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Existence and uniqueness results for fractional q-difference equation with p-Laplacian

Öykûm ÜLKE*

National Defence University, Air NCO Vocational HE School, Gaziemir-İzmir,

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Abstract

The aim of this paper is to obtain some results on the existence and uniqueness of solutions to the boundary value problem of the fractional q-difference equation with p-Laplacian using Schaefer's and Banach's fixed point theorems. As an application, an example is presented to illustrate the main result.

Keywords: *Fractional q-difference equation, fixed point theorems, existence, uniqueness, Schaefer's fixed point theorem, Banach fixed point theorem.*

p-Laplasyenli kesirli q-fark denkleminin varlık ve teklik sonuçları

Öz

Bu çalışmanın amacı, Schaefer ve Banach'ın sabit nokta teoremlerini kullanarak p-Laplasyenli kesirli q-fark denkleminin sınır değer probleminin çözümlerinin varlığı ve tekliği üzerine bazı sonuçlar elde etmektir. Uygulama olarak, ana sonucu göstermek için bir örnek sunulmuştur.

Anahtar kelimeler: *Kesirli q-fark denklem, sabit nokta teoeremleri, varlık, teklik, Schaefer sabit nokta teoremi, Banach sabit nokta teoremi.*

1. Introduction

Fractional calculus generalizes integer-order analysis by considering derivatives and integrals of non-integer order and has found many applications in various fields of applied sciences and engineering. The q-difference calculus was first developed by Jackson [1, 2]. Fractional q-difference equations have recently attracted the attention of several

*Öykûm ÜLKE, oykum.ulke@msu.edu.tr, http://orcid.org/0000-0002-1104-854X

researchers for the applications of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, electricity, etc. Therefore, fractional q-difference calculus has been of great interest, and many good results can be found in [3-21] and references therein. Some recent works obtained many results regarding the existence and uniqueness of solutions, positive solutions, negative solutions, and extremal solutions by applying some well-known tools of fixed point theory, such as the Banach contraction principle, the Guo–Krasnosel'skii fixed point theorem on cones, monotone iterative methods, and Leray–Schauder degree theory.

Fractional differential equations with p-Laplacian operators have been extensively used in many fields of science and engineering. There are some studies concerning the existence of solutions for fractional differential equations and fractional q-difference equation with p-Laplacian operators; see, e.g., the papers [22-25].

In 2015, Zhao [26] studied the existence of positive solutions for the following q-fractional boundary value problems with p-Laplacian:

$$\begin{cases} D_q^{\beta}\left(\varphi_p\left(D_q^{\alpha}u(t)\right)\right) = f\big(t,u(t)\big), \qquad t \in (0,1), \\ u(0)=0,\ u(1)=\int_0^1 h(t)u(t)d_qt,\ D_q^{\alpha}u(0)=0,\ D_q^{\alpha}u(1)=bD_q^{\alpha}u(\eta), \end{cases}$$

where $D_q^{\alpha}, D_q^{\beta}$ are the fractional q-derivative of Riemann-Liouville type with $1<\alpha,\beta\le 2,\ 0\le b<1,\ 0<\eta<1,\ \varphi_p(s)=|s|^{p-2}s,\ \varphi_p^{-1}=\varphi_r,\ p^{-1}+r^{-1}=1,$ $p>1,\ r>1$ and $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+),\ h\in C([0,1],\mathbb{R}^+)\big(\mathbb{R}^+ := [0,+\infty)\big)$.

In 2021, Wang et al. [27] used the Green function and the Guo-Krasnoselskii fixed point theorem on cones to study the existence of at least one or two positive solutions in terms of different eigenvalue intervals for the BVP of fractional q-difference equation with φ-Laplacian:

$$\begin{cases} D_q^{\beta}\left(\varphi\left(D_q^{\alpha}u(t)\right)\right) = \lambda f\big(u(t)\big), \qquad 0<t<1, \\ u(0)=D_qu(0)=D_qu(1)=0,\ \ \varphi\left(D_q^{\alpha}u(0)\right)=D_q\left(\varphi\left(D_q^{\alpha}u(1)\right)\right)=0, \end{cases}$$

where $0<q<1,\ 2<\alpha\le 3,\ 1<\beta\le 2,\ \lambda>0$ is a parameter, and $D_q^{\alpha}, D_q^{\beta}$ are the standard Riemann-Liouville fractional q-derivatives.

