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On New Spinor Sequences of Jacobsthal and Jacobsthal-Lucas Quaternions

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Abstract

In this study, two new spinor sequences using spinor representations of Jacobsthal and Jacobsthal-Lucas quaternions are defined. Moreover, some formulas such that Binet, Cassini, summation formulas and generating functions of these spinor sequences, which are called as Jacobsthal and Jacobsthal-Lucas spinor sequences, are expressed. Then, some relationships between Jacobsthal and Jacobsthal-Lucas spinors are obtained. Therefore, an easier and more interesting representation of Jacobsthal and Jacobsthal-Lucas quaternions, which are generalization of Jacobsthal and Jacobsthal-Lucas number sequences, are obtained. We believe that these new spinor sequences will be useful and advantageable in many branches of science, such as geometry, algebra and physics.

Keywords: Jacobsthal numbers; Jacobsthal-Lucas numbers; Quaternions; Spinors.

Jacobsthal ve Jacobsthal-Lucas Kuaterniyonlarının Yeni Spinor Dizileri Üzerine

Öz

Bu çalışmada Jacobsthal ve Jacobsthal-Lucas kuaterniyonlarının spinör gösterimleri kullanılarak iki yeni spinor dizisi tanımlanmıştır. Ayrıca, Jacobsthal ve Jacobsthal-Lucas spinor dizileri olarak adlandırılan bu spinor dizilerinin Binet, Cassini, toplam formülleri ve üreteç fonksiyonları gibi bazı formüller ifade edilmiştir. Daha sonra Jacobsthal ve Jacobsthal-Lucas spinorları arasındaki bazı ilişkiler elde edilmiştir. Böylece Jacobsthal ve Jacobsthal-Lucas sayı

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dizilerinin genellemesi olan Jacobsthal ve Jacobsthal-Lucas kuaterniyonlarının daha kolay ve ilginç bir temsili elde edilmiş olur. Bu yeni spinor dizilerinin geometri, cebir ve fizik gibi birçok bilim dalında faydalı ve avantajlı olacağına inanmaktayız.

Anahtar Kelimeler: Jacobsthal sayıları; Jacobsthal-Lucas sayıları; Kuaterniyonlar; Spinorlar.

1. Introduction

The first known number sequence is the Fibonacci number sequence expressed by Fibonacci (1170-1250), which is frequently encountered in nature [1-3]. The Lucas number sequence is another example of a number sequence. In addition, there are many number sequences in the literature. Moreover, considering different characteristic equations and initial values, different number sequences can be obtained in [4-6]. Other studies of this subject are [7-11] Horadam discussed Pell numbers and their properties [5]. On the other hand, Horadam gave Jacobsthal and Jacobsthal-Lucas number sequences [4]. Daşdemir studied on the Jacobsthal numbers in [12]. In [13] the Jacobsthal quaternions were expressed. Then, a new approach to Jacobsthal quaternions were obtained in [14]. Halıcı expressed bicomplex Jacobsthal-Lucas numbers [15]. Moreover, in the other study [16], the new recurrences were obtained. Arslan obtained the complex gaussian Jacobsthal quaternions [17]. Özkan end Uysal expressed the higher order Jacobsthal quaternions [18]. Moreover, the J(r, n)-Jacobsthal quaternions were obtained in [19]. Other studies can be given in [20,21].

Spinor whose transformation is associated to spins in physics can be defined as vectors of a geometric space basically. Geometrically, Cartan introduced spinors [22]. Cartan's study [22] is an admirable study in spinor geometry because in that study, the spinor representations of some geometric expressions were expressed in an easy and understandable way. Another inspiring study on the spinors in geometry was done by Vivarelli [23]. In [23], the relationship between quaternions and spinor representations of rotations with three dimensional were obtained. In the study [24], the spinor representations of the Frenet frame and curvatures of any curve in E3 were given. Darboux spinor equations in E3 were obtained [25]. Moreover, in [26], the spinor Bishop frame in E3 was expressed. The spinor equations for some special curves such as Bertrand, involute-evolute, successor, Mannheim curves, Sabban frames, and Lie groups were obtained in [27-32]. Then, for any Minkowski space, the hyperbolic spinor equations were given [33-36]. In addition to that, Fibonacci and Lucas spinors were expressed in [37].

2. Materials and Methods

Now, the spinors, real quaternions, relationships between them spinors, and Jacobsthal, Jacobsthal-Lucas quaternions are given.

Suppose that any isotropic vector is $v=(v_1,v_2,v_3)\in C^3$ where $v_1^2+v_2^2+v_3^2=0$ and the complex vector 3-space is C^3. We can express the set of isotropic vectors in C^3 with the aid of a two-dimensional surface in C^2. Suppose that this surface has coordinates ϖ_1 and ϖ_2. So, we can write $v_1=\varpi_1^2-\varpi_2^2$, $v_2=\mathbf{i}(\varpi_1^2+\varpi_2^2)$, $v_3=-2\varpi_1\varpi_2$ and $\varpi_1=\pm\sqrt{\frac{v_1-\mathbf{i}v_2}{2}}$, $\varpi_2=\pm\sqrt{\frac{-v_1-\mathbf{i}v_2}{2}}$. This two-dimensional complex vector is called as spinor

$$\varpi=(\varpi_1,\varpi_2)=\begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix}$$

[22].

Suppose that any real quaternion is $r=r_0+\boldsymbol{i}r_1+\boldsymbol{j}r_2+\boldsymbol{k}r_3$ where $r_0,r_1,r_2,r_3\in R$. $\{\boldsymbol{1},\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is called the quaternion basis such that

$$\boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=-1,\ \boldsymbol{ij}=-\boldsymbol{ji}=\boldsymbol{k},\ \boldsymbol{jk}=-\boldsymbol{kj}=\boldsymbol{i},\ \boldsymbol{ki}=-\boldsymbol{ik}=\boldsymbol{j}$$

[38]. We can write $r=S_r+\mathbf{V}_r$ where $r_0=S_r$ and $\mathbf{V}_r=\boldsymbol{i}r_1+\boldsymbol{j}r_2+\boldsymbol{k}r_3$ is called scalar and vector parts of r, respectively [38]. Assume that two any real quaternions $p=S_p+\mathbf{V}_p$, $r=S_r+\mathbf{V}_r$. So, the quaternion product is also

$$r\times p=S_rS_p-\langle\mathbf{V}_r,\mathbf{V}_p\rangle+S_r\mathbf{V}_p+S_p\mathbf{V}_r+\mathbf{V}_r\wedge\mathbf{V}_p$$

[38]. We know that this operation is non-commutative. In addition to that, the quaternion conjugate and the norm of r are expressed as $r^*=S_q-V_r$ and $N(r)=\sqrt{r_1^2+r_2^2+r_3^2+r_4^2}$. The norm of r be $N(r)=1$ then, r is called as unit quaternion [38].

