \tag{2.20}$$

for all $X, Y, Z, W \in \Gamma(TM)$ and $U, V \in \Gamma\left(TM^{\perp}\right)$, where $R^{\perp}$ is the Riemannian curvature tensor of $\nabla^{\perp}$. Moreover, if $\widehat{R}(X,Y)Z$ belongs to $\Gamma(TM)$ for all $X, Y, Z \in \Gamma(TM)$, then M is termed a *curvature invariant submanifold* [37].

The *Ricci tensor* Ric of M is given by

$$Ric(X,Y) = \sum_{i=1}^{m} \widehat{g}(R(B_i, X)Y, B_i) \tag{2.21}$$

for all $X, Y \in \Gamma(TM)$, so the *scalar curvature* ρ of M is defined by

$$\rho = \sum_{i=1}^{m} Ric(B_i, B_i), \tag{2.22}$$

where $\{B_1, \ldots, B_m\}$ is a local orthonormal frame for TM [1].

If M is a space of constant curvature c, then R is written in the following form:

$$R(X,Y)Z = c\{\widehat{g}(Y,Z)X - \widehat{g}(X,Z)Y\} \tag{2.23}$$

for all $X, Y, Z \in \Gamma(TM)$ [37].

We denote by $T_P^1 M$ the set of unit tangent vectors at a point $P \in M$ in $T_P M$, that is, $T_P^1 M = \{X_P \in T_P M : \widehat{g}(X_P, X_P) = 1\}$. If $\{b_1, \ldots, b_m\}$ is an orthonormal basis of $T_P M$, then for a fixed index $i \in \{1, \ldots, m\}$, the *Ricci curvature* of the basis element b_i, denoted by $Ric(b_i)$, is given by

$$Ric(b_i) = \sum_{j \neq i}^{m} K_{ij},$$

where K_{ij} stands for the sectional curvature of the 2-plane section Π_2 spanned by the basis elements b_i and b_j at $P \in M$ for any $i, j \in \{1, \ldots, m\}$ [20].

Let Π_k be a k-plane section of $T_P M$ at a point $P \in M$. The *k-th Ricci curvature* of Π_k at a unit tangent vector X_P is defined by

$$Ric_{\Pi_k}(X_P) = K_{12} + K_{13} + \cdots + K_{1k},$$

where X_P is determined by a chosen orthonormal basis $\{b_1, \ldots, b_m\}$ of $T_P M$ such that $b_1 = X_P$ [7]. We note that if $k = m$, then $\Pi_m = T_P M$ and $Ric_{\Pi_m}(X_P)$ is the usual Ricci curvature of $X_P \in T_P^1 M$, denoted by $Ric(X_P)$.

Finally, we recall the concept of a bi-slant submanifold in metallic Riemannian manifolds.

Any isometrically immersed submanifold M of a metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ is called *bi-slant* [23, 24] if there is a pair of orthogonal differentiable distributions D^{θ_1} and D^{θ_2} on M such that $TM = D^{\theta_1} \oplus D^{\theta_2}$ and D^{θ_1}, D^{θ_2} are slant distributions with the Wirtinger angles θ_1, θ_2, respectively.

A bi-slant submanifold M of a metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ is called *proper* if its Wirtinger angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$. Otherwise,

(a) If $\theta_1 = \theta_2 = 0$, then M is an *invariant submanifold* [3, 23, 24, 27],

(b) If $\theta_1 = \theta_2 = \frac{\pi}{2}$, then M is an *anti-invariant submanifold* [3, 23, 24],
(c) If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is a *semi-invariant submanifold* [23, 24],
(d) If $\theta_1 = \theta_2 = \theta$ and $\widehat{g}\left(\widehat{F}X, Y\right) = 0$ for all $X \in \Gamma\left(D^{\theta_1}\right)$ and $Y \in \Gamma\left(D^{\theta_2}\right)$, then M is a *slant submanifold* with the Wirtinger angle θ [23, 24],
(e) If $\theta_1 = 0$ and $\theta_2 \neq 0$, then M is a *semi-slant submanifold* [23, 24],
(f) If $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq \frac{\pi}{2}$, then M is a *hemi-slant submanifold* [24].

We consider a bi-slant submanifold M of a metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$. Let π_1 and π_2 be orthogonal projection operators on the slant distributions $D^{\theta_1}$and D^{θ_2} with the Wirtinger angles θ_1, θ_2, respectively. For any $X \in \Gamma(TM)$, we get

$$X = \pi_1 X + \pi_2 X.$$

Thus, (2.4) turns into

$$\widehat{F}X = \pi_1 TX + \pi_2 TX + NX = T\pi_1 X + T\pi_2 X + N\pi_1 X + N\pi_2 X$$

for any $X \in \Gamma(TM)$. Furthermore, taking account of [23, Proposition 22], we have

$$(\pi_A T)^2 \pi_A X = \cos^2\theta_A (p\pi_A T\pi_A X + q\pi_A X),\ A = 1, 2 \tag{2.24}$$

for any $X \in \Gamma(TM)$.

Now, we present some non-trivial examples of bi-slant submanifolds in metallic Riemannian manifolds.

Example 2.1. We consider 10-dimensional Euclidean space $\mathbb{R}^{10}$ equipped with the usual inner product $\langle,\rangle$. Let $i : M \to \mathbb{R}^{10}$ be the immersion by

$$i(u_1, u_2, u_3, v_1, v_2) = (u_1 \cos t, u_2 \cos t, u_3 \cos t, u_1 \sin t, u_2 \sin t, u_3 \sin t, v_1 - v_2, v_1 + v_2, v_1, v_2),$$

where $M = \left\{(u_1, u_2, u_3, v_1, v_2) : u_1, u_2, u_3, v_1, v_2 \in \mathbb{R}, t \in \left[0, \frac{\pi}{2}\right]\right\}$. Thus, we find a local orthonormal frame $\{B_1, B_2, B_3, B_4\}$ for TM such that

$$B_1 = \cos t\frac{\partial}{\partial x_1} + \sin t\frac{\partial}{\partial x_4},\ B_2 = \cos t\frac{\partial}{\partial x_2} + \sin t\frac{\partial}{\partial x_5},\ B_3 = \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial x_6},$$

$$B_4 = \frac{1}{\sqrt{3}}\left(\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}\right), B_5 = \frac{1}{\sqrt{3}}\left(-\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}}\right).$$

Let us define a tensor field $\widehat{F}$ of type $(1,1)$ on $\mathbb{R}^{10}$ by

$$\widehat{F}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (\sigma x_1, \sigma x_2, \sigma x_3, \widehat{\sigma} x_4, \widehat{\sigma} x_5, \widehat{\sigma} x_6, \sigma x_7, \sigma x_8, \widehat{\sigma} x_9, \widehat{\sigma} x_{10}),$$

where σ is the (p,q)-metallic number and $\widehat{\sigma} = p - \sigma$. In this situation, it is clear that the triple $(\mathbb{R}^{10}, \langle,\rangle, \widehat{F})$ is a metallic Riemannian manifold.

If we take $D^1 = Span\{B_1, B_2, B_3\}$ and $D^2 = Span\{B_4, B_5\}$, then D^1 and D^2 are slant distributions with the Wirtinger angles θ_1 and θ_2, respectively, where $\cos\theta_1 = \frac{\sigma\cos^2 t + \widehat{\sigma}\sin^2 t}{\sqrt{\sigma^2\cos^2 t + \widehat{\sigma}^2\sin^2 t}}$ and $\cos\theta_2 = \frac{p+\sigma}{\sqrt{3(2\sigma^2+\widehat{\sigma}^2)}}$. Hence, M is a 5-dimensional bi-slant submanifold. Furthermore, it is obvious to verify that if we take $t = \frac{1}{2}\arccos\frac{1}{3}$, then the Wirtinger angles of D_1 and D_2 are both equal, i.e., $\theta_1 = \theta_2$, so M is a slant submanifold with the Wirtinger angle $\theta_1 = \theta_2 = \arccos\left(\frac{p+\sigma}{\sqrt{3(2\sigma^2+\widehat{\sigma}^2)}}\right)$.

Remark 2.1. Like in Example 2.1, we note that if the Wirtinger angles of the slant distributions are equal in any bi-slant submanifold of a metallic Riemannian manifold, then it may be a slant submanifold; however, this fact isn't always true.