In the light of the above studies, this paper will consider the following boundary value problem of fractional q-difference equations:

$$\begin{cases} D_q^{\beta}\left(\varphi_p\big(D_q^{\alpha}u(t)\big)\right) + f\big(t,u(t)\big) = 0\,, \qquad t\in(0,1), & (1.1) \\ u(0)=0,\ u(1)=c\int_0^1 h(t)u(t)d_qt,\ \ D_q^{\alpha}u(0)=aD_q^{\alpha}u(\xi),\ \ D_q^{\alpha}u(1)=bD_q^{\alpha}u(\eta), & (1.2) \end{cases}$$

where $D_q^{\alpha}, D_q^{\beta}$ are the fractional q-derivative of the Riemann-Liouville type with

$1 < \alpha, \beta \leq 2,\ 0 \leq b \leq a \leq 1,\ 0 < \eta \leq \xi < 1,\ \varphi_p(s) = |s|^{p-2}s,\ \varphi_p^{-1} = \varphi_r,$
$p^{-1} + r^{-1} = 1,\ p > 1,\ r > 1$ and $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+),\ h \in C([0,1], \mathbb{R}^+).$

Suppose that the following condition is satisfied:

$$(H_1) \qquad k := 1 - c\int_0^1 h(t)t^{\alpha-1}d_qt > 0,$$

throughout this paper. The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of our main results. In Section 3, we investigate the existence and uniqueness of a solution $u(t)$ for the boundary value problem (BVP) (1.1) - (1.2) by using Banach and Schaefer's fixed point theorems. In Section 4, we present an example to illustrate our main results.

2. Preliminaries

In this section, some useful definitions and preliminaries, which are essential for the proof of the main results, are listed. Some fundamental definitions of q-calculus are given in [28, 29].

Definition 2.1 [29] Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is

$(I_q^0 f)(x) = f(x)$ and
$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x - qt)^{(\alpha-1)} f(t)d_qt$ where $x \in [0,1]$.

Definition 2.2 [30] The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by
$(D_q^\alpha f)(x) = f(x)$ and

$(D_q^\alpha f)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \quad \alpha > 0,$

where p is the smallest integer greater than or equal to α.

Lemma 2.1 [31]
(1) If f and g are q-integral on the interval $[a, b\,]$, $\alpha \in \mathbb{R}$, and $c \in [a, b]$, then:

- *(i)* $\int_a^b \big(f(t) + g(t)\big)d_qt = \int_a^b f(t)d_qt + \int_a^b g(t)d_qt$
- *(ii)* $\int_a^b \alpha f(t)d_qt = \alpha\int_a^b f(t)d_qt$
- *(iii)* $\int_a^b f(t)\, d_qt = \int_a^c f(t)\, d_qt + \int_c^b f(t)\, d_qt$
- *(iv)* $\int x^\alpha\, d_qs = \frac{x^{\alpha+1}}{[\alpha+1]}$ for $\alpha \neq -1$,

(2) If $|f|$ is q-integral on the interval $[0, x]$, then $\left|\int_0^x f(t)\, d_qt\right| \leq \int_0^x |f(t)| d_qt,$

(3) If f and g are q-integral on the interval $[0, x]$ and $f(t) \leq g(t)$ for all $t \in [0, x]$, then $\int_0^x f(t)\, d_qt \leq \int_0^x g(t)\, d_qt.$

Lemma 2.2 [32] Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds.

$$\left(I_q^{\alpha} D_q^{p} f\right)(x) = \left(D_q^{p} I_q^{\alpha} f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^{k} f\right)(0).$$

Theorem 2.1 (Schaefer's Fixed Point Theorem) [33] Let $(X, \|.\|)$ be a normed space, H a continuous mapping of X into X which is compact on each bounded subset D of X. Then either:

(i) $x = \lambda H x$ has a solution in X for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Theorem 2.2 (Banach Fixed Point Theorem) [34] Let (X, d) be a complete metric space, and let $f: X \to X$ be a contraction operator, i.e., there exists a $\lambda \in (0,1)$ such that $d(f(x), f(y)) \leq k.d(x, y)$ for any $x, y \in X$. Then there exists a unique $p \in X$ such that $f(p) = p$.