Vivarelli expressed a relation between spinors and quaternions with following transformation

$$f : H \rightarrow S$$
$$r \rightarrow f(r_0 + \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3) \cong \begin{bmatrix} r_3 + \mathbf{i}r_0 \\ r_1 + \mathbf{i}r_2 \end{bmatrix} \equiv \varpi \tag{1}$$

where $r = r_0 + \boldsymbol{i}r_1 + \boldsymbol{j}r_2 + \boldsymbol{k}r_3$ is any real quaternion [18]. The spinor representation of $r \times p$ such that

$$r \times p \rightarrow -\mathbf{i}\hat{\varpi}\rho \tag{2}$$

where the spinor ρ corresponds to the p considering f in the equation (1) and the complex, unitary, square matrix $\hat{\varpi}$ can be written as

$$\hat{\varpi} = \begin{bmatrix} r_3 + \mathbf{i}r_0 & r_1 - \mathbf{i}r_2 \\ r_1 + \mathbf{i}r_2 & -r_3 + \mathbf{i}r_0 \end{bmatrix} \tag{3}$$

[23]. In addition, the spinor matrix $\varpi_L = -\mathbf{i}\hat{\varpi}$, namely

$$\varpi_L = \begin{bmatrix} r_0 - \mathbf{i}r_3 & -r_2 - \mathbf{i}r_1 \\ r_2 - \mathbf{i}r_1 & r_0 + \mathbf{i}r_3 \end{bmatrix} \tag{4}$$

was called the fundamental spinor matrix or the left Hamilton spinor matrix of r [23, 39].

Now, some equalities about the Jacobsthal and Jacobsthal-Lucas quaternions given in [13, 14] can be expressed. For $n \geq 2$ the nth Jacobsthal and Jacobsthal-Lucas quaternions are defined that

$$JQ_n = J_n + \boldsymbol{i}J_{n+1} + \boldsymbol{j}J_{n+2} + \boldsymbol{k}J_{n+3}$$

and

$$JLQ_n = JL_n + \boldsymbol{i}JL_{n+1} + \boldsymbol{j}JL_{n+2} + \boldsymbol{k}JL_{n+3}$$

where the nth Jacobsthal number and Jacobsthal-Lucas number $J_n = J_{n-1} + 2J_{n-2}$ $(J_0 = 0,\ J_1 = 1)$ and $JL_n = JL_{n-1} + 2JL_{n-2}$ $(JL_0 = 2,\ JL_1 = 1)$ [13, 14]. Therefore, the recurrence relations of the Jacobsthal and Jacobsthal-Lucas quaternions for $n \geq 2$ are

$$JQ_n = JQ_{n-1} + 2JQ_{n-2}$$

with initial conditions $JQ_0 = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, $JQ_1 = 1 + \boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$ and

$$JLQ_n = JLQ_{n-1} + 2JLQ_{n-2}$$

with initial conditions $JLQ_0 = 2 + \boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}$, $JLQ_1 = 1 + 5\boldsymbol{i} + 7\boldsymbol{j} + 17\boldsymbol{k}$ [13, 14].

Now, we write some relationships between the Jacobsthal and Jacobsthal-Lucas quaternions with the aid of [13, 14]. Therefore, the Binet formulas for the Jacobsthal and Jacobsthal-Lucas quaternions are given that

$$JQ_n = \frac{1}{\alpha - \beta}\left(\alpha^n \underline{\alpha} - \beta^n \underline{\beta}\right)$$

and

$$JLQ_n = \alpha^n \underline{\alpha} + \beta^n \underline{\beta}$$

where the quaternions $\underline{\alpha}$ and $\underline{\beta}$ are $\underline{\alpha} = 1 + \alpha\boldsymbol{i} + \alpha^2\boldsymbol{j} + \alpha^3\boldsymbol{k} = 1 + 2\boldsymbol{i} + 4\boldsymbol{j} + 8\boldsymbol{k}$ and $\underline{\beta} = 1 + \beta\boldsymbol{i} + \beta^2\boldsymbol{j} + \beta^3\boldsymbol{k} = 1 - \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\alpha = 2$, $\beta = -1$ are roots of the characteristic equation $x^2 - x - 2 = 0$. On the other hand, we give Cassini formulas for the Jacobsthal and Jacobsthal-Lucas quaternions can be given that

$$JQ_{n-1}JQ_{n+1} - (JQ_n)^2 = (-1)^n 2^{n-1}\left(7 + 5\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}\right)$$

and

$$JLQ_{n-1}JLQ_{n+1} - (JLQ_n)^2 = (-2)^{n-1} 3^2\left(7 + 5\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}\right),$$

respectively [13,14].

3. Results and Discussion

In this section, we define relationships between Jacobsthal, Jacobsthal-Lucas quaternions and spinors and, express the spinors corresponding to Jacobsthal and Jacobsthal-Lucas quaternions. Therefore, we call as Jacobsthal spinor and Jacobsthal-Lucas spinor associated with Jacobsthal and Jacobsthal-Lucas quaternions. Then, we give some relationships between Jacobsthal spinor and Jacobsthal-Lucas spinor. We obtain some formulas such that Binet, Cassini, summation formulas and generating functions for these spinors and some theorems.

Definition 1. Let $JQ_n = J_n + \boldsymbol{i}J_{n+1} + \boldsymbol{j}J_{n+2} + \boldsymbol{k}J_{n+3}$ be nth Jacobsthal quaternion where J_n is nth Jacobsthal number and the set of Jacobsthal quaternions be H_J. Then, the following linear transformation

$$f_J : H_J \rightarrow S$$
$$JQ_n \rightarrow f_J(JQ_n) \cong SJ_n = \begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix} \tag{5}$$

can be obtain where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coincide with basis vectors in $\sim^3$ and $\mathbf{i}^2 = -1$. So, a new spinor sequence corresponding Jacobsthal quaternions is called as "Jacobsthal Spinor Sequence" such as

$$\{SJ_n\}_{n\in N}^{\infty} = \left\{ \begin{bmatrix} 3 \\ 1+\mathbf{i} \end{bmatrix}, \begin{bmatrix} 5+\mathbf{i} \\ 1+3\mathbf{i} \end{bmatrix}, \begin{bmatrix} 11+\mathbf{i} \\ 3+5\mathbf{i} \end{bmatrix}, \begin{bmatrix} 21+3\mathbf{i} \\ 5+11\mathbf{i} \end{bmatrix}, \ldots \right\}$$

where $SJ_n = \begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix}$ is nth Jacobsthal spinor and J_n is nth Jacobsthal number.

Similarly, we can give the following definition of Jacobsthal-Lucas spinor sequence.