Example 2.2. Let $\widehat{F}$ be a tensor field of type $(1,1)$ on 8-dimensional Euclidean space $\left(\mathbb{R}^8, \langle,\rangle\right)$ by

$$\begin{aligned}\widehat{F}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \quad = \quad & \left(\frac{p}{2}x_1 + \frac{\sqrt{\Delta}}{2}x_6, \frac{p}{2}x_2 + \frac{\sqrt{\Delta}}{2}x_5, \frac{p}{2}x_3 + \frac{\sqrt{\Delta}}{2}x_8, \frac{p}{2}x_4 + \frac{\sqrt{\Delta}}{2}x_7,\right.\\ & \left.\frac{p}{2}x_5 + \frac{\sqrt{\Delta}}{2}x_2, \frac{p}{2}x_6 + \frac{\sqrt{\Delta}}{2}x_1, \frac{p}{2}x_7 + \frac{\sqrt{\Delta}}{2}x_4, \frac{p}{2}x_8 + \frac{\sqrt{\Delta}}{2}x_3\right),\end{aligned}$$

where $\langle , \rangle$ is the dot product on $\mathbb{R}^8$, $\Delta = p^2 + 4q$ and $p, q \in \mathbb{Z}^+$. It is explicit to check that the triple $(\mathbb{R}^8, \langle , \rangle, \widehat{F})$ is a metallic Riemannian manifold.

We consider the immersion i from M into $\mathbb{R}^8$ given by

$$\begin{aligned} i(u_1, u_2, v_1, v_2) = \; & (u_1 \cos t_1 - u_2 \sin t_1, u_1 \sin t_1 + u_2 \cos t_1, u_1 \cos t_2 - u_2 \sin t_2, u_1 \sin t_2 + u_2 \cos t_2, \\ & v_1 \cos t_2 - v_2 \sin t_2, v_1 \sin t_2 + v_2 \cos t_2, v_1 \cos t_1 - v_2 \sin t_1, v_1 \sin t_1 + v_2 \cos t_1), \end{aligned}$$

where $M = \left\{(u_1, u_2, v_1, v_2) : u_1, u_2, v_1, v_2 \in \mathbb{R}, t_1, t_2 \in \left(0, \frac{\pi}{2}\right]\right\}$. In this case, we find a local orthonormal frame $\{B_1, B_2, B_3, B_4\}$ for TM such that

$$B_1 = \frac{1}{\sqrt{2}} \left(\cos t_1 \frac{\partial}{\partial x_1} + \sin t_1 \frac{\partial}{\partial x_2} + \cos t_2 \frac{\partial}{\partial x_3} + \sin t_2 \frac{\partial}{\partial x_4} \right),$$

$$B_2 = \frac{1}{\sqrt{2}} \left(-\sin t_1 \frac{\partial}{\partial x_1} + \cos t_1 \frac{\partial}{\partial x_2} - \sin t_2 \frac{\partial}{\partial x_3} + \cos t_2 \frac{\partial}{\partial x_4} \right),$$

$$B_3 = \frac{1}{\sqrt{2}} \left(\cos t_2 \frac{\partial}{\partial x_5} + \sin t_2 \frac{\partial}{\partial x_6} + \cos t_1 \frac{\partial}{\partial x_7} + \sin t_1 \frac{\partial}{\partial x_8} \right),$$

$$B_4 = \frac{1}{\sqrt{2}} \left(-\sin t_2 \frac{\partial}{\partial x_5} + \cos t_2 \frac{\partial}{\partial x_6} - \sin t_1 \frac{\partial}{\partial x_7} + \cos t_1 \frac{\partial}{\partial x_8} \right).$$

If we take $D^1 = Span\{B_1, B_4\}$ and $D^2 = Span\{B_2, B_3\}$, then D^1 and D^2 are slant distributions with the same Wirtinger angle θ, where $\cos\theta = \sqrt{\frac{\Delta}{\Delta + p^2}} \cos(t_1 + t_2)$. Thus, M is a 4-dimensional bi-slant submanifold; however, it isn't a slant submanifold.

Example 2.3. Let $\mathbb{R}^8$ be 8-dimensional Euclidean space equipped with the usual inner product $\langle , \rangle$. We define a tensor field $\widehat{F}$ of type $(1,1)$ on $\mathbb{R}^8$ by

$$\begin{aligned} \widehat{F}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \; & \Big(\frac{p}{2}x_1 + \frac{\sqrt{\Delta}}{2}x_5, \frac{p}{2}x_2 + \frac{\sqrt{\Delta}}{2}x_6, \frac{p}{2}x_3 + \frac{\sqrt{\Delta}}{2}x_7, \frac{p}{2}x_4 + \frac{\sqrt{\Delta}}{2}x_8, \\ & \frac{p}{2}x_5 + \frac{\sqrt{\Delta}}{2}x_1, \frac{p}{2}x_6 + \frac{\sqrt{\Delta}}{2}x_2, \frac{p}{2}x_7 + \frac{\sqrt{\Delta}}{2}x_3, \frac{p}{2}x_8 + \frac{\sqrt{\Delta}}{2}x_4 \Big), \end{aligned}$$

where $\Delta = p^2 + 4q$ and $p, q \in \mathbb{Z}^+$. In this case, the triple $(\mathbb{R}^8, \langle , \rangle, \widehat{F})$ is a metallic Riemannian manifold.

We consider a submanifold M of $(\mathbb{R}^8, \langle , \rangle, \widehat{F})$ determined by the following immersion $i : M \to \mathbb{R}^8$:

$$i(u_1, u_2, v_1, v_2) = (u_1, v_1, v_1, u_1, u_2 \cos t_1, u_2 \sin t_1, v_2 \cos t_2, v_2 \sin t_2),$$

where $M = \left\{(u_1, u_2, v_1, v_2) : u_1, u_2, v_1, v_2 \in \mathbb{R}, t_1, t_2 \in \left(0, \frac{\pi}{2}\right]\right\}$. Hence, TM has a local orthonormal frame $\{B_1, B_2, B_3, B_4\}$ such that

$$B_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4} \right), B_2 = \left(\cos t_1 \frac{\partial}{\partial x_5} + \sin t_1 \frac{\partial}{\partial x_6} \right),$$

$$B_3 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right), B_4 = \left(\cos t_2 \frac{\partial}{\partial x_7} + \sin t_2 \frac{\partial}{\partial x_8} \right).$$

If we put $D^1 = Span\{B_1, B_2\}$ and $D^2 = Span\{B_3, B_4\}$, then D^1 and D^2 are slant distributions with the Wirtinger angles θ_1 and θ_2, respectively, where $\cos\theta_1 = \sqrt{\frac{\Delta}{2(\Delta + p^2)}} \cos t_1$ and $\cos\theta_2 = \sqrt{\frac{\Delta}{2(\Delta + p^2)}} \cos t_2$. Thus, M is a 4-dimensional bi-slant submanifold. In particular, if we take $\theta_1 = \theta_2$, then it is seen that M isn't a slant submanifold.

Remark 2.2. As it is understood from the above two examples, it is worth noting that if both distributions are slant ones with the same Wirtinger angle in any bi-slant submanifold of a metallic Riemannian manifold, then it isn't necessarily a slant submanifold.

3. Metallic structures on product manifolds

In this section, we examine locally decomposable metallic Riemannian manifolds. We also obtain some important results on their invariant and anti-invariant submanifolds.

A metallic structure $\widehat{F}$ on a differentiable manifold $\widehat{M}$ induces two maps denoted by $\widehat{r_1}$ and $\widehat{r_2}$, respectively, as follows:

$$\widehat{r_1} = -\frac{1}{2\sigma_{p,q} - p}\left((p - \sigma_{p,q})\, I - \widehat{F}\right) \tag{3.1}$$

and

$$\widehat{r_2} = \frac{1}{2\sigma_{p,q} - p}\left(\sigma_{p,q} I - \widehat{F}\right), \tag{3.2}$$

where $\sigma_{p,q}$ is the (p,q)-metallic number and I is the identity operator on $T\widehat{M}$. Hence, the maps $\widehat{r_1}$ and $\widehat{r_2}$ verify that the following relations:

$$\widehat{r_1} + \widehat{r_2} = I,\, \widehat{r_1}^2 = \widehat{r_1},\, \widehat{r_2}^2 = \widehat{r_2},\, \widehat{r_1}\,\widehat{r_2} = \widehat{r_2}\,\widehat{r_1} = 0. \tag{3.3}$$

That is, $\widehat{r_1}$ and $\widehat{r_2}$ are projection operators. Moreover, $\widehat{F}$ is stated in the following form:

$$\widehat{F} = \sigma_{p,q}\widehat{r_1} + (p - \sigma_{p,q})\,\widehat{r_2}, \tag{3.4}$$

from which we have

$$\widehat{F}\,\widehat{r_1} = \widehat{r_1}\,\widehat{F} = \sigma_{p,q}\widehat{r_1} \tag{3.5}$$

and

$$\widehat{F}\,\widehat{r_2} = \widehat{r_2}\,\widehat{F} = (p - \sigma_{p,q})\,\widehat{r_2}. \tag{3.6}$$