Lemma 2.3 Suppose that (H_1) holds. Let $y \in C[0,1]$ and $1 < \alpha \leq 2$. Then

$$\left(D_q^{\alpha} u\right)(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = 0, \qquad u(1) = c \int_0^1 h(t) u(t) d_q t, \tag{2.2}$$

is equivalent to

$$u(t) = \int_{s=0}^{1} G(t, qs) y(s) d_q s,$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{k} c \int_0^1 h(t) g(t,s) d_q t,$$

$$g(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \left(t(1-s)\right)^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ \left(t(1-s)\right)^{(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

<u>Proof:</u>

$$u(t) = -I_q^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \qquad c_1, c_2 \in \mathbb{R}$$

$$u(0) = 0 \Rightarrow c_2 = 0.$$

$$\Rightarrow u(t) = -I_q^{\alpha} y(t) + c_1 t^{\alpha-1} = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s) d_q s + c_1 t^{\alpha-1}$$

$$u(1) = -\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} y(s) d_q s + c_1 = c \int_0^1 h(t) u(t) d_q t$$

$$c_1 = c\int_0^1 h(t)u(t)d_qt + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}y(s)d_qs.$$

$$u(t) = -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}y(s)d_qs + t^{\alpha-1}\,c\int_0^1 h(t)u(t)d_qt + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-1)}y(s)d_qs$$

$$= -\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}y(s)d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (1-qs)^{(\alpha-1)}y(s)d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_t^1 (1-qs)^{(\alpha-1)}y(s)d_qs + t^{\alpha-1}\,c\int_0^1 h(t)u(t)d_qt$$

$$= \int_0^1 g(t,qs)y(s)d_qs + t^{\alpha-1}\,c\int_0^1 h(t)u(t)d_qt$$

where

$$g(t,s) = \frac{1}{\Gamma_q(\alpha)}\begin{cases} \left(t(1-s)\right)^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & s \le t \\ (t(1-s))^{(\alpha-1)}, & t \le s. \end{cases}$$

Since

$$u(t) = \int_0^1 g(t,qs)y(s)d_qs + t^{\alpha-1}\,c\int_0^1 h(t)u(t)d_qt\,, \tag{2.3}$$

we get

$$\int_0^1 h(t)u(t)d_qt = \int_0^1 h(t)\int_0^1 g(t,qs)y(s)d_qs\,d_qt + \int_0^1 h(t)\;t^{\alpha-1}\,c\int_0^1 h(t)u(t)d_qt\,d_qt$$

$$\int_0^1 h(t)u(t)d_qt\,.\left\{\underbrace{1 - c\int_0^1 h(t)t^{\alpha-1}d_qt}_{k}\right\} = \int_0^1 h(t)\int_0^1 g(t,qs)y(s)d_qs\,d_qt$$

$$\Rightarrow \int_0^1 h(t)u(t)d_qt = \frac{1}{k}\int_0^1 h(t)\int_0^1 g(t,qs)y(s)d_qs\,d_qt.$$

From (2.3),

$$u(t) = \int_{s=0}^1 g(t,qs)y(s)d_qs + \frac{t^{\alpha-1}\,c}{k}\int_{t=0}^1 h(t)\int_{s=0}^1 g(t,qs)y(s)d_qs\,d_qt$$

$$u(t) = \int_{s=0}^1 g(t,qs)y(s)d_qs + \frac{t^{\alpha-1}\,c}{k}\int_{s=0}^1 \left\{\int_{t=0}^1 h(t)g(t,qs)d_qt\right\}y(s)d_qs$$

$$u(t) = \int_0^1 G(t,qs)y(s)d_qs,$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{k} c \int_0^1 h(t) g(t,s) d_q t\,.$$

□

Lemma 2.4 Suppose that (H_1) holds. Let $y \in C[0,1]$, $1 < \alpha, \beta \leq 2$, $0 \leq b \leq a \leq 1$, $0 < \eta \leq \xi < 1$. Then

$$D_q^\beta \left(\varphi_p \left(D_q^\alpha u(t) \right) \right) = -y(t),$$
$$u(0) = 0, \qquad u(1) = c \int_0^1 h(t) u(t) d_q t,$$
$$D_q^\alpha u(0) = a D_q^\alpha u(\xi), \quad D_q^\alpha u(1) = b D_q^\alpha u(\eta),$$

is equivalent to

$$u(t) = \int_0^1 G(t,qs) \varphi_r \left(\int_0^1 H(s,q\eta) y(\eta) d_q \eta \right) d_q s,$$

where

$$H(t,s) = \begin{cases} H_1(t,s), & s \leq t, \\ H_2(t,s), & s \geq t, \end{cases}$$