Definition 2. Assume that $JLQ_n = JL_n + \boldsymbol{i}JL_{n+1} + \boldsymbol{j}JL_{n+2} + \boldsymbol{k}JL_{n+3}$ is nth Jacobsthal-Lucas quaternion where JL_n is nth Jacobsthal-Lucas number. The nth Jacobsthal-Lucas quaternion JLQ_n matches the spinor $JLQ_n \rightarrow SJL_n = \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_n \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix}$. Then, a new spinor sequence corresponding Jacobsthal-Lucas quaternions is defined as "Jacobsthal-Lucas Spinor Sequence" $\{SJL_n\}_{n\in N}^{\infty} = \left\{ \begin{bmatrix} 7+2\mathbf{i} \\ 1+5\mathbf{i} \end{bmatrix}, \begin{bmatrix} 17+\mathbf{i} \\ 5+7\mathbf{i} \end{bmatrix}, \begin{bmatrix} 31+5\mathbf{i} \\ 7+17\mathbf{i} \end{bmatrix}, \begin{bmatrix} 65+7\mathbf{i} \\ 17+31\mathbf{i} \end{bmatrix}, \ldots \right\}$ where $SJL_n = \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_n \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix}$ is nth Jacobsthal-Lucas spinor and JL_n is nth Jacobsthal-Lucas number.

Definition 3. The conjugate of Jacobsthal quaternion JQ_n is JQ_n^*, and Jacobsthal spinor corresponding to this conjugate is defined as

$$SJ_n^* = \begin{bmatrix} -J_{n+3} + \mathbf{i}J_n \\ -J_{n+1} - \mathbf{i}J_{n+2} \end{bmatrix}.$$

Similarly, the Jacobsthal-Lucas spinor corresponding to the conjugate of Jacobsthal-Lucas quaternion JLQ_n is defined as

$$SJL_n^* = \begin{bmatrix} -JL_{n+3} + \mathbf{i}JL_n \\ -JL_{n+1} - \mathbf{i}JL_{n+2} \end{bmatrix}.$$

Definition 4. The Jacobsthal spinor representation of the norm of Jacobsthal quaternion JQ_n is $\overline{SJ_n}^t SJ_n$. Similarly, the Jacobsthal-Lucas spinor representation of the norm of Jacobsthal-Lucas quaternion JLQ_n is $\overline{SJL_n}^t SJL_n$.

Now, the recurrence relations of Jacobsthal and Jacobsthal-Lucas spinor sequences with the following equations should be stated.

Theorem 5. The recurrence relation of Jacobsthal spinors is

$$SJ_{n+2} = SJ_{n+1} + 2SJ_n$$

where nth, $(n+1)th$ and $(n+2)th$ Jacobsthal spinors are SJ_n, SJ_{n+1}, and SJ_{n+2}, respectively. The recurrence relation for Jacobsthal-Lucas spinor is

$$SJL_{n+2} = SJL_{n+1} + 2SJL_n$$

where nth, $(n+1)th$ and $(n+2)th$ Jacobsthal-Lucas spinors are SJL_n, SJL_{n+1}, and SJL_{n+2}, respectively.

Proof: We show the recurrence relation for Jacobsthal spinors in first. Therefore, if we calculate $SJ_{n+1} + 2SJ_n$, then we obtain

$$SJ_{n+1} + 2SJ_n = \begin{bmatrix} J_{n+4} + \mathbf{i}J_{n+1} \\ J_{n+2} + \mathbf{i}J_{n+3} \end{bmatrix} + 2\begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix} = \begin{bmatrix} J_{n+4} + 2J_{n+3} + \mathbf{i}(J_{n+1} + 2J_n) \\ J_{n+2} + 2J_{n+1} + \mathbf{i}(J_{n+3} + 2J_{n+2}) \end{bmatrix}.$$

Since the recurrence relation for Jacobsthal number sequence is $SJ_{n+2} = SJ_{n+1} + 2SJ_n$ in [4], we have

$$SJ_{n+1} + 2SJ_n = \begin{bmatrix} J_{n+5} + \mathbf{i}J_{n+2} \\ J_{n+3} + \mathbf{i}J_{n+4} \end{bmatrix} = SJ_{n+2}.$$

Similarly, we can easily obtain for Jacobsthal-Lucas spinor sequence such that

$$SJL_{n+1}+2SJL_n=\begin{bmatrix} JL_{n+4}+2JL_{n+3}+\mathbf{i}(JL_{n+1}+2JL_n) \\ JL_{n+2}+2JL_{n+1}+\mathbf{i}(JL_{n+3}+2JL_{n+2}) \end{bmatrix}=\begin{bmatrix} JL_{n+5}+\mathbf{i}JL_{n+2} \\ JL_{n+3}+\mathbf{i}JL_{n+4} \end{bmatrix}=SJL_{n+2}.$$

where we use the recurrence relation of the Jacobsthal Lucas number sequence $SJL_{n+2}=SJL_{n+1}+2SJL_n$ in [4].

Now, we can give the some equations about Jacobsthal and Jacobsthal-Lucas spinors.

Theorem 6. Suppose that nth, $(n+1)th$, $(n+r)th$, and $(n-r)th$ Jacobsthal spinors are SJ_n, SJ_{n+1}, SJ_{n+r}, and SJ_{n-r}, respectively. In this case, for $n\geq 1$, $r\geq 1$ there are the following relations;

$$\begin{aligned}
&i)\ SJ_{n+1}+SJ_n=2^n S_\alpha,\\
&ii)\ SJ_{n+1}-SJ_n=\frac{1}{3}\left[2^n S_\alpha+2(-1)^n S_\beta\right],\\
&iii)\ SJ_{n+r}+SJ_{n-r}=\frac{1}{3}\left[2^{n-r}\left(2^{2r}+1\right)S_\alpha-2(-1)^{n-r}S_\beta\right],\\
&iv)\ SJ_{n+r}-SJ_{n-r}=\frac{1}{3}2^{n-r}\left(2^{2r}-1\right)S_\alpha
\end{aligned}$$

where the spinors S_α and S_β are $S_\alpha=\begin{bmatrix} \alpha^3+\mathbf{i} \\ \alpha+\mathbf{i}\alpha^2 \end{bmatrix}=\begin{bmatrix} 8+\mathbf{i} \\ 2+4\mathbf{i} \end{bmatrix}$ and $S_\beta=\begin{bmatrix} \beta^3+\mathbf{i} \\ \beta+\mathbf{i}\beta^2 \end{bmatrix}=\begin{bmatrix} -1+\mathbf{i} \\ -1+\mathbf{i} \end{bmatrix}=(-1+\mathbf{i})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

Proof:

i) Assume that nth and $(n+1)th$ Jacobsthal spinors are SJ_n and SJ_{n+1}, respectively. Then, we can write the equation

$$SJ_{n+1}+SJ_n=\begin{bmatrix} J_{n+4}+\mathbf{i}J_{n+1} \\ J_{n+2}+\mathbf{i}J_{n+3} \end{bmatrix}+\begin{bmatrix} J_{n+3}+\mathbf{i}J_n \\ J_{n+1}+\mathbf{i}J_{n+2} \end{bmatrix}=\begin{bmatrix} J_{n+4}+J_{n+3}+\mathbf{i}(J_{n+1}+J_n) \\ J_{n+2}+J_{n+1}+\mathbf{i}(J_{n+3}+J_{n+2}) \end{bmatrix}.$$