We denote by $\widehat{D^1}$ and $\widehat{D^2}$ the distributions associated with $\widehat{r_1}$ and $\widehat{r_2}$, respectively. That is, we have

$$\widehat{D^1} = \bigcup_{\widehat{P}\in\widehat{M}} \widehat{D^1}_{\widehat{P}},\, \widehat{D^1}_{\widehat{P}} = \left\{X_{\widehat{P}} \in T_{\widehat{P}}\widehat{M} : \widehat{F}X_{\widehat{P}} = \sigma_{p,q}X_{\widehat{P}}\right\} \tag{3.7}$$

and

$$\widehat{D^2} = \bigcup_{\widehat{P}\in\widehat{M}} \widehat{D^2}_{\widehat{P}},\, \widehat{D^2}_{\widehat{P}} = \left\{X_{\widehat{P}} \in T_{\widehat{P}}\widehat{M} : \widehat{F}X_{\widehat{P}} = (p - \sigma_{p,q})\,X_{\widehat{P}}\right\}. \tag{3.8}$$

Theorem 3.1. *Let $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ be a metallic Riemannian manifold. Then $\widehat{M}$ is a locally decomposable metallic Riemannian manifold such that $\widehat{M} = \widehat{M_1} \times \widehat{M_2}$ if and only if the distributions $\widehat{D^1}$ and $\widehat{D^2}$ are parallel with respect to the Riemannian connection $\widehat{\nabla}$, where $\widehat{M_1}$ and $\widehat{M_2}$ are integral manifolds of $\widehat{D^1}$ and $\widehat{D^2}$, respectively.*

Proof. We assume that $\widehat{M}$ is a locally decomposable metallic Riemannian manifold. Let $Y \in \Gamma\left(\widehat{D^1}\right)$. Because of the fact that $\widehat{\nabla}\,\widehat{F} = 0$, we obtain from (3.7) that

$$\widehat{F}\,\widehat{\nabla}_X Y = \widehat{\nabla}_X \widehat{F} Y = \sigma_{p,q}\widehat{\nabla}_X Y,$$

or equivalently

$$\widehat{\nabla}_X Y \in \Gamma\left(\widehat{D^1}\right) \tag{3.9}$$

for all $X \in \Gamma\left(T\widehat{M}\right)$. In other words, $\widehat{D^1}$ is parallel with respect to $\widehat{\nabla}$. Similarly, it can be proven that $\widehat{D^2}$ is so.

The converse can be easily shown by use of (3.4). Consequently, the proof has been obtained. □

Let $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ be a locally decomposable metallic Riemannian manifold such that $\widehat{M} = \widehat{M_1} \times \widehat{M_2}$. In this case, the Riemannian metric $\widehat{g}$ can be expressed by

$$\widehat{g}\,(X, Y) = \widehat{g_1}\,(\widehat{r_1}X, \widehat{r_1}Y) + \widehat{g_2}\,(\widehat{r_2}X, \widehat{r_2}Y)$$

for all $X, Y \in \Gamma\left(T\widehat{M}\right)$, where $\widehat{g_1}$ and $\widehat{g_2}$ are the Riemannian metrics of $\widehat{M_1}$ and $\widehat{M_2}$, respectively. As well, it is readily concluded from (3.5) and (3.6) that

$$\widehat{g}\left(\widehat{F}X, Y\right) = \sigma_{p,q}\widehat{g_1}\left(\widehat{r_1}X, \widehat{r_1}Y\right) + \left(p - \sigma_{p,q}\right)\widehat{g_2}\left(\widehat{r_2}X, \widehat{r_2}Y\right)$$

and

$$\widehat{g}\left(\widehat{F}X, \widehat{F}Y\right) = \sigma_{p,q}^2\widehat{g_1}\left(\widehat{r_1}X, \widehat{r_1}Y\right) + \left(p - \sigma_{p,q}\right)^2\widehat{g_2}\left(\widehat{r_2}X, \widehat{r_2}Y\right)$$

for all $X, Y \in \Gamma\left(T\widehat{M}\right)$. We also put $\dim\widehat{M_1} = \widehat{m_1}$ and $\dim\widehat{M_2} = \widehat{m_2}$.

Theorem 3.2. *Let $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ be a locally decomposable metallic Riemannian manifold, where $\widehat{M} = \widehat{M_1} \times \widehat{M_2}$ and $\widehat{m_1}, \widehat{m_2} > 2$. Then $\left(\widehat{M_1}, \widehat{g_1}\right)$ and $\left(\widehat{M_2}, \widehat{g_2}\right)$ are Einstein manifolds with the Ricci curvatures $\widehat{k_1}$ and $\widehat{k_2}$, respectively, if and only if the Ricci tensor $\widehat{Ric}$ of $\widehat{M}$ has the form*

$$\widehat{Ric}\left(X, Y\right) = \lambda\widehat{g}\left(X, Y\right) + \mu\widehat{g}\left(\widehat{F}X, Y\right) \tag{3.10}$$

for all $X, Y \in \Gamma\left(T\widehat{M}\right)$, where $\lambda = \frac{-(p-\sigma_{p,q})\widehat{k_1} + \sigma_{p,q}\widehat{k_2}}{2\sigma_{p,q} - p}$ and $\mu = \frac{\widehat{k_1} - \widehat{k_2}}{2\sigma_{p,q} - p}$.

Proof. If $\widehat{M_1}$ and $\widehat{M_2}$ are Einstein manifolds with the Ricci curvatures $\widehat{k_1}$ and $\widehat{k_2}$, respectively, then we have

$$\widehat{Ric}^{\widehat{M_1}} = \widehat{k_1}\,\widehat{g_1} \tag{3.11}$$

and

$$\widehat{Ric}^{\widehat{M_2}} = \widehat{k_2}\,\widehat{g_2}, \tag{3.12}$$

where $\widehat{Ric}^{\widehat{M_1}}$ and $\widehat{Ric}^{\widehat{M_2}}$ are the Ricci tensors of $\widehat{M_1}$ and $\widehat{M_2}$, respectively. In this case, it is laborless to deduce from (3.5) and (3.6) that the relations in (3.11) and (3.12) can be expressed in the above form (3.10).

The converse follows directly from (3.5) and (3.6) by a simple computation. Consequently, the proof has been completed. □

Theorem 3.3. *Let $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ be a locally decomposable metallic Riemannian manifold, where $\widehat{M} = \widehat{M_1} \times \widehat{M_2}$ and $\widehat{m_1}, \widehat{m_2} > 2$. Then $\left(\widehat{M_1}, \widehat{g_1}\right)$ and $\left(\widehat{M_2}, \widehat{g_2}\right)$ are spaces of constant curvatures $\widehat{c_1}$ and $\widehat{c_2}$, respectively, if and only if the Riemannian curvature tensor $\widehat{R}$ of $\widehat{M}$ has the form*

$$\begin{aligned}\widehat{R}\left(X, Y\right)Z \;=\;& a\left\{\widehat{g}\left(Y, Z\right)X - \widehat{g}\left(X, Z\right)Y + \widehat{g}\left(\widehat{F}Y, Z\right)\widehat{F}X - \widehat{g}\left(\widehat{F}X, Z\right)\widehat{F}Y\right\} \\ &+ b\left\{\widehat{g}\left(\widehat{F}Y, Z\right)X - \widehat{g}\left(\widehat{F}X, Z\right)Y + \widehat{g}\left(Y, Z\right)\widehat{F}X - \widehat{g}\left(X, Z\right)\widehat{F}Y\right\}\end{aligned} \tag{3.13}$$

for all $X, Y, Z \in \Gamma\left(T\widehat{M}\right)$, where $a = \frac{-(p-\sigma_{p,q})\widehat{c_1} + \sigma_{p,q}\widehat{c_2}}{(q+1)(2\sigma_{p,q} - p)}$ and $b = \frac{\left(1 + (p-\sigma_{p,q})^2\right)\widehat{c_1} - \left(1 + \sigma_{p,q}^2\right)\widehat{c_2}}{2(q+1)(2\sigma_{p,q} - p)}$.

Proof. If $\widehat{M_1}$ and $\widehat{M_2}$ are spaces of constant curvatures $\widehat{c_1}$ and $\widehat{c_2}$, respectively, then we have

$$\widehat{R}^{\widehat{M_1}}\left(\widehat{r_1}X, \widehat{r_1}Y\right)\widehat{r_1}Z = \widehat{c_1}\left\{\widehat{g_1}\left(\widehat{r_1}Y, \widehat{r_1}Z\right)\widehat{r_1}X - \widehat{g_1}\left(\widehat{r_1}X, \widehat{r_1}Z\right)\widehat{r_1}Y\right\} \tag{3.14}$$

and

$$\widehat{R}^{\widehat{M_2}}\left(\widehat{r_2}X, \widehat{r_2}Y\right)\widehat{r_2}Z = \widehat{c_2}\left\{\widehat{g_2}\left(\widehat{r_2}Y, \widehat{r_2}Z\right)\widehat{r_2}X - \widehat{g_2}\left(\widehat{r_2}X, \widehat{r_2}Z\right)\widehat{r_2}Y\right\} \tag{3.15}$$

for all $X, Y, Z \in \Gamma\left(T\widehat{M}\right)$, where $\widehat{R}^{\widehat{M_1}}$ and $\widehat{R}^{\widehat{M_2}}$ are the Riemannian curvature tensors of $\widehat{M_1}$ and $\widehat{M_2}$, respectively. By a direct computation, it seems from (3.5) and (3.6) that the relations in (3.14) and (3.15) can be worded in the above form (3.13).