such that

$$H_1(t,s) = \begin{cases} -\Delta(t-s)^{(\beta-1)} + \left\{ \begin{array}{c} a^{p-1} t^{\beta-2} \left[\left(b^{p-1} \eta^{\beta-2} - 1 \right) t - \left(b^{p-1} \eta^{\beta-1} - 1 \right) \right] (\xi - s)^{(\beta-1)} \\ + b^{p-1} t^{\beta-2} \left[-\left(a^{p-1} \xi^{\beta-2} - 1 \right) t + a^{p-1} \xi^{\beta-1} \right] (\eta - s)^{(\beta-1)} \\ + t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)} \end{array} \right\}, & s \leq \eta, \\ -\Delta(t-s)^{(\beta-1)} + \left\{ \begin{array}{c} a^{p-1} t^{\beta-2} \left[\left(b^{p-1} \eta^{\beta-2} - 1 \right) t - \left(b^{p-1} \eta^{\beta-1} - 1 \right) \right] (\xi - s)^{(\beta-1)} \\ + t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)} \end{array} \right\}, & \eta \leq s \leq \xi, \\ -\Delta(t-s)^{(\beta-1)} + t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)}, & \xi \leq s, \end{cases}$$

and

$$H_2(t,s) = \begin{cases} \begin{array}{c} a^{p-1} t^{\beta-2} \left[\left(b^{p-1} \eta^{\beta-2} - 1 \right) t - \left(b^{p-1} \eta^{\beta-1} - 1 \right) \right] (\xi - s)^{(\beta-1)} \\ + b^{p-1} t^{\beta-2} \left[-\left(a^{p-1} \xi^{\beta-2} - 1 \right) t + a^{p-1} \xi^{\beta-1} \right] (\eta - s)^{(\beta-1)} \\ + t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)} \end{array}, & s \leq \eta, \\ \begin{array}{c} a^{p-1} t^{\beta-2} \left[\left(b^{p-1} \eta^{\beta-2} - 1 \right) t - \left(b^{p-1} \eta^{\beta-1} - 1 \right) \right] (\xi - s)^{(\beta-1)} \\ + t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)} \end{array}, & \eta \leq s \leq \xi, \\ t^{\beta-2} \left[\left(a^{p-1} \xi^{\beta-2} - 1 \right) t - a^{p-1} \xi^{\beta-1} \right] (1-s)^{(\beta-1)}, & \xi \leq s. \end{cases}$$

Proof:

$$D_q^\beta \left(\varphi_p \left(D_q^\alpha u(t) \right) \right) = -y(t)$$

$$\varphi_p \left(D_q^\alpha u(t) \right) = -I_q^\beta y(t) + c_3 t^{\beta-1} + c_4 t^{\beta-2}\,, \qquad c_3, c_4 \in \mathbb{R}^+$$

$$D_q^\alpha u(0) = a D_q^\alpha u(\xi) \Rightarrow \varphi_p \left(D_q^\alpha u(0) \right) = a^{p-1} \varphi_p \left(D_q^\alpha u(\xi) \right)$$

$$\Rightarrow c_4 = a^{p-1} \left\{ -I_q^\beta y(\xi) + c_3 \xi^{\beta-1} + c_4 \xi^{\beta-2} \right\}$$

$$a^{p-1}\xi^{\beta-1}c_3 + \left(a^{p-1}\xi^{\beta-2} - 1\right)c_4 = a^{p-1}I_q^{\beta}y(\xi) = \frac{a^{p-1}}{\Gamma_q(\beta)}\int_0^{\xi}(\xi - qs)^{(\beta-1)}y(s)d_qs$$

$$D_q^{\alpha}u(1) = bD_q^{\alpha}u(\eta) \;\Rightarrow\; \varphi_p\left(D_q^{\alpha}u(1)\right) = b^{p-1}\varphi_p\left(D_q^{\alpha}u(\eta)\right)$$

$$-I_q^{\beta}y(1) + c_3 + c_4 = b^{p-1}\left\{-I_q^{\beta}y(\eta) + c_3\eta^{\beta-1} + c_4\eta^{\beta-2}\right\}$$

$$\left(b^{p-1}\eta^{\beta-1} - 1\right)c_3 + \left(b^{p-1}\eta^{\beta-2} - 1\right)c_4 = b^{p-1}I_q^{\beta}y(\eta) - I_q^{\beta}y(1)$$

$$= \frac{b^{p-1}}{\Gamma_q(\beta)}\int_0^{\eta}(\eta - qs)^{(\beta-1)}y(s)d_qs - \frac{1}{\Gamma_q(\beta)}\int_0^{1}(1 - qs)^{(\beta-1)}y(s)d_qs$$