On the other hand, we know that $J_{n+1}+J_n=2^n$ in [4]. Therefore, we obtain

$$SJ_{n+1}+SJ_n=\begin{bmatrix} 2^{n+3}+\mathbf{i}2^n \\ 2^{n+1}+\mathbf{i}2^{n+2} \end{bmatrix}=2^n\begin{bmatrix} 8+\mathbf{i} \\ 2+4\mathbf{i} \end{bmatrix}=2^n S_\alpha$$

where $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$.

ii) We suppose that nth and $(n+1)th$ Jacobsthal spinors are SJ_n and SJ_{n+1}, respectively. We can write

$$SJ_{n+1} - SJ_n = \begin{bmatrix} J_{n+4} + \mathbf{i}J_{n+1} \\ J_{n+2} + \mathbf{i}J_{n+3} \end{bmatrix} - \begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix} = \begin{bmatrix} J_{n+4} - J_{n+3} + \mathbf{i}(J_{n+1} - J_n) \\ J_{n+2} - J_{n+1} + \mathbf{i}(J_{n+3} - J_{n+2}) \end{bmatrix}.$$

If we use the equation $J_{n+1} - J_n = \frac{1}{3}\left(2^n - 2(-1)^{n+1}\right)$ for Jacobsthal numbers in [4] then, we get

$$SJ_{n+1} + SJ_n = \begin{bmatrix} \frac{1}{3}\left(2^{n+3} - 2(-1)^{n+4}\right) + \mathbf{i}\frac{1}{3}\left(2^n - 2(-1)^{n+1}\right) \\ \frac{1}{3}\left(2^{n+1} - 2(-1)^{n+2}\right) + \mathbf{i}\frac{1}{3}\left(2^{n+2} - 2(-1)^{n+3}\right) \end{bmatrix}$$
$$= \frac{1}{3}\left(2^n \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} - 2(-1)^{n+1} \begin{bmatrix} -1 + \mathbf{i} \\ -1 + \mathbf{i} \end{bmatrix}\right) = \frac{1}{3}\left(2^n S_\alpha + 2(-1)^n S_\beta\right)$$

where the spinors S_α and S_β are $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix} = \begin{bmatrix} -1 + \mathbf{i} \\ -1 + \mathbf{i} \end{bmatrix} = (-1 + \mathbf{i})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

iii) Let $(n+r)th$ and $(n-r)th$ Jacobsthal spinors be SJ_{n+r} and SJ_{n-r}, respectively. Moreover, we know that there is the equation $J_{n+r} + J_{n-r} = \frac{1}{3}\left(2^{n-r}\left(2^{2r} + 1\right) - 2(-1)^{n-r}\right)$ for Jacobsthal numbers in [4]. Therefore, we obtain that

$$SJ_{n+r} + SJ_{n-r} = \begin{bmatrix} J_{n+r+3} + J_{n-r+3} + \mathbf{i}(J_{n+r} + J_{n-r}) \\ J_{n+r+1} + J_{n-r+1} + \mathbf{i}(J_{n+r+2} - J_{n-r+2}) \end{bmatrix}$$
$$= \frac{1}{3}\left(2^{n-r}\left(2^{2r} + 1\right)\begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} - 2(-1)^{n-r}\begin{bmatrix} -1 + \mathbf{i} \\ -1 + \mathbf{i} \end{bmatrix}\right) = \frac{1}{3}\left(2^{n-r}\left(2^{2r} + 1\right)S_\alpha - 2(-1)^{n-r} S_\beta\right)$$

where the spinors S_α and S_β are $S_\alpha = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = (-1 + \mathbf{i})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

iv) Similarly, we can easily obtain the equation

$$SJ_{n+r} - SJ_{n-r} = \frac{1}{3} 2^{n-r} \left(2^{2r} - 1\right) \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} = \frac{1}{3} 2^{n-r} \left(2^{2r} - 1\right) S_\alpha$$

where we have $J_{n+r} - J_{n-r} = \frac{1}{3} 2^{n-r} \left(2^{2r} - 1\right)$ in [4].

Similar to Theorem 6, we can easily give the following demonstrable theorem.

Theorem 7. Let nth, $(n+1)th$, $(n+r)th$, and $(n-r)th$ Jacobsthal-Lucas spinors are SJL_n, SJL_{n+1}, SJL_{n+r}, and SJL_{n-r}, respectively. In this case, for $n \geq 1$, $r \geq 1$ there are the following relations;

$$\begin{aligned} &i)\ SJL_{n+1} + SJL_n = 3.2^n S_\alpha, \\ &ii)\ SJL_{n+1} - SJL_n = 2^n S_\alpha - 2(-1)^n S_\beta, \\ &iii)\ SJL_{n+r} + SJL_{n-r} = 2^{n-r}\left(2^{2r} + 1\right) S_\alpha + 2(-1)^{n-r} S_\beta, \\ &iv)\ SJL_{n+r} - SJL_{n-r} = 2^{n-r}\left(2^{2r} - 1\right) S_\alpha \end{aligned}$$

where the spinors S_α and S_β are $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix} = \begin{bmatrix} -1 + \mathbf{i} \\ -1 + \mathbf{i} \end{bmatrix} = (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively.

Theorem 8. Let nth Jacobsthal and Jacobsthal-Lucas spinors be SJ_n and SJL_n, respectively. So, the relationships between Jacobsthal and Jacobsthal-Lucas spinors are

$$\begin{aligned} &i)\ SJ_n + SJL_n = 2SJ_{n+1}, \\ &ii)\ SJ_{n+1} + 2SJ_{n-1} = SJL_n, \\ &iii)\ 3SJ_n + SJL_n = 2^{n+1} S_\alpha \end{aligned}$$

where $S_\alpha = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$.