Conversely, if (3.13) holds, then we conclude from (3.5) and (3.6) that $\widehat{M_1}$ and $\widehat{M_2}$ are spaces of constant curvatures $\widehat{c_1}$ and $\widehat{c_2}$, respectively. Therefore, the proof has been shown. □

Theorem 3.4. *Let M be any invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1} \times \widehat{M_2}$. Then there exist two submanifolds M_1 and M_2 being totally geodesic in M such that $M = M_1 \times M_2$ is a locally decomposable metallic Riemannian manifold, where M_1 and M_2 are submanifolds of $\widehat{M_1}$ and $\widehat{M_2}$, respectively.*

Proof. We define two subspaces of $T_P M$ at each point $P \in M$ as follows:

$$D^1_P = \{X_P \in T_P M : JX_P = \sigma_{p,q} X_P\} \tag{3.16}$$

and

$$D^2_P = \{X_P \in T_P M : JX_P = (p - \sigma_{p,q}) X_P\}, \tag{3.17}$$

where J denotes the metallic structure on M. Then the subspaces given in (3.16) and (3.17) define two distributions $D^1 = \bigcup_{P \in M} D^1_P$ and $D^2 = \bigcup_{P \in M} D^2_P$, respectively. Considering the fact that $\nabla J = 0$ for invariant submanifolds of locally decomposable metallic Riemannian manifolds [3, Corollary 3.13], the parallelism of D^1 and D^2 can be shown in similar way to the proof of Theorem 3.1. Let us denote by M_1 and M_2 the integral manifolds of D^1 and D^2, respectively. Hence, from [1, Theorem 4.4], M_1 and M_2 are totally geodesic in M. Now, we show that M_1 and M_2 are submanifolds of $\widehat{M_1}$ and $\widehat{M_2}$, respectively. If $X \in \Gamma\left(D^1\right)$, then we infer from (3.1) and (3.2) that $\widehat{r_1} X = X$ and $\widehat{r_2} X = 0$. Thus, we get that X pertains to $\Gamma\left(T\widehat{M_1}\right)$, i.e., M_1 is a submanifold of $\widehat{M_1}$. By a similar argument as above, it can be demonstrated that M_2 is a submanifold of $\widehat{M_2}$. Therefore, the proof has been completed. □

From now on unless otherwise stated, we suppose that $\widehat{M_1}$ and $\widehat{M_2}$ are spaces of constant curvatures $\widehat{c_1}$ and $\widehat{c_2}$, respectively, with $\widehat{m_1}, \widehat{m_2} > 2$.

Theorem 3.5. *Any invariant submanifold M of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$ is curvature invariant, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$.*

Proof. In the light of that M is invariant, it follows from (3.13) that the Codazzi equation takes the form

$$\left(\widehat{R}(X,Y)Z\right)^{\perp} = 0$$

for all $X, Y, Z \in \Gamma(TM)$, which refers that $\widehat{R}(X,Y)Z \in \Gamma(TM)$, i.e., M is a curvature invariant submanifold. □

Lemma 3.1. *Let M be an arbitrary m-dimensional submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$. Then the Ricci tensor Ric of M is given by*

$$\begin{aligned} Ric(X,Y) \;=\; & a\left\{(m-1-q)\,\widehat{g}(X,Y) + \left(tr\widehat{F} - p\right)\widehat{g}\left(\widehat{F}X, Y\right)\right\} \\ & + b\left\{tr\widehat{F}\widehat{g}(X,Y) + (m-2)\,\widehat{g}\left(\widehat{F}X, Y\right)\right\} \\ & + m\widehat{g}\left(h(X,Y), H\right) - \sum_{i=1}^{m} \widehat{g}\left(h(X,B_i), h(Y,B_i)\right) \end{aligned} \tag{3.18}$$

for all $X, Y \in \Gamma(TM)$, where $\{B_1, \ldots, B_m\}$ is a local orthonormal frame for TM. Also, it concludes that the scalar curvature ρ of M is given by

$$\rho = a\left(m(m-1-q) + \left(tr\widehat{F} - p\right)tr\widehat{F}\right) + 2b(m-1)\,tr\widehat{F} + m^2\|H\|^2 - \|h\|^2. \tag{3.19}$$

Proof. It follows from (3.13) that the Gauss equation for M in $\widehat{M}$ is given by

$$\begin{aligned} R(X,Y)Z \;=\; & a\left\{\widehat{g}(Y,Z)X - \widehat{g}(X,Z)Y + \widehat{g}\left(\widehat{F}Y, Z\right)\widehat{F}X - \widehat{g}\left(\widehat{F}X, Z\right)\widehat{F}Y\right\} \\ & + b\left\{\widehat{g}\left(\widehat{F}Y, Z\right)X - \widehat{g}\left(\widehat{F}X, Z\right)Y + \widehat{g}(Y,Z)\widehat{F}X - \widehat{g}(X,Z)\widehat{F}Y\right\} \\ & + A_{h(Y,Z)}X - A_{h(X,Z)}Y \end{aligned} \tag{3.20}$$

for all $X, Y, Z \in \Gamma(TM)$. Hence, using (2.2), (2.3), (2.14), (2.17) and (3.20), a direct computation shows us that the Ricci tensor Ric defined by (2.21) is equal to (3.18). Besides, taking account of the expression of the scalar curvature ρ in (2.22), we obtain from (2.16), (2.17) and (3.18) that (3.19) holds. □

Lemma 3.2. *Let M be any m-dimensional anti-invariant submanifold of an $\widehat{m}$-dimensional locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$. Then we have $A_{\widehat{F}X}Y = 0$ for all $X, Y \in \Gamma(TM)$. Additionally, if $\widehat{m} = 2m$, then M is totally geodesic.*

Proof. Using the fact that $\widehat{\nabla}\widehat{F} = 0$, we obtain from the Gauss and Weingarten formulas that

$$-A_{\widehat{F}Y}X + \nabla^{\perp}_{X}\widehat{F}Y = \widehat{F}\nabla_X Y + \widehat{F}h(X, Y)$$

for all $X, Y \in \Gamma(TM)$, from which we have

$$\widehat{g}\left(A_{\widehat{F}Y}X, Z\right) = -\widehat{g}\left(\widehat{F}h(X, Y), Z\right) \tag{3.21}$$

for all $Z \in \Gamma(TM)$. Since h is symmetric, we get from (3.21) that

$$A_{\widehat{F}Y}X = A_{\widehat{F}X}Y \tag{3.22}$$

for all $X, Y \in \Gamma(TM)$. If (3.21) is used again, then the self adjointness of A states that

$$A_{\widehat{F}Y}X = -A_{\widehat{F}X}Y \tag{3.23}$$

for all $X, Y \in \Gamma(TM)$. Therefore, it results from (3.22) and (3.23) that

$$A_{\widehat{F}Y}X = 0. \tag{3.24}$$

Besides, if $\widehat{m} = 2m$, then $\left\{\widehat{F}X : X \in \Gamma(TM)\right\}$ is a local frame for $TM^{\perp}$. Hence, we obtain

$$\widehat{g}\left(h(X, Z), \widehat{F}Y\right) = \widehat{g}\left(A_{\widehat{F}Y}X, Z\right) = 0,$$

which implies from (3.24) that $h = 0$, in other words, M is totally geodesic. □

Theorem 3.6. *Let M be any m-dimensional anti-invariant submanifold of a $2m$-dimensional locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M}_1(\widehat{c}_1) \times \widehat{M}_2(\widehat{c}_2)$. Then M is a space of constant curvature a.*

Proof. By means of the anti-invariance of M, Lemma 3.2 tells us that the Gauss equation given in (2.18) is reduced to the form

$$R(X, Y)Z = a\{\widehat{g}(Y, Z)X - \widehat{g}(X, Z)Y\}$$

for all $X, Y, Z \in \Gamma(TM)$, which implies from (2.23) that M is a space of constant curvature a. □

Theorem 3.7. *Let M be any m-dimensional anti-invariant submanifold of a $2m$-dimensional locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M}_1(\widehat{c}_1) \times \widehat{M}_2(\widehat{c}_2)$. Then the Ricci tensor Ric and the scalar curvature ρ of M are given by*

$$Ric = a(m - 1 - q)\widehat{g}$$

and

$$\rho = am(m - 1 - q),$$

respectively.

Proof. Using Lemmas 3.1 and 3.2, the proof can be easily obtained. □

4. Chen-Ricci inequalities for submanifolds of metallic Riemannian manifolds

In this section, we give the Chen-Ricci inequalities for invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds of a locally decomposable metallic Riemannian manifold whose components are spaces of constant curvature.

We start by recalling the main theorem for Riemannian manifolds regarding the Chen-Ricci inequalities given by Hong and Tripathi in [21].