$$\Delta = \begin{vmatrix} a^{p-1}\xi^{\beta-1} & a^{p-1}\xi^{\beta-2} - 1 \\ b^{p-1}\eta^{\beta-1} - 1 & b^{p-1}\eta^{\beta-2} - 1 \end{vmatrix} = a^{p-1}\xi^{\beta-1}\left(b^{p-1}\eta^{\beta-2} - 1\right) - \left(a^{p-1}\xi^{\beta-2} - 1\right)\left(b^{p-1}\eta^{\beta-1} - 1\right)$$

$$= a^{p-1}b^{p-1}\xi^{\beta-1}\eta^{\beta-2} - a^{p-1}\xi^{\beta-1} - a^{p-1}b^{p-1}\xi^{\beta-2}\eta^{\beta-1} + a^{p-1}\xi^{\beta-2} + b^{p-1}\eta^{\beta-1} - 1$$
$$= a^{p-1}b^{p-1}\xi^{\beta-2}\eta^{\beta-2}(\xi - \eta) + a^{p-1}\xi^{\beta-2}(1 - \xi) + b^{p-1}\eta^{\beta-1} - 1.$$

$$\Rightarrow a^{p-1}\xi^{\beta-1} \geq b^{p-1}\eta^{\beta-1} \Rightarrow b^{p-1}\eta^{\beta-1} - a^{p-1}\xi^{\beta-1} \leq 0 \quad \text{and} \quad a^{p-1}\xi^{\beta-2} - 1 \leq 0$$

$$c_3 = \frac{1}{\Delta}\begin{vmatrix} \frac{a^{p-1}}{\Gamma_q(\beta)}\int_0^{\xi}(\xi - qs)^{(\beta-1)}y(s)d_qs & a^{p-1}\xi^{\beta-2} - 1 \\ \frac{b^{p-1}}{\Gamma_q(\beta)}\int_0^{\eta}(\eta - qs)^{(\beta-1)}y(s)d_qs - \frac{1}{\Gamma_q(\beta)}\int_0^{1}(1 - qs)^{(\beta-1)}y(s)d_qs & b^{p-1}\eta^{\beta-2} - 1 \end{vmatrix}$$

$$= \frac{1}{\Delta\Gamma_q(\beta)}\left\{a^{p-1}\left(b^{p-1}\eta^{\beta-2} - 1\right)\int_0^{\xi}(\xi - qs)^{(\beta-1)}y(s)d_qs\right.$$
$$- b^{p-1}\left(a^{p-1}\xi^{\beta-2} - 1\right)\int_0^{\eta}(\eta - qs)^{(\beta-1)}y(s)d_qs$$
$$\left. + \left(a^{p-1}\xi^{\beta-2} - 1\right)\int_0^{1}(1 - qs)^{(\beta-1)}y(s)d_qs\right\}.$$

$$c_4 = \frac{1}{\Delta}\begin{vmatrix} a^{p-1}\xi^{\beta-1} & \frac{a^{p-1}}{\Gamma_q(\beta)}\int_0^{\xi}(\xi - qs)^{(\beta-1)}y(s)d_qs \\ b^{p-1}\eta^{\beta-1} - 1 & \frac{b^{p-1}}{\Gamma_q(\beta)}\int_0^{\eta}(\eta - qs)^{(\beta-1)}y(s)d_qs - \frac{1}{\Gamma_q(\beta)}\int_0^{1}(1 - qs)^{(\beta-1)}y(s)d_qs \end{vmatrix}$$

$$= \frac{1}{\Delta\Gamma_q(\beta)}\Big\{a^{p-1}b^{p-1}\xi^{\beta-1}\int_0^{\eta}(\eta-qs)^{(\beta-1)}y(s)d_qs$$
$$- a^{p-1}\xi^{\beta-1}\int_0^{1}(1-qs)^{(\beta-1)}y(s)d_qs$$
$$- a^{p-1}\big(b^{p-1}\eta^{\beta-1}-1\big)\int_0^{\xi}(\xi-qs)^{(\beta-1)}y(s)d_qs\Big\}.$$