Proof: Assume that nth Jacobsthal and Jacobsthal-Lucas spinors are SJ_n and SJL_n, respectively. Now, we can give proofs.

i) We demonstrate the summation of Jacobsthal and Jacobsthal-Lucas spinors $SJ_n + SJL_n$. In this case, we obtain

$$SJ_n + SJL_n = \begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix} + \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_n \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix} = \begin{bmatrix} J_{n+3} + JL_{n+3} + \mathbf{i}\left(J_n + JL_n\right) \\ J_{n+1} + JL_{n+1} + \mathbf{i}\left(J_{n+2} + JL_{n+2}\right) \end{bmatrix}.$$

On the other hand, we know that $J_n + JL_n = 2J_{n+1}$ for Jacobsthal and Jacobsthal-Lucas numbers from [4]. Consequently, we get

$$SJ_n + SJL_n = \begin{bmatrix} 2J_{n+4} + \mathbf{i}2J_{n+1} \\ 2J_{n+2} + \mathbf{i}2J_{n+3} \end{bmatrix} = 2SJ_{n+1}.$$

This completes the proof.

ii) $(n-1)th$, $(n+1)th$ Jacobsthal spinors are SJ_{n-1}, SJ_{n+1} and nth Jacobsthal-Lucas spinor is SJL_n. Therefore, we have

$$SJ_{n+1} + 2SJ_{n-1} = \begin{bmatrix} J_{n+4} + 2J_{n+2} + \mathbf{i}\left(J_{n+1} + 2J_{n-1}\right) \\ J_{n+2} + 2J_n + \mathbf{i}\left(J_{n+3} + 2J_{n+1}\right) \end{bmatrix} = \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_n \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix} = SJL_n$$

where we use the equation $J_{n+1} + 2J_{n-1} = JL_n$ in [4].

iii) Similarly, we can easily obtain that

$$3SJ_n + SJL_n = \begin{bmatrix} 3J_{n+3} + JL_{n+3} + \mathbf{i}\left(3J_n + JL_n\right) \\ 3J_{n+1} + JL_{n+1} + \mathbf{i}\left(3J_{n+2} + JL_{n+2}\right) \end{bmatrix} = \begin{bmatrix} 2^{n+4} + \mathbf{i}2^{n+1} \\ 2^{n+2} + \mathbf{i}2^{n+3} \end{bmatrix} = 2^{n+1}\begin{bmatrix} 8 + 2\mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} = 2^{n+1}S_\alpha.$$

We have the equation $3J_n + JL_n = 2^{n+1}$ from [4] in here. The proof is completed.

Theroem 9. Assume that nth Jacobsthal and Jacobsthal-Lucas spinors are SJ_n and SJL_n, respectively. Therefore, Binet Formulas for these spinors are the following equations.

i) Binet formula for Jacobsthal spinors is

$$SJ_n = \frac{1}{3}\left(2^n S_\alpha - (-1)^n S_\beta\right), \tag{6}$$

ii) Binet formula for Jacobsthal-Lucas spinors is

$$SJL_n = 2^n S_\alpha + (-1)^n S_\beta \tag{7}$$

where $\alpha = 2$, $\beta = -1$ are the roots of characteristic equation $x^2 - x - 2 = 0$ and $S_\alpha = \begin{bmatrix} \alpha^3 + \mathbf{i} \\ \alpha + \mathbf{i}\alpha^2 \end{bmatrix} = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = \begin{bmatrix} \beta^3 + \mathbf{i} \\ \beta + \mathbf{i}\beta^2 \end{bmatrix} = \begin{bmatrix} -1 + \mathbf{i} \\ -1 + \mathbf{i} \end{bmatrix} = (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Proof:

i) We know that Binet formula for the Jacobsthal number sequence is $J_n = \dfrac{2^n - (-1)^n}{3}$ in [4]. Therefore, if we write the last equation in the nth Jacobsthal spinor we obtain

$$SJ_n = \begin{bmatrix} J_{n+3} + \mathbf{i}J_n \\ J_{n+1} + \mathbf{i}J_{n+2} \end{bmatrix} = \begin{bmatrix} \dfrac{2^{n+3} - (-1)^{n+3}}{3} + \mathbf{i}\dfrac{2^n - (-1)^n}{3} \\ \dfrac{2^{n+1} - (-1)^{n+1}}{3} + \mathbf{i}\dfrac{2^{n+2} - (-1)^{n+2}}{3} \end{bmatrix} = \frac{1}{3}\left(\begin{bmatrix} 2^{n+3} + \mathbf{i}2^n \\ 2^{n+1} + \mathbf{i}2^{n+2} \end{bmatrix} - \begin{bmatrix} (-1)^{n+3} + \mathbf{i}(-1)^n \\ (-1)^{n+1} + \mathbf{i}(-1)^{n+2} \end{bmatrix} \right)$$

$$= \frac{1}{3}\left(2^n \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} - (-1)^n (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

or

$$SJ_n = \frac{1}{3}\left(2^n S_\alpha - (-1)^n S_\beta \right)$$

where $S_\alpha = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

ii) Now, we give the Binet formula for Jacobsthal-Lucas spinors. We know that the Binet formula for Jacobsthal-Lucas number sequence is $JL_n = 2^n + (-1)^n$ in [4]. In this case, we can obtain

$$SJL_n = \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_n \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix} = \begin{bmatrix} 2^{n+3} + (-1)^{n+3} + \mathbf{i}\left(2^n + (-1)^n\right) \\ 2^{n+1} + (-1)^{n+1} + \mathbf{i}\left(2^{n+2} + (-1)^{n+2}\right) \end{bmatrix} = \begin{bmatrix} 2^{n+3} + \mathbf{i}2^n \\ 2^{n+1} + \mathbf{i}2^{n+2} \end{bmatrix} + \begin{bmatrix} (-1)^{n+3} + \mathbf{i}(-1)^n \\ (-1)^{n+1} + \mathbf{i}(-1)^{n+2} \end{bmatrix}$$

$$= 2^n \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix} + (-1)^n (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

or

$$SJL_n = 2^n S_\alpha + (-1)^n S_\beta$$

where $S_\alpha = \begin{bmatrix} 8 + \mathbf{i} \\ 2 + 4\mathbf{i} \end{bmatrix}$ and $S_\beta = (-1 + \mathbf{i}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Theorem 10. Let nth Jacobsthal, Jacobsthal-Lucas spinors be SJ_n, SJL_n and fundamental Jacobsthal, Jacobsthal-Lucas spinor matrices (left Hamilton Jacobsthal, Jacobsthal-Lucas spinor matrices) be $(SJ_n)_L$, $(SJL_n)_L$, respectively. Therefore, for fundamental Jacobsthal and Jacobsthal-Lucas spinor matrices there are the following equations;

$$i)\ (SJ_n)_L = \frac{1}{3}\left(2^n (S_\alpha)_L - (-1)^n (S_\beta)_L\right), \tag{8}$$

and

$$ii)\ (SJL_n)_L = 2^n (S_\alpha)_L + (-1)^n (S_\beta)_L \tag{9}$$

where $(S_\alpha)_L = \begin{bmatrix} 1-8\mathbf{i} & -4-2\mathbf{i} \\ 4-2\mathbf{i} & 1+8\mathbf{i} \end{bmatrix}$ and $(S_\beta)_L = \begin{bmatrix} 1+\mathbf{i} & -1+\mathbf{i} \\ 1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}$ are left Hamilton spinor matrices (fundamental spinor matrices) corresponding to the spinors S_α and S_β, respectively.