Theorem 4.1. *Let M be any m-dimensional submanifold of a Riemannian manifold $\left(\widehat{M}, \widehat{g}\right)$. Then for any point $P \in M$, the following assertions hold:*

(a) *For all $X_P \in T_P^1 M$, we have*

$$Ric\left(X_P\right) \leq \frac{1}{4} m^2 \left\|H_P\right\|^2 + \widehat{Ric}_{(T_P M)}\left(X_P\right), \tag{4.1}$$

where $\widehat{Ric}_{(T_P M)}\left(X_P\right)$ is the m-th Ricci curvature of $T_P M$ at $X_P \in T_P^1 M$ with respect to $\widehat{M}$.

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.1) is valid if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$.*

(c) *The equality case in (4.1) is satisfied if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Let $\{b_1, \ldots, b_m\}$ be an orthonormal basis for $T_P M$ at a point P. We denote by $\widehat{K_{ij}}$ the sectional curvature of the 2-plane section spanned by the basis elements b_i and b_j at P in $\widehat{M}$ for any $i, j \in \{1, \ldots, m\}$. Then by the help of (3.13), a direct computation gives us that

$$\widehat{K_{ij}} = \widehat{K}(b_i \wedge b_j) = a\left\{1 + \widehat{g}\left(Tb_i, b_i\right) \widehat{g}\left(Tb_j, b_j\right) - \widehat{g}^2\left(Tb_i, b_j\right)\right\} + b\left\{\widehat{g}\left(Tb_i, b_i\right) + \widehat{g}\left(Tb_j, b_j\right)\right\} \tag{4.2}$$

for any $i, j \in \{1, \ldots, m\}$. Also, the squared norm of T at P is given by

$$\left\|Tb_i\right\|^2 = \sum_{j=1}^{m} \widehat{g}^2\left(Tb_i, b_j\right). \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\begin{aligned}
\widehat{Ric}_{(T_P M)}\left(b_i\right) &= \sum_{j=1,\, j \neq i}^{m} \widehat{K_{ij}} \\
&= a\left\{m - 1 + \widehat{g}\left(Tb_i, b_i\right) \sum_{j=1}^{m} \widehat{g}\left(Tb_j, b_j\right) - \widehat{g}^2\left(Tb_i, b_i\right) - \sum_{j=1}^{m} \widehat{g}^2\left(Tb_i, b_j\right)\right. \\
&\quad \left. + \widehat{g}^2\left(Tb_i, b_i\right)\right\} + b\left\{(m-1)\widehat{g}\left(Tb_i, b_i\right) + \sum_{j=1}^{m} \widehat{g}\left(Tb_j, b_j\right) - \widehat{g}\left(Tb_i, b_i\right)\right\} \\
&= a\left\{m - 1 + \widehat{g}\left(Tb_i, b_i\right) trT - \left\|Tb_i\right\|^2\right\} + b\left\{(m-2)\widehat{g}\left(Tb_i, b_i\right) + trT\right\}.
\end{aligned}$$

Hence, we obtain that $\widehat{Ric}_{(T_P M)}\left(X_P\right)$ is written in the following form:

$$\widehat{Ric}_{(T_P M)}\left(X_P\right) = a\left\{m - 1 + \widehat{g}\left(TX_P, X_P\right) trT - \left\|TX_P\right\|^2\right\} + b\left\{(m-2)\widehat{g}\left(TX_P, X_P\right) + trT\right\} \tag{4.4}$$

for all $X_P \in T_P^1 M$.

Hereafter we denote by $\{b_1, \ldots, b_m\}$ the orthonormal basis of $T_P M$ corresponding to a local orthonormal frame $\{B_1, \ldots, B_m\}$ of TM at a point P.

Theorem 4.2. *Let M be any m-dimensional submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M}_1\left(\widehat{c}_1\right) \times \widehat{M}_2\left(\widehat{c}_2\right)$. Then for any point $P \in M$, the following assertions are correct:*

(a) *For all $X_P \in T_P^1M$, we have*

$$Ric(X_P) \leq \frac{1}{4}m^2 \|H_P\|^2 + a\left\{m-1+\widehat{g}(TX_P,X_P)\,trT - \|TX_P\|^2\right\} + b\left\{(m-2)\,\widehat{g}(TX_P,X_P) + trT\right\}. \tag{4.5}$$

(b) *For a fixed $X_P \in T_P^1M$, the equality case in (4.5) holds if and only if $h(X_P,X_P) = \frac{1}{2}mH_P$ and $h(X_P,Y_P) = 0$ for every $Y_P \in T_PM$ such that $\widehat{g}(X_P,Y_P) = 0$. Furthermore, provided that $H_P = 0$, then $X_P \in T_P^1M$ satisfies the equality case in (4.5) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.5) is valid if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Proof. Using the expression of $\widehat{Ric}_{(T_PM)}(X_P)$ in (4.4), it can be easily obtained from Theorem 4.1 that the assertions (a), (b) and (c) are correct. □

Theorem 4.3. *Let M be any m-dimensional bi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}(\widehat{c_1}) \times \widehat{M_2}(\widehat{c_2})$. Then for any point $P \in M$, the following assertions are valid:*

(a) *For all $X_P \in T_P^1M$, we have*

$$\begin{aligned} Ric(X_P) \leq{} & \frac{1}{4}m^2\|H_P\|^2 + a\Bigg\{m-1-q\sum_{A=1}^{2}\cos^2\theta_A + 2\widehat{g}(\pi_1T\pi_2X_P,\pi_1X_P)\,trT \\ & + \sum_{A=1}^{2}\Big(\widehat{g}(\pi_AT\pi_AX_P,\pi_AX_P)\left(trT - p\cos^2\theta_A\right) + q\cos^2\theta_A\|\pi_{3-A}X_P\|^2 \\ & - 2\widehat{g}(\pi_AT\pi_AX_P,\pi_AT\pi_{3-A}X_P) - \|\pi_AT\pi_{3-A}X_P\|^2\Big)\Bigg\} \\ & + b\left\{(m-2)\left(\sum_{A=1}^{2}\widehat{g}(\pi_AT\pi_AX_P,\pi_AX_P) + 2\widehat{g}(\pi_1T\pi_2X_P,\pi_1X_P)\right) + trT\right\}, \end{aligned} \tag{4.6}$$

where π_1 and π_2 are the projection operators of TM onto the slant distributions D^{θ_1} and D^{θ_2} with the Wirtinger angles θ_1 and θ_2, respectively.

(b) *For a fixed $X_P \in T_P^1M$, the equality case in (4.6) holds if and only if $h(X_P,X_P) = \frac{1}{2}mH_P$ and $h(X_P,Y_P) = 0$ for every $Y_P \in T_PM$ such that $\widehat{g}(X_P,Y_P) = 0$. Furthermore, provided that $H_P = 0$, then $X_P \in T_P^1M$ yields the equality case in (4.6) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.6) is satisfied if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Proof. Using the main characterizations of the slant distributions D^{θ_1} and D^{θ_2} given in (2.24), we deduce from (4.4) that $\widehat{Ric}_{(T_PM)}(X_P)$ has the following form:

$$\begin{aligned} \widehat{Ric}_{(T_PM)}(X_P) ={} & a\Bigg\{m-1-q\sum_{A=1}^{2}\cos^2\theta_A + 2\widehat{g}(\pi_1T\pi_2X_P,\pi_1X_P)\,trT \\ & + \sum_{A=1}^{2}\Big(\widehat{g}(\pi_AT\pi_AX_P,\pi_AX_P)\left(trT - p\cos^2\theta_A\right) + q\cos^2\theta_A\|\pi_{3-A}X_P\|^2 \\ & - 2\widehat{g}(\pi_AT\pi_AX_P,\pi_AT\pi_{3-A}X_P) - \|\pi_AT\pi_{3-A}X_P\|^2\Big)\Bigg\} \\ & + b\left\{(m-2)\left(\sum_{A=1}^{2}\widehat{g}(\pi_AT\pi_AX_P,\pi_AX_P) + 2\widehat{g}(\pi_1T\pi_2X_P,\pi_1X_P)\right) + trT\right\} \end{aligned}$$

for all $X_P \in T_P^1M$, where π_1 and π_2 are the projection operators of TM onto the slant distributions D^{θ_1} and D^{θ_2}, respectively. Thus, the proof follows immediately from Theorem 4.2. □

Theorem 4.4. *Let M be any m-dimensional bi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. If we establish a local orthonormal frame $\{B_1,\ldots,B_m\}$ at a point $P\in M$ for TM such that*

$$D^{\theta_1}=Span\left\{B_\alpha=\pm\frac{\sec\theta_1\pi_1TB_\alpha}{\left\|\widehat{F}B_\alpha\right\|}\right\}_{1\leq\alpha\leq k}\quad\text{and } D^{\theta_2}=Span\left\{B_{k+\beta}=\pm\frac{\sec\theta_2\pi_2TB_{k+\beta}}{\left\|\widehat{F}B_{k+\beta}\right\|}\right\}_{1\leq\beta\leq l},$$

where $TM=D^{\theta_1}\oplus D^{\theta_2}$ and D^{θ_1}, D^{θ_2} are the slant distributions with the Wirtinger angles θ_1, θ_2, respectively, then the following assertions hold:

(a) *For all $X_P\in T_P^1M$, we have*

$$\begin{aligned}Ric\left(X_P\right)\quad\leq\quad&\frac{1}{4}m^2\left\|H_P\right\|^2+a\left\{m-1-\frac{1}{2}\sum_{A=1}^{2}\left(p\sigma^{\theta_A}\cos^2\theta_A+q\cos^2\theta_A\right.\right.\\&\left.+4\widehat{g}\left(\pi_AT\pi_AX_P,\pi_AT\pi_{3-A}X_P\right)+2\left\|\pi_AT\pi_{3-A}X_P\right\|^2\right)\\&\left.+\frac{1}{2}\left(\sum_{A=1}^{2}\sigma^{\theta_A}+4\widehat{g}\left(\pi_1T\pi_2X_P,\pi_1X_P\right)\right)\left(\sum_{\alpha=1}^{k}\sigma_\alpha^{\theta_1}+\sum_{\beta=1}^{l}\sigma_{k+\beta}^{\theta_2}\right)\right\}\\&+b\left\{\frac{1}{2}\left(m-2\right)\left(\sum_{A=1}^{2}\sigma^{\theta_A}+4\widehat{g}\left(\pi_1T\pi_2X_P,\pi_1X_P\right)\right)+\sum_{\alpha=1}^{k}\sigma_\alpha^{\theta_1}+\sum_{\beta=1}^{l}\sigma_{k+\beta}^{\theta_2}\right\},\end{aligned}\tag{4.7}$$

where $\sigma^{\theta_1},\sigma_\alpha^{\theta_1}\in\left\{\frac{p-\sqrt{p^2+4q\sec^2\theta_1}}{2\sec^2\theta_1},\frac{p+\sqrt{p^2+4q\sec^2\theta_1}}{2\sec^2\theta_1}\right\}$ and $\sigma^{\theta_2},\sigma_{k+\beta}^{\theta_2}\in\left\{\frac{p-\sqrt{p^2+4q\sec^2\theta_2}}{2\sec^2\theta_2},\frac{p+\sqrt{p^2+4q\sec^2\theta_2}}{2\sec^2\theta_2}\right\}$.

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.7) is satisfied if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Moreover, provided that $H_P=0$, then $X_P\in T_P^1M$ verifies the equality case in (4.7) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.7) is valid if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

Proof. Taking account of the chosen of local orthonormal frames of the slant distributions D^{θ_1} and D^{θ_2}, a direct computation gives us that $\widehat{g}\left(Tb_\alpha,b_\alpha\right)=\frac{p\pm\sqrt{p^2+4q\sec^2\theta_1}}{2\sec^2\theta_1}$ for any $\alpha\in\{1,\ldots,k\}$ and $\widehat{g}\left(Tb_{k+\beta},b_{k+\beta}\right)=\frac{p\pm\sqrt{p^2+4q\sec^2\theta_2}}{2\sec^2\theta_2}$ for any $\beta\in\{1,\ldots,l\}$. Hence, putting $X_P=\frac{1}{\sqrt{2}}\left(b_1+b_{k+1}\right)$ in (4.6), Theorem 4.4 tells us that the assertions (a), (b) and (c) are true. □

Remark 4.1. Taking into consideration that invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds are a non-proper bi-slant submanifold in terms of the Wirtinger angles of the distributions involved in its definition, with the help of Theorem 4.3, we can obtain the Chen-Ricci inequalities for such types of submanifolds of metallic Riemannian manifolds whose components are spaces of constant curvature.

Choosing $\theta_1=\theta_2=0$ in Theorem 4.3, we have the following two results for invariant submanifolds:

Theorem 4.5. *Let M be any m-dimensional invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P\in M$, the following assertions hold:*

(a) *For all $X_P\in T_P^1M$, we have*

$$Ric\left(X_P\right)\leq\frac{1}{4}m^2\left\|H_P\right\|^2+a\left\{m-1-q+\widehat{g}\left(TX_P,X_P\right)\left(trT-p\right)\right\}+b\left\{\left(m-2\right)\widehat{g}\left(TX_P,X_P\right)+trT\right\}.\tag{4.8}$$

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.8) is valid if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Moreover, provided that $H_P=0$, then $X_P\in T_P^1M$ yields the equality case in (4.8) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.8) is verified if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

Theorem 4.6. *Let M be any m-dimensional invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. If we constitute a local orthonormal frame $\{B_1,\ldots,B_m\}$ at a point $P\in M$ for TM such that*

$$\left\{B_i=\pm\frac{\widehat{F}B_i}{\left\|\widehat{F}B_i\right\|}\right\}_{1\leq i\leq m},$$

then the following assertions are true:

(a) *For all $X_P\in T_P^1M$, we have*

$$Ric\left(X_P\right)\leq\frac{1}{4}m^2\left\|H_P\right\|^2+a\left\{m-1-q+\sigma\left(\sum_{i=1}^{m}\sigma_i-p\right)\right\}+b\left\{\sigma\left(m-2\right)+\sum_{i=1}^{m}\sigma_i\right\}, \tag{4.9}$$

where $\sigma,\sigma_i\in\{p-\sigma_{p,q},\sigma_{p,q}\}$.

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.9) is verified if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Additionally, provided that $H_P=0$, then $X_P\in T_P^1M$ satisfies the equality case in (4.9) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.9) is verified if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

Taking $\theta_1=\theta_2=\frac{\pi}{2}$ in Theorem 4.3, then the following result holds for anti-invariant submanifolds:

Theorem 4.7. *Let M be any m-dimensional anti-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P\in M$, the following assertions are valid:*

(a) *For all $X_P\in T_P^1M$, we have*

$$Ric\left(X_P\right)\leq\frac{1}{4}m^2\left\|H_P\right\|^2+a\left(m-1\right). \tag{4.10}$$

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.10) holds if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Furthermore, provided that $H_P=0$, then $X_P\in T_P^1M$ verifies the equality case in (4.10) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.10) is satisfied if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

Corollary 4.1. *Let M be any m-dimensional anti-invariant submanifold of a $2m$-dimensional locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P\in M$, the following assertions are verified:*

(a) *For all $X_P\in T_P^1M$, we have*

$$Ric\left(X_P\right)=a\left(m-1\right). \tag{4.11}$$

(b) *P is a totally geodesic point.*

Proof. The proof is a direct consequence of Lemma 3.2 and Theorem 4.7. □

Putting $\theta_1=0$ and $\theta_2=\frac{\pi}{2}$ in Theorem 4.3, we get the following two results for semi-invariant submanifolds:

Theorem 4.8. *Let M be any m-dimensional proper semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P\in M$, the following assertions are true:*

(a) *For all $X_P \in T_P^1 M$, we have*

$$\begin{aligned} Ric\left(X_P\right) \quad \leq \quad & \frac{1}{4} m^2 \left\| H_P \right\|^2 + a \left\{ m - 1 - q + \widehat{g}\left(TX_P, X_P\right)\left(trT - p\right) + q \left\| \pi_2 X_P \right\|^2 \right\} \\ & + b \left\{ \left(m-2\right) \widehat{g}\left(TX_P, X_P\right) + trT \right\}, \end{aligned} \tag{4.12}$$

where π_2 denotes the projection operator of TM onto the anti-invariant distribution $D^{\perp}$.

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.12) is valid if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$. Moreover, provided that $H_P = 0$, then $X_P \in T_P^1 M$ satisfies the equality case in (4.12) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.12) is verified if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Theorem 4.9. *Let M be any m-dimensional proper semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$. If we found a local orthonormal frame $\{B_1, \ldots, B_m\}$ at a point $P \in M$ for TM such that*

$$D = Span \left\{ B_\alpha = \pm \frac{\widehat{F} B_\alpha}{\left\| \widehat{F} B_\alpha \right\|} \right\}_{1 \leq \alpha \leq k} \quad \text{and } D^{\perp} = Span \left\{ B_{k+\beta} \right\}_{1 \leq \beta \leq l},$$

where D and $D^{\perp}$ are the invariant and anti-invariant distributions, respectively, then the following assertions are satisfied:

(a) *For all $X_P \in T_P^1 M$, we have*

$$Ric\left(X_P\right) \leq \frac{1}{4} m^2 \left\| H_P \right\|^2 + a \left\{ m - 1 - \frac{q}{2} + \frac{1}{2} \sigma \left(\sum_{\alpha=1}^{k} \sigma_\alpha - p \right) \right\} + b \left\{ \frac{1}{2} \sigma \left(m-2\right) + \sum_{\alpha=1}^{k} \sigma_\alpha \right\}, \tag{4.13}$$

where $\sigma, \sigma_\alpha \in \{p - \sigma_{p,q}, \sigma_{p,q}\}$.