$$\varphi_p\left(D_q^{\alpha}u(t)\right) = -\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}y(s)d_qs$$
$$+\frac{t^{\beta-1}}{\Delta\Gamma_q(\beta)}\Big\{a^{p-1}\big(b^{p-1}\eta^{\beta-2}-1\big)\int_0^{\xi}(\xi-qs)^{(\beta-1)}y(s)d_qs$$
$$- b^{p-1}\big(a^{p-1}\xi^{\beta-2}-1\big)\int_0^{\eta}(\eta-qs)^{(\beta-1)}y(s)d_qs$$
$$+ \big(a^{p-1}\xi^{\beta-2}-1\big)\int_0^{1}(1-qs)^{(\beta-1)}y(s)d_qs\Big\}$$
$$+\frac{t^{\beta-2}}{\Delta\Gamma_q(\beta)}\Big\{a^{p-1}b^{p-1}\xi^{\beta-1}\int_0^{\eta}(\eta-qs)^{(\beta-1)}y(s)d_qs$$
$$- a^{p-1}\xi^{\beta-1}\int_0^{1}(1-qs)^{(\beta-1)}y(s)d_qs$$
$$- a^{p-1}\big(b^{p-1}\eta^{\beta-1}-1\big)\int_0^{\xi}(\xi-qs)^{(\beta-1)}y(s)d_qs\Big\}$$
$$= -\frac{1}{\Gamma_q(\beta)}\int_0^t (t-qs)^{(\beta-1)}y(s)d_qs$$
$$+\frac{t^{\beta-2}}{\Delta\Gamma_q(\beta)}\int_0^{\eta}\{a^{p-1}[\big(b^{p-1}\eta^{\beta-2}-1\big)t-\big(b^{p-1}\eta^{\beta-1}-1\big)](\xi-qs)^{(\beta-1)}$$
$$+ b^{p-1}[-\big(a^{p-1}\xi^{\beta-2}-1\big)t+a^{p-1}\xi^{\beta-1}](\eta-qs)^{(\beta-1)}$$
$$+ [\big(a^{p-1}\xi^{\beta-2}-1\big)t-a^{p-1}\xi^{\beta-1}](1-qs)^{(\beta-1)}\}\ y(s)d_qs$$
$$+\frac{t^{\beta-2}}{\Delta\Gamma_q(\beta)}\int_{\eta}^{\xi}\{a^{p-1}[\big(b^{p-1}\eta^{\beta-2}-1\big)t-\big(b^{p-1}\eta^{\beta-1}-1\big)](\xi-qs)^{(\beta-1)}$$
$$+ [\big(a^{p-1}\xi^{\beta-2}-1\big)t-a^{p-1}\xi^{\beta-1}](1-qs)^{(\beta-1)}\}\ y(s)d_qs$$
$$+\frac{t^{\beta-2}}{\Delta\Gamma_q(\beta)}\int_{\xi}^{1}\{[\big(a^{p-1}\xi^{\beta-2}-1\big)t-a^{p-1}\xi^{\beta-1}](1-qs)^{(\beta-1)}\}\ y(s)d_qs$$
$$= \frac{1}{\Delta\Gamma_q(\beta)}\int_0^1 H(t,qs)y(s)d_qs, \qquad \Delta=-\Omega$$

where

$$H(t,s) = \begin{cases} H_1(t,s), & s \le t, \\ H_2(t,s), & s \ge t, \end{cases}$$

$$H_1(t,s)=\begin{cases}-\Delta(t-s)^{(\beta-1)}+\left\{\begin{matrix}a^{p-1}t^{\beta-2}\left[\left(b^{p-1}\eta^{\beta-2}-1\right)t-\left(b^{p-1}\eta^{\beta-1}-1\right)\right](\xi-s)^{(\beta-1)}\\+b^{p-1}t^{\beta-2}\left[-\left(a^{p-1}\xi^{\beta-2}-1\right)t+a^{p-1}\xi^{\beta-1}\right](\eta-s)^{(\beta-1)}\\+t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}\end{matrix}\right\}, & s\le\eta,\\ -\Delta(t-s)^{(\beta-1)}+\left\{\begin{matrix}a^{p-1}t^{\beta-2}\left[\left(b^{p-1}\eta^{\beta-2}-1\right)t-\left(b^{p-1}\eta^{\beta-1}-1\right)\right](\xi-s)^{(\beta-1)}\\+t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}\end{matrix}\right\}, & \eta\le s\le\xi,\\ -\Delta(t-s)^{(\beta-1)}+t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}, & \xi\le s,\end{cases}$$

and

$$H_2(t,s)=\begin{cases}\begin{matrix}a^{p-1}t^{\beta-2}\left[\left(b^{p-1}\eta^{\beta-2}-1\right)t-\left(b^{p-1}\eta^{\beta-1}-1\right)\right](\xi-s)^{(\beta-1)}\\+b^{p-1}t^{\beta-2}\left[-\left(a^{p-1}\xi^{\beta-2}-1\right)t+a^{p-1}\xi^{\beta-1}\right](\eta-s)^{(\beta-1)}\\+t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}\end{matrix}, & s\le\eta,\\ \begin{matrix}a^{p-1}t^{\beta-2}\left[\left(b^{p-1}\eta^{\beta-2}-1\right)t-\left(b^{p-1}\eta^{\beta-1}-1\right)\right](\xi-s)^{(\beta-1)}\\+t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}\end{matrix}, & \eta\le s\le\xi,\\ t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2}-1\right)t-a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}, & \xi\le s.\end{cases}$$