Proof:

i) We assume nth Jacobsthal spinor SJ_n. If we use the equation (4), then we get

$$(SJ_n)_L = \begin{bmatrix} J_n - \mathbf{i}J_{n+3} & -J_{n+2} - \mathbf{i}J_{n+1} \\ J_{n+2} - \mathbf{i}J_{n+1} & J_n + \mathbf{i}J_{n+3} \end{bmatrix}.$$

Now, we use Binet formula for Jacobsthal numbers $J_n = \dfrac{2^n - (-1)^n}{3}$ in [4]. Therefore, we have

$$\begin{aligned} (SJ_n)_L &= \frac{1}{3}\begin{bmatrix} 2^n - (-1)^n - \mathbf{i}\left(2^{n+3} - (-1)^{n+3}\right) & -\left(2^{n+2} - (-1)^{n+2}\right) - \mathbf{i}\left(2^{n+1} - (-1)^{n+1}\right) \\ 2^{n+2} - (-1)^{n+2} - \mathbf{i}\left(2^{n+1} - (-1)^{n+1}\right) & 2^n - (-1)^n + \mathbf{i}\left(2^{n+3} - (-1)^{n+3}\right) \end{bmatrix} \\ &= \frac{1}{3}\left(2^n \begin{bmatrix} 1-8\mathbf{i} & -4-2\mathbf{i} \\ 4-2\mathbf{i} & 1+8\mathbf{i} \end{bmatrix} - (-1)^n \begin{bmatrix} 1+\mathbf{i} & -1+\mathbf{i} \\ 1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}\right). \end{aligned}$$

If we use again the equation (4), then we obtain

$$(SJ_n)_L = \frac{1}{3}\left(2^n (S_\alpha)_L - (-1)^n (S_\beta)_L\right)$$

where $\left(S_\alpha\right)_L = \begin{bmatrix} 1-8\mathbf{i} & -4-2\mathbf{i} \\ 4-2\mathbf{i} & 1+8\mathbf{i} \end{bmatrix}$ and $\left(S_\beta\right)_L = \begin{bmatrix} 1+\mathbf{i} & -1+\mathbf{i} \\ 1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}$.

ii) Let nth Jacobsthal-Lucas spinor SJL_n. If we use the equation (4) for Jacobsthal-Lucas spinor, then we obtain $\left(SJL_n\right)_L = \begin{bmatrix} JL_n - \mathbf{i}JL_{n+3} & -JL_{n+2} - \mathbf{i}JL_{n+1} \\ JL_{n+2} - \mathbf{i}JL_{n+1} & JL_n + \mathbf{i}JL_{n+3} \end{bmatrix}$. On the other hand, we use Binet formula for Jacobsthal-Lucas numbers $JL_n = 2^n + (-1)^n$ in [4]. So, we have

$$\begin{aligned}\left(SJL_n\right)_L &= \begin{bmatrix} 2^n + (-1)^n - \mathbf{i}\left(2^{n+3} + (-1)^{n+3}\right) & -\left(2^{n+2} + (-1)^{n+2}\right) - \mathbf{i}\left(2^{n+1} + (-1)^{n+1}\right) \\ 2^{n+2} + (-1)^{n+2} - \mathbf{i}\left(2^{n+1} + (-1)^{n+1}\right) & 2^n + (-1)^n + \mathbf{i}\left(2^{n+3} + (-1)^{n+3}\right) \end{bmatrix} \\ &= 2^n \begin{bmatrix} 1-8\mathbf{i} & -4-2\mathbf{i} \\ 4-2\mathbf{i} & 1+8\mathbf{i} \end{bmatrix} + (-1)^n \begin{bmatrix} 1+\mathbf{i} & -1+\mathbf{i} \\ 1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}.\end{aligned}$$

If we use the equation (4), then we get $\left(S_\alpha\right)_L = \begin{bmatrix} 1-8\mathbf{i} & -4-2\mathbf{i} \\ 4-2\mathbf{i} & 1+8\mathbf{i} \end{bmatrix}$ and $\left(S_\beta\right)_L = \begin{bmatrix} 1+\mathbf{i} & -1+\mathbf{i} \\ 1+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}$ which are left Hamilton spinor matrices corresponding to the spinors S_α and S_β, respectively. Consequently, we find

$$\left(SJL_n\right)_L = 2^n \left(S_\alpha\right)_L + (-1)^n \left(S_\beta\right)_L.$$

Now, we express the Cassini Formula for Jacobsthal and Jacobsthal-Lucas spinors.

Theorem 11. Assume that $(n-1)th$, nth and $(n+1)th$ Jacobsthal spinors are SJ_{n-1}, SJ_n and SJ_{n+1}. In this case, Cassini formula for Jacobsthal spinors is

$$(SJ_{n-1})_L SJ_{n+1} - (SJ_n)_L SJ_n = -(-2)^{n-1}(5+7\mathbf{i})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and considering $(n-1)th$, nth and $(n+1)th$ Jacobsthal-Lucas spinors are SJL_{n-1}, SJL_n and SJL_{n+1} for Jacobsthal-Lucas spinors the similar formula is

$$(SJL_{n-1})_L SJL_{n+1} - (SJL_n)_L SJL_n = 9(-2)^{n-1}(5+7\mathbf{i})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $n \geq 1$.

Proof: Firstly, we give Cassini formula for Jacobsthal spinors. Jacobsthal spinor product corresponding to the product of Jacobsthal quaternions $JQ_{n-1}JQ_{n+1}-(JQ_n)^2$ is $(SJ_{n-1})_L SJ_{n+1}-(SJ_n)_L SJ_n$. In this case, if we use the Binet formula $SJ_n=\frac{1}{3}\left(2^n S_\alpha-(-1)^n S_\beta\right)$ in the equation (6) and the equation (8) for Jacobsthal spinors, then we get

$$(SJ_{n-1})_L SJ_{n+1}-(SJ_n)_L SJ_n=\frac{1}{9}\begin{pmatrix}\left(-2^{n-1}(-1)^{n+1}+2^n(-1)^n\right)(S_\alpha)_L S_\beta\\+\left(-2^{n+1}(-1)^{n-1}+2^n(-1)^n\right)(S_\beta)_L S_\alpha\end{pmatrix}. \tag{10}$$

If we make necessary arrangements in the last equation, then we have

$$(SJ_{n-1})_L SJ_{n+1}-(SJ_n)_L SJ_n=-\frac{1}{3}(-2)^{n-1}\left((S_\alpha)_L S_\beta+2(S_\beta)_L S_\alpha\right).$$

Now, we calculate the spinor product $(S_\alpha)_L S_\beta+2(S_\beta)_L S_\alpha$. Therefore, we obtain

$$\begin{aligned}(S_\alpha)_L S_\beta+2(S_\beta)_L S_\alpha&=\begin{bmatrix}1-8\mathbf{i} & -4-2\mathbf{i}\\4-2\mathbf{i} & 1+8\mathbf{i}\end{bmatrix}\begin{bmatrix}-1+\mathbf{i}\\-1+\mathbf{i}\end{bmatrix}+2\begin{bmatrix}1+\mathbf{i} & -1+\mathbf{i}\\1+\mathbf{i} & 1-\mathbf{i}\end{bmatrix}\begin{bmatrix}8+\mathbf{i}\\2+4\mathbf{i}\end{bmatrix}\\&=\begin{bmatrix}13+7\mathbf{i}\\-11-\mathbf{i}\end{bmatrix}+2\begin{bmatrix}1+7\mathbf{i}\\13+11\mathbf{i}\end{bmatrix}=\begin{bmatrix}15+21\mathbf{i}\\15+21\mathbf{i}\end{bmatrix}.\end{aligned} \tag{11}$$