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.13) is valid if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$. Additionally, provided that $H_P = 0$, then $X_P \in T_P^1 M$ yields the equality case in (4.13) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.13) holds if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

If we take $\theta_1 = \theta_2 = \theta$ and $\widehat{g}\left(\widehat{F} X, Y\right) = 0$ for all $X \in \Gamma\left(D^{\theta_1}\right)$ and $Y \in \Gamma\left(D^{\theta_2}\right)$ in Theorem 4.3, then the following two results are derived for slant submanifolds:

Theorem 4.10. *Let M be any m-dimensional proper slant submanifold with the Wirtinger angle θ of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P \in M$, the following assertions are correct:*

(a) *For all $X_P \in T_P^1 M$, we have*

$$\begin{aligned} Ric\left(X_P\right) \quad \leq \quad & \frac{1}{4} m^2 \left\| H_P \right\|^2 + a \left\{ m - 1 - q \cos^2 \theta + \widehat{g}\left(TX_P, X_P\right)\left(trT - p \cos^2 \theta\right) \right\} \\ & + b \left\{ \left(m-2\right) \widehat{g}\left(TX_P, X_P\right) + trT \right\}. \end{aligned} \tag{4.14}$$

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.14) holds if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$. Furthermore, provided that $H_P = 0$, then $X_P \in T_P^1 M$ verifies the equality case in (4.14) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.14) is valid if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Theorem 4.11. *Let M be any m-dimensional proper slant submanifold with the Wirtinger angle θ of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. If we create a local orthonormal frame $\{B_1,\ldots,B_m\}$ at a point $P\in M$ for TM such that*

$$\left\{B_i=\pm\frac{\sec\theta TB_i}{\left\|\widehat{F}B_i\right\|}\right\}_{1\leq i\leq m},$$

then the following assertions hold:

(a) *For all $X_P\in T_P^1M$, we have*

$$\begin{aligned} Ric\left(X_P\right) &\leq \frac{1}{4}m^2\left\|H_P\right\|^2+a\left\{m-1-q\cos^2\theta+\sigma^\theta\left(\sum_{i=1}^m\sigma_i^\theta-p\cos^2\theta\right)\right\} \\ &\quad +b\left\{\sigma^\theta\left(m-2\right)+\sum_{i=1}^m\sigma_i^\theta\right\}, \end{aligned} \tag{4.15}$$

where $\sigma^\theta,\sigma_i^\theta\in\left\{\frac{p-\sqrt{p^2+4q\sec^2\theta}}{2\sec^2\theta},\frac{p+\sqrt{p^2+4q\sec^2\theta}}{2\sec^2\theta}\right\}$.

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.15) is verified if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Moreover, provided that $H_P=0$, then $X_P\in T_P^1M$ yields the equality case in (4.15) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.15) is satisfied if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

If we put $\theta_1=0$ and $\theta_2=\theta\neq0$ in Theorem 4.3, then the following two results are obtained for semi-slant submanifolds:

Theorem 4.12. *Let M be any m-dimensional proper semi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P\in M$, the following assertions are verified:*

(a) *For all $X_P\in T_P^1M$, we have*

$$\begin{aligned} Ric\left(X_P\right) &\leq \frac{1}{4}m^2\left\|H_P\right\|^2+a\left\{m-1-q\left(1+\cos^2\theta\right)+\widehat{g}\left(T\pi_1X_P,X_P\right)\left(trT-p\right)\right. \\ &\quad \left.+\widehat{g}\left(T\pi_2X_P,X_P\right)\left(trT-p\cos^2\theta\right)+q\left(\left\|\pi_2X_P\right\|^2+\cos^2\theta\left\|\pi_1X_P\right\|^2\right)\right\} \\ &\quad +b\left\{\left(m-2\right)\sum_{A=1}^2\widehat{g}\left(T\pi_AX_P,X_P\right)+trT\right\}, \end{aligned} \tag{4.16}$$

where π_1 and π_2 are the projection operators of TM onto the invariant distribution D and the slant distribution D^θ with the Wirtinger angle $\theta\neq0$, respectively.

(b) *For a fixed $X_P\in T_P^1M$, the equality case in (4.16) holds if and only if $h\left(X_P,X_P\right)=\frac{1}{2}mH_P$ and $h\left(X_P,Y_P\right)=0$ for every $Y_P\in T_PM$ such that $\widehat{g}\left(X_P,Y_P\right)=0$. Additionally, provided that $H_P=0$, then $X_P\in T_P^1M$ verifies the equality case in (4.16) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.16) is valid if and only if either P is a totally geodesic point or $m=2$ and P is a totally umbilical point.*

Theorem 4.13. *Let M be any m-dimensional proper semi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M},\widehat{g},\widehat{F}\right)$, where $\widehat{M}=\widehat{M_1}\left(\widehat{c_1}\right)\times\widehat{M_2}\left(\widehat{c_2}\right)$. If we form a local orthonormal frame $\{B_1,\ldots,B_m\}$ at a point $P\in M$ for TM such that*

$$D=Span\left\{B_\alpha=\pm\frac{\widehat{F}B_\alpha}{\left\|\widehat{F}B_\alpha\right\|}\right\}_{1\leq\alpha\leq k}\quad\text{and }D^\theta=\left\{B_{k+\beta}=\pm\frac{\sec\theta TB_{k+\beta}}{\left\|\widehat{F}B_{k+\beta}\right\|}\right\}_{1\leq\beta\leq l},$$

where D and D^{θ} are the invariant distribution and the slant distribution with the Wirtinger angle $\theta \neq 0$, respectively, then the following assertions are correct:

(a) *For all $X_P \in T_P^1 M$, we have*

$$Ric\left(X_P\right) \leq \frac{1}{4} m^2 \left\|H_P\right\|^2 + a \left\{ m - 1 - \frac{q}{2}\left(1+\cos^2\theta\right) + \frac{1}{2}\left(\sigma + \sigma^{\theta}\right)\left(\sum_{\alpha=1}^{k} \sigma_{\alpha} + \sum_{\beta=1}^{l} \sigma_{k+\beta}^{\theta}\right) \right.$$
$$\left. - \frac{p}{2}\left(\sigma + \sigma^{\theta}\cos^2\theta\right) \right\} + b\left\{ \frac{1}{2}\left(m-2\right)\left(\sigma+\sigma^{\theta}\right) + \sum_{\alpha=1}^{k}\sigma_{\alpha} + \sum_{\beta=1}^{l}\sigma_{k+\beta}^{\theta} \right\}, \tag{4.17}$$

where $\sigma, \sigma_{\alpha} \in \{p - \sigma_{p,q}, \sigma_{p,q}\}$ and $\sigma^{\theta}, \sigma_{k+\beta}^{\theta} \in \left\{ \frac{p - \sqrt{p^2 + 4q\sec^2\theta}}{2\sec^2\theta}, \frac{p + \sqrt{p^2+4q\sec^2\theta}}{2\sec^2\theta} \right\}$.

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.17) is valid if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$. Furthermore, provided that $H_P = 0$, then $X_P \in T_P^1 M$ yields the equality case in (4.17) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.17) holds if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

If we choose $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta \neq \frac{\pi}{2}$ in Theorem 4.3, then the following two results are deduced for hemi-slant submanifolds.

Theorem 4.14. *Let M be any m-dimensional proper hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$. Then for any point $P \in M$, the following assertions are satisfied:*

(a) *For all $X_P \in T_P^1 M$, we have*

$$Ric\left(X_P\right) \leq \frac{1}{4} m^2 \left\|H_P\right\|^2 + a\left\{ m - 1 - q\cos^2\theta + \widehat{g}\left(T\pi_2 X_P, X_P\right)\left(trT - p\cos^2\theta\right) \right.$$
$$\left. + q\cos^2\theta \left\|\pi_1 X_P\right\|^2 \right\} + b\left\{ (m-2)\,\widehat{g}\left(T\pi_2 X_P, X_P\right) + trT \right\}, \tag{4.18}$$

where π_1 and π_2 are the projection operators of TM onto the anti-invariant distribution $D^{\perp}$ and the slant distribution D^{θ} with the Wirtinger angle θ, respectively.