□

Also, we get

$$\varphi_p\left(D_q^\alpha u(t)\right)=\frac{1}{\Delta\Gamma_q(\beta)}\int_0^1 H(t,qs)y(s)d_qs,$$

and so

$$D_q^\alpha u(t)=\varphi_r\left(\frac{1}{\Delta\Gamma_q(\beta)}\int_0^1 H(t,qs)y(s)d_qs\right)=-\varphi_r\left(\frac{1}{\Omega\Gamma_q(\beta)}\int_0^1 H(t,qs)y(s)d_qs\right)$$

$$=\frac{1}{\left(\Delta\Gamma_q(\beta)\right)^{r-1}}\varphi_r\left(\int_0^1 H(t,qs)y(s)d_qs\right)=-\frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\varphi_r\left(\int_0^1 H(t,qs)y(s)d_qs\right).$$

In other words,

$$D_q^\alpha u(t)-\frac{1}{\left(\Delta\Gamma_q(\beta)\right)^{r-1}}\varphi_r\left(\int_0^1 H(t,qs)y(s)d_qs\right)=0,$$

$$D_q^\alpha u(t)+\frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\varphi_r\left(\int_0^1 H(t,qs)y(s)d_qs\right)=0,$$

with

$$u(0)=0,\qquad u(1)=c\int_0^1 h(t)u(t)d_qt.$$

So, we get

$$u(t)=\int_0^1 G(t,qs)\left[-\frac{1}{\left(\Delta\Gamma_q(\beta)\right)^{r-1}}\varphi_r\left(\int_0^1 H(s,q\eta)y(\eta)d_q\eta\right)\right]d_qs$$

$$= -\frac{1}{\left(\Delta\Gamma_q(\beta)\right)^{r-1}}\int_0^1 G(t,qs)\left[\varphi_r\left(\int_0^1 H(s,q\eta)y(\eta)d_q\eta\right)\right]d_qs$$
$$= \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\int_0^1 G(t,qs)\left[\varphi_r\left(\int_0^1 H(s,q\eta)y(\eta)d_q\eta\right)\right]d_qs.$$

Lemma 2.5 For $t,s \in [0,1]$, the functions $G(t,s)$ and $H(t,s)$ satisfy the followings:

$$|G(t,s)| \le \frac{1}{\Gamma_q(\alpha)}.A \quad \text{and} \quad |H(t,s)| \le B,$$

where $B = \Omega\left\{1 + a^{p-1}\left(1 - b^{p-1}\eta^{\beta-1}\right) + (1 + b^{p-1})\left(1 + a^{p-1}\xi^{\beta-2}\right)\right\}$,

and $A = 1 + \frac{1}{k}|c|\int_0^1 |h(t)|d_qt$.

<u>*Proof:*</u>

$$|G(t,s)| = \left|g(t,s) + \frac{t^{\alpha-1}}{k}c\int_0^1 h(t)g(t,s)d_qt\right| \le |g(t,s)| + \frac{1}{k}|c|\int_0^1 |h(t)||g(t,s)|d_qt,$$

$$|G(t,s)| \le g(s,s)\left\{1 + \frac{1}{k}|c|\int_0^1 |h(t)|d_qt\right\} := g(s,s)\,A \le \frac{1}{\Gamma_q(\alpha)}A.$$