Considering the equations (10) and (11) consequently, we obtain

$$(SJ_{n-1})_L SJ_{n+1}-(SJ_n)_L SJ_n=-(-2)^{n-1}(5+7\mathbf{i})\begin{bmatrix}1\\1\end{bmatrix}.$$

Similarly, for Jacobsthal-Lucas spinors considering Binet formula $(SJL_n)_L=2^n(S_\alpha)_L+(-1)^n(S_\beta)_L$ in the equation (7) and $(SJL_n)_L=2^n(S_\alpha)_L+(-1)^n(S_\beta)_L$ in the equation (9), we have

$$\begin{aligned}(SJL_{n-1})_L SJL_{n+1}-(SJL_n)_L SJL_n&=\begin{pmatrix}\left(2^{n-1}(S_\alpha)_L+(-1)^{n-1}(S_\beta)_L\right)\left(2^{n+1}S_\alpha+(-1)^{n+1}S_\beta\right)\\-\left(2^n(S_\alpha)_L+(-1)^n(S_\beta)_L\right)\left(2^n S_\alpha+(-1)^n S_\beta\right)\end{pmatrix}\\&=3(-2)^{n-1}\left((S_\alpha)_L S_\beta+2(S_\beta)_L S_\alpha\right).\end{aligned}$$

If we use the equation (11) in the last equation, consequently we have

$$(SJL_{n-1})_L SJL_{n+1} - (SJL_n)_L SJL_n = 9(-2)^{n-1}(5+7\mathbf{i})\begin{bmatrix}1\\1\end{bmatrix}.$$

Theorem 12. The generating function for Jacobsthal spinors is

$$G_{SJ}(t) = \frac{1}{1-t-2t^2}\begin{bmatrix}3+2t+\mathbf{i}t\\1+\mathbf{i}(1+2t)\end{bmatrix}$$

and the generating function for Jacobsthal-Lucas spinors is

$$G_{SJL}(t) = \frac{1}{1-t-2t^2}\begin{bmatrix}7+10t+\mathbf{i}(2-t)\\1+4t+\mathbf{i}(5+2t)\end{bmatrix}.$$

Proof: We take *nth* Jacobsthal spinor as SJ_n. Therefore, for *nth* Jacobsthal spinor the generating function is calculated with the aid of the equation $G_{SJ}(t) = \sum_{n=0}^{\infty} SJ_n t^n$. In this case, using $G_{SJ}(t)$, $tG_{SJ}(t)$ and $2t^2 G_{SJ}(t)$ we obtain that

$$\begin{aligned}
&G_{SJ}(t) = SJ_0 + SJ_1 t + SJ_2 t^2 + SJ_3 t^3 + SJ_4 t^4 + SJ_5 t^5 + ...\\
&-tG_{SJ}(t) = -SJ_0 t - SJ_1 t^2 - SJ_2 t^3 - SJ_3 t^4 - SJ_4 t^5 - SJ_5 t^6 + ...\\
&-2t^2 G_{SJ}(t) = -2SJ_0 t^2 - 2SJ_1 t^3 - 2SJ_2 t^4 - 2SJ_3 t^5 - 2SJ_4 t^6 - 2SJ_5 t^7 + ...
\end{aligned}$$

and

$$G_{SJ}(t) = \frac{1}{(1-t-2t^2)}\left(SJ_0 + \left(SJ_1 - SJ_0\right)t\right)$$

where

$$SJ_0 + (SJ_1 - 2SJ_0)t = \begin{bmatrix}3\\1+\mathbf{i}\end{bmatrix} + \left(\begin{bmatrix}5+\mathbf{i}\\1+3\mathbf{i}\end{bmatrix} - \begin{bmatrix}3\\1+\mathbf{i}\end{bmatrix}\right)t = \begin{bmatrix}3+2t+\mathbf{i}t\\1+\mathbf{i}(1+2t)\end{bmatrix}.$$

Consequently, we get

$$G_{SJ}(t) = \frac{1}{1-t-2t^2}\begin{bmatrix}3+2t+\mathbf{i}t\\1+\mathbf{i}(1+2t)\end{bmatrix}.$$

Now, we calculate the generating function for Jacobsthal-Lucas spinors. Therefore, if we consider the function $G_{SJL}(t)=\sum_{n=0}^{\infty} SJL_n t^n$, we have

$$G_{SJL}(t)=\frac{1}{(1-t-2t^2)}\left(SJL_0+\left(SJL_1-SJL_0\right)t\right)$$

considering $G_{SJL}(t)$, $tG_{SJL}(t)$ and $2t^2G_{SJL}(t)$. Finally, we obtain

$$G_{SJL}(t)=\frac{1}{1-t-2t^2}\begin{bmatrix} 7+10t+\mathbf{i}\left(2-t\right) \\ 1+4t+\mathbf{i}(5+2t) \end{bmatrix}.$$

This completes the proof.

Theorem 13. Assume that $(-n)th$ Jacobsthal and Jacobsthal-Lucas spinors are SJ_{-n} and SJL_{-n}.In this case these spinors are calculated as follows;
for Jacobsthal spinors

$$SJ_{-n}=\left(-\frac{1}{2}\right)^n\begin{bmatrix} 8J_{n-3}-\mathbf{i}J_n \\ 2J_{n-1}-\mathbf{i}4J_{n-2} \end{bmatrix},$$

for Jacobsthal-Lucas spinors

$$SJL_{-n}=-\left(-\frac{1}{2}\right)^n\begin{bmatrix} 8JL_{n-3}-\mathbf{i}JL_n \\ 2JL_{n-1}-\mathbf{i}4JL_{n-2} \end{bmatrix}.$$

Proof: We know that the Jacobsthal spinor $SJ_n=\begin{bmatrix} J_{n+3}+\mathbf{i}J_n \\ J_{n+1}+\mathbf{i}J_{n+2} \end{bmatrix}$ with the aid of the equation (5).