(b) *For a fixed $X_P \in T_P^1 M$, the equality case in (4.18) holds if and only if $h\left(X_P, X_P\right) = \frac{1}{2} m H_P$ and $h\left(X_P, Y_P\right) = 0$ for every $Y_P \in T_P M$ such that $\widehat{g}\left(X_P, Y_P\right) = 0$. Moreover, provided that $H_P = 0$, then $X_P \in T_P^1 M$ verifies the equality case in (4.18) if and only if it belongs to $\mathcal{N}_P$.*

(c) *The equality case in (4.18) is verified if and only if either P is a totally geodesic point or $m = 2$ and P is a totally umbilical point.*

Theorem 4.15. *Let M be any m-dimensional proper hemi-slant submanifold of a locally decomposable metallic Riemannian manifold $\left(\widehat{M}, \widehat{g}, \widehat{F}\right)$, where $\widehat{M} = \widehat{M_1}\left(\widehat{c_1}\right) \times \widehat{M_2}\left(\widehat{c_2}\right)$. If we select a local orthonormal frame $\{B_1, \ldots, B_m\}$ at a point $P \in M$ for TM such that*

$$D^{\perp} = Span\left\{B_{\alpha}\right\}_{1 \leq \alpha \leq k} \text{ and } D^{\theta} = Span\left\{ B_{k+\beta} = \pm \frac{\sec\theta T B_{k+\beta}}{\left\|\widehat{F} B_{k+\beta}\right\|} \right\}_{1 \leq \beta \leq l},$$

where $D^{\perp}$ and D^{θ} are the anti-invariant distribution and the slant distribution with the Wirtinger angle θ, respectively, then the following assertions hold:

(a) *For all* $X_P \in T_P^1 M$, *we have*

$$Ric\left(X_P\right) \leq \frac{1}{4}m^2 \left\|H_P\right\|^2 + a\left\{m-1-\frac{q}{2}\cos^2\theta + \frac{1}{2}\sigma^\theta\left(\sum_{\beta=1}^{l}\sigma_{k+\beta}^\theta - p\right)\right\} + b\left\{\frac{1}{2}\left(m-2\right)\sigma^\theta + \sum_{\beta=1}^{l}\sigma_{k+\beta}^\theta\right\}, \tag{4.19}$$

where $\sigma^\theta, \sigma_{k+\beta}^\theta \in \left\{\frac{p-\sqrt{p^2+4q\sec^2\theta}}{2\sec^2\theta}, \frac{p+\sqrt{p^2+4q\sec^2\theta}}{2\sec^2\theta}\right\}$.

(b) *For a fixed* $X_P \in T_P^1 M$, *the equality case in (4.19) is verified if and only if* $h\left(X_P, X_P\right) = \frac{1}{2}mH_P$ *and* $h\left(X_P, Y_P\right) = 0$ *for every* $Y_P \in T_P M$ *such that* $\widehat{g}\left(X_P, Y_P\right) = 0$. *Additionally, provided that* $H_P = 0$, *then* $X_P \in T_P^1 M$ *yields the equality case in (4.19) if and only if it belongs to* $\mathcal{N}_P$.

(c) *The equality case in (4.19) is valid if and only if either* P *is a totally geodesic point or* $m = 2$ *and* P *is a totally umbilical point.*

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References

[1] Bejancu, A.: Geometry of CR Submanifolds. D. Reidel Publishing Company, Dordrecht (1986).
[2] Blaga, A. M.: *The geometry of golden conjugate connections*. Sarajevo J. Math. **10** (23), 237-245 (2014).
[3] Blaga, A. M., Hreţcanu, C. E.: *Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold*. Novi Sad J. Math. **48** (2), 55-80 (2018).
[4] Blaga, A. M., Hreţcanu, C. E.: *Metallic conjugate connections*. Rev. Un. Mat. Argentina **59** (1), 179-192 (2018).
[5] Blaga, A. M., Hreţcanu, C. E.: *Remarks on metallic warped product manifolds*. Facta Univ. Ser. Math. Inform. **33** (1), 269-277 (2018).
[6] Chen, B. Y.: *Mean curvature and shape operator of isometric immersions in real space forms*. Glasgow. Math. J. **38** (1), 87-97 (1996).
[7] Chen, B. Y.: *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension*. Glasgow Math. J. **41** (1), 33-41 (1999).
[8] Choudhary, M. A., Blaga, A. M.: *Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in metallic Riemannian space forms*. J. Geom. **111** (3), Article ID 39, 18 pages (2020).
[9] Crâşmăreanu, M. C., Hreţcanu, C. E.: *Golden differential geometry*. Chaos Soliton. Fract. **38** (5), 1229-1238 (2008).
[10] de Spinadel, V. W.: *The family of metallic means*. Vis. Math. **1** (3), (1999).
[11] de Spinadel, V. W.: *The metallic means family and multifractal spectra*. Nonlinear Anal. **36** (6), 721-745 (1999).
[12] de Spinadel, V. W.: *The metallic means family and renormalization group techniques*. Proc. Steklov Inst. Math. (suppl. 1), 194-209 (2000).
[13] de Spinadel, V. W.: *The metallic means family and forbidden symmetries*. Int. Math. J. **2** (3), 279-288 (2002).
[14] Eken, Ş., Gülbahar, M., Kılıç, E.: *Some inequalities for Riemannian submersions*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **63** (3), 471-482 (2017).
[15] Etayo, F., Santamaría, R., Upadhyay, A.: *On the geometry of almost golden Riemannian manifolds*. Mediterr. J. Math. **14** (5), Article ID 187, 14 pages (2017).
[16] Gezer, A., Karaman, Ç.: *On metallic Riemannian structures*. Turk. J. Math. **3** (6), 954-962 (2015).
[17] Gök, M., Keleş, S., Kılıç, E.: *Schouten and Vrănceanu connections on golden manifolds*. Int. Electron. J. Geom. **12** (2), 169-181 (2019).
[18] Gök, M., Keleş, S., Kılıç, E.: *Some characterizations of semi-invariant submanifolds of golden Riemannian manifolds*. Mathematics **7** (12), Article ID 1209, 12 pages (2019).
[19] Gülbahar, M., Eken, Ş., Kılıç, E.: *Sharp inequalities involving the Ricci curvature for Riemannian submersions*. Kragujevac J. Math. **41** (2), 279-293 (2017).
[20] Hong, S., Matsumoto, K., Tripathi, M. M.: *Certain basic inequalities for submanifolds of locally conformal Kahler space forms*. SUT J. Math. **41** (1), 75-94 (2005).

[21] Hong, S., Tripathi, M. M.: *On Ricci curvature of submanifolds*. Int. J. Pure Appl. Math. Sci. **2** (2), 227-245 (2005).
[22] Hreţcanu, C. E., Blaga, A. M.: *Submanifolds in metallic Riemannian manifolds*. Differ. Geom. Dyn. Syst. **20**, 83-97 (2018).
[23] Hreţcanu, C. E., Blaga, A. M.: *Slant and semi-slant submanifolds in metallic Riemannian manifolds*. J. Funct. Spaces **2018** (3), Article ID 2864263, 13 pages (2018).
[24] Hreţcanu, C. E., Blaga, A. M.: *Hemi-slant submanifolds in metallic Riemannian manifolds*. Carpathian J. Math. **35** (1), 59-68 (2019).
[25] Hreţcanu, C. E., Crâşmăreanu, M. C.: *On some invariant submanifolds in a Riemannian manifold with golden structure*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **53** (suppl. 1), 199-211 (2007).
[26] Hreţcanu, C. E., Crâşmăreanu, M. C.: *Applications of the golden ratio on Riemannian manifolds*. Turk. J. Math. **33** (2), 179-191 (2009).
[27] Hreţcanu, C. E., Crâşmăreanu, M. C.: *Metallic structures on Riemannian manifolds*. Rev. Un. Mat. Argentina **54** (2), 15-27 (2013).
[28] Kılıç, E., Tripathi, M. M., Gülbahar, M.: *Chen-Ricci inequalities for submanifolds of Riemannian and Kaehlerian product manifolds*. Ann. Pol. Math. **116** (1), 37-56 (2016).
[29] Kim, J. S., Tripathi, M. M., Choi, J.: *Ricci curvature of submanifolds in locally conformal almost cosyplectic manifolds*. Indian J. Pure Appl. Math. **35** (3), 259-271 (2004).
[30] Mihai, A.: *Inequalities on the Ricci curvature*. J. Math. Inequal. **9** (3), 811-822 (2015).
[31] Mihai, I.: *Ricci curvature of submanifolds in Sasakian space forms*. J. Aust. Math. Soc. **72** (2), 247-256 (2002).
[32] Mustafa, A., Uddin, S., Al-Solamy, F. R.: *Chen-Ricci inequality for warped products in Kenmotsu space forms and its applications*. RACSAM **113** (4), 3585-3602 (2019).
[33] Özgür, C., Özgür, N. Y.: *Classification of metallic shaped hypersurfaces in real space forms*. Turk. J. Math. **39** (5), 784-794 (2015).
[34] Özgür, C., Özgür, N. Y.: *Metallic shaped hypersurfaces in Lorentzian space forms*. Rev. Un. Mat. Argentina **58** (2), 215-226 (2017).
[35] Tripathi, M. M.: *Chen-Ricci inequality for submanifolds of contact metric manifolds*. J. Adv. Math. Stud. **1** (1-2), 111-134 (2008).
[36] Vîlcu, G. E.: *B.-Y. Chen inequalities for slant submanifolds in quaternionic space forms*. Turk. J. Math. **34** (1), 115-128 (2010).
[37] Yano, K., Kon, M.: Structures on Manifolds. World Scientific, Singapore (1984).
[38] Yoon, D. W.: *Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms*. Turk. J. Math. **30** (1), 43-56 (2006).

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