In this case, for $(-n)th$ the Jacobsthal spinor we can write

$$SJ_{-n}=\begin{bmatrix} J_{-n+3}+\mathbf{i}J_{-n} \\ J_{-n+1}+\mathbf{i}J_{-n+2} \end{bmatrix}.$$

On the other hand, the equation for negative subscript $(-n)th$ Jacobsthal number is known as $J_{-n}=\frac{(-1)^{n+1}}{2^n}J_n$. Therefore, we obtain

$$SJ_{-n} = \begin{bmatrix} \frac{(-1)^{n-2}}{2^{n-3}} J_{n-3} + \mathbf{i} \frac{(-1)^{n+1}}{2^{n}} J_{n} \\ \frac{(-1)^{n}}{2^{n-1}} J_{n-1} + \mathbf{i} \frac{(-1)^{n-1}}{2^{n-2}} J_{n-2} \end{bmatrix} = \frac{(-1)^{n+1}}{2^{n}} \begin{bmatrix} -8J_{n-3} + \mathbf{i}J_{n} \\ -2J_{n-1} + \mathbf{i}4J_{n-2} \end{bmatrix}$$

and consequently

$$SJ_{-n} = \left(-\frac{1}{2}\right)^{n} \begin{bmatrix} 8J_{n-3} - \mathbf{i}J_{n} \\ 2J_{n-1} - \mathbf{i}4J_{n-2} \end{bmatrix}.$$

Now, we calculate the Jacobsthal-Lucas spinor SJL_{-n} for negative subscript $(-n)th$. We know that the Jacobsthal-Lucas spinor $SJL_{n} = \begin{bmatrix} JL_{n+3} + \mathbf{i}JL_{n} \\ JL_{n+1} + \mathbf{i}JL_{n+2} \end{bmatrix}$ from the equation (5). So, for $(-n)th$ the Jacobsthal-Lucas spinor we can write

$$SJL_{-n} = \begin{bmatrix} JL_{-n+3} + \mathbf{i}JL_{-n} \\ JL_{-n+1} + \mathbf{i}JL_{-n+2} \end{bmatrix}.$$

On the other hand, the equation for negative subscript $(-n)th$ Jacobsthal-Lucas number is known as $JL_{-n} = \frac{(-1)^{n}}{2^{n}} JL_{n}$. Consequently, we get

$$SJL_{-n} = \begin{bmatrix} \frac{(-1)^{n-3}}{2^{n-3}} JL_{n-3} + \mathbf{i} \frac{(-1)^{n}}{2^{n}} JL_{n} \\ \frac{(-1)^{n-1}}{2^{n-1}} JL_{n-1} + \mathbf{i} \frac{(-1)^{n-2}}{2^{n-2}} J_{n-2} \end{bmatrix} = -\left(-\frac{1}{2}\right)^{n} \begin{bmatrix} 8JL_{n-3} - \mathbf{i}JL_{n} \\ 2JL_{n-1} - \mathbf{i}4JL_{n-2} \end{bmatrix}.$$

Theorem 14. Let *nth* Jacobsthal spinor be SJ_{n}. The summation formulas for Jacobsthal spinors are the following options;

$$i) \sum_{s=1}^{n} SJ_{s} = \frac{1}{2}\left[SJ_{n+2} - SJ_{2}\right],$$

$$ii) \sum_{s=0}^{p} SJ_{n+s} = \frac{1}{2}\left[SJ_{n+p+2} - SJ_{n+1}\right].$$

Proof:

i) We know that for Jacobsthal spinors the Binet formula is

$$\sum_{s=1}^{n} SJ_s = \sum_{s=1}^{n}\left(\frac{1}{3}\left(2^n S_\alpha - (-1)^n S_\beta\right)\right) = \frac{1}{3}\left(\left(\sum_{s=1}^{n} 2^n\right) S_\alpha - \left(\sum_{s=1}^{n} (-1)^n\right) S_\beta\right).$$

On the other hand, we know that there is the equation $\sum_{s=1}^{n} x^s = \frac{x - x^{n+1}}{1-x}$ for geometric sequences. Then we have

$$\sum_{s=1}^{n} SJ_s = \frac{1}{3}\left(\left(\frac{2-2^{n+1}}{1-2}\right) S_\alpha - \left(\frac{-1-(-1)^{n+1}}{1-(-1)}\right) S_\beta\right) = \frac{1}{6}\left(\left(-4+2^{n+2}\right) S_\alpha + \left(1+(-1)^{n+1}\right) S_\beta\right).$$

Now, we make necessary arrangements and use Binet formula for Jacobsthal spinors in the equation (6). Therefore we get

$$\sum_{s=1}^{n} SJ_s = \frac{1}{6}\left(-4S_\alpha + 2^{n+2} S_\alpha + S_\beta + (-1)^{n+1} S_\beta\right) = \frac{1}{2}\left(-\frac{1}{3}\left(2^2 S_\alpha - S_\beta\right) + \frac{1}{3}\left(2^{n+2} S_\alpha - (-1)^{n+2} S_\beta\right)\right)$$

and consequently

$$\sum_{s=1}^{n} SJ_s = \frac{1}{2}\left(SJ_{n+2} - SJ_2\right).$$

ii) Now, we find the summation formulas for Jacobsthal spinors with subscript $(n+s)$*th.* With the aid of the equation (6) is

$$\sum_{s=0}^{p} SJ_{n+s} = \sum_{s=1}^{n}\left(\frac{1}{3}\left(2^{n+s} S_\alpha - (-1)^{n+s} S_\beta\right)\right) = \frac{1}{3}\left(\left(\sum_{s=1}^{n} 2^{n+s}\right) S_\alpha - \left(\sum_{s=1}^{n} (-1)^{n+s}\right) S_\beta\right).$$

Moreover, there is the equation $\sum_{s=0}^{n} x^{n+s} = \frac{x^n - x^{n+p+1}}{1-x}$ for geometric sequences. Therefore, we have

$$\sum_{s=0}^{p} SJ_{n+s} = \frac{1}{3}\left(\left(\frac{2^n - 2^{n+p+1}}{1-2}\right) S_\alpha - \left(\frac{(-1)^n - (-1)^{n+p+1}}{1-(-1)}\right) S_\beta\right) = \frac{1}{6}\left(\left(-2^{n+1} + 2^{n+p+2}\right) S_\alpha - \left((-1)^n - (-1)^{n+p+1}\right) S_\beta\right).$$

If we make necessary arrangements and use Binet formula for Jacobsthal spinors in the equation (6), then we get

$$\sum_{s=0}^{p} SJ_{n+s} = \frac{1}{2}\left(-\frac{1}{3}\left(2^{n+1} S_\alpha - (-1)^{n+1} S_\beta\right) + \frac{1}{3}\left(2^{n+p+2} S_\alpha - (-1)^{n+p+2} S_\beta\right)\right)$$

and consequently

$$\sum_{s=0}^{p} SJ_{n+s} = \frac{1}{2}\left[SJP_{n+p+2} - SJ_{n+1}\right].$$

Similar to Theorem 13, we can express the following demonstrable theorem.

Theorem 14. Assume that nth Jacobsthal-Lucas spinor be SJL_n. The summation formulas for Jacobsthal-Lucas spinors are the following equations;

$$i)\ \sum_{s=1}^{n} SJL_s = \frac{1}{2}\left[SJL_{n+2} - SJL_2\right],$$
$$ii)\ \sum_{s=0}^{p} SJL_{n+s} = \frac{1}{2}\left[SJL_{n+p+2} - SJL_{n+1}\right].$$

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