$$\begin{aligned}|H(t,s)| &= \left|-\Delta(t-s)^{(\beta-1)} + a^{p-1}t^{\beta-2}\left[\left(b^{p-1}\eta^{\beta-2} - 1\right)t - \left(b^{p-1}\eta^{\beta-1} - 1\right)\right](\xi - s)^{(\beta-1)}\right.\\ &\quad + b^{p-1}t^{\beta-2}\left[-\left(a^{p-1}\xi^{\beta-2} - 1\right)t + a^{p-1}\xi^{\beta-1}\right](\eta - s)^{(\beta-1)}\\ &\quad \left. + t^{\beta-2}\left[\left(a^{p-1}\xi^{\beta-2} - 1\right)t - a^{p-1}\xi^{\beta-1}\right](1-s)^{(\beta-1)}\right|\\ &\le \Omega\left\{1 + a^{p-1}\left(1 - b^{p-1}\eta^{\beta-1}\right) + b^{p-1}\left(1 - a^{p-1}\xi^{\beta-2} + a^{p-1}\xi^{\beta-1}\right) + 1 - a^{p-1}\xi^{\beta-2}\right.\\ &\quad \left. + a^{p-1}\xi^{\beta-1}\right\}(1-s)^{(\beta-1)}\end{aligned}$$

$$1 - a^{p-1}\xi^{\beta-2} + a^{p-1}\xi^{\beta-1} = 1 - a^{p-1}\xi^{\beta-2}(\xi - 1) \le 1 + a^{p-1}\xi^{\beta-2}$$

$$|H(t,s)| \le \Omega\left\{1 + a^{p-1}\left(1 - b^{p-1}\eta^{\beta-1}\right) + (1 + b^{p-1})\left(1 + a^{p-1}\xi^{\beta-2}\right)\right\} := B. \qquad \square$$

3. Main Result

In this section, the main results of the present study will be stated and proved. Transform the problem (1.1) - (1.2) into a fixed point problem. The operator can be defined as $T: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$Tu(t) := \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\eta)f\big(\eta,u(\eta)\big)d_q\eta\right)d_qs. \tag{3.1}$$

We consider the Banach space $C([0,1])$ endowed with norm $\|u\| = \max_{t\in[0.1]}|u(t)|$. Denote

$X := \left\{ u \in C([0,1]) \colon D_q^\alpha u, \varphi_p\left(D_q^\alpha u\right), D_q^\beta u, \varphi_p\left(D_q^\beta u\right) \in C([0,1]) \right\}.$

Suppose that the following conditions are satisfied:

(H_2): There exists a real constant $k > 0$ such that

$$|f(t,u) - f(t,v)| \leq k|u - v| \text{ for all } t \in [0,1] \text{ and } u, v \in [0,\delta].$$

(H_3): There exist nonnegative functions $g, h \in L[0,1]$ and

$$M_g := \int_0^1 g(t) d_q t > 0, \qquad M_h := \int_0^1 h(t) d_q t > 0,$$

such that $f(t,u) \leq g(t) + h(t)u^{p-1}$ for any $t \in [0,1]$ and $u \in [0,\delta]$, where

$$\delta = \left(\frac{A^{p-1} M_g B}{N - M_h B A^{p-1}} \right)^{r-1},$$

and $N := \left(\Gamma_q(\alpha)\right)^{p-1} \Omega\, \Gamma_q(\beta)$.

Lemma 3.1 If (H_3) holds, then $T(D) \subset D$, where $D = \{u \in X \colon \|u\| < \delta\}$.

<u>*Proof:*</u> We need to show that $\|Tu\| < \delta$ for any $u \in D$. By Lemma 2.5 and (H_3), we get

$$|Tu(t)| = \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \left| \int_0^1 G(t,qs) \varphi_r \left(\int_0^1 H(s,q\eta) f(\eta, u(\eta)) d_q\eta \right) d_q s \right|$$

$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \int_0^1 |G(t,qs)| \varphi_r \left(\int_0^1 |H(s,q\eta)| |f(\eta,u(\eta))| d_q\eta \right) d_q s$$

$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \int_0^1 \frac{1}{\Gamma_q(\alpha)} A . \varphi_r \left(\int_0^1 B . \left(g(\eta) + h(\eta)(u(\eta))^{p-1}\right) d_q\eta \right) d_q s$$

$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \cdot \frac{1}{\Gamma_q(\alpha)} A . \varphi_r \left(B \left(\int_0^1 g(\eta) d_q\eta + \delta^{p-1} \int_0^1 h(\eta)\, d_q\eta \right) \right)$$

$$= \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \cdot \frac{1}{\Gamma_q(\alpha)} A . B^{r-1} . \varphi_r\left(M_g + \delta^{p-1} M_h\right)$$

$$= \left(\frac{\left(M_g + \delta^{p-1} M_h\right) B}{\Omega\Gamma_q(\beta)} \right)^{r-1} . \frac{A}{\Gamma_q(\alpha)} = \delta.$$

This shows that $\|Tu\| < \delta$. □

Lemma 3.2 If (H_3) holds, there exists an $R_0 := B.\left(M_g + \delta^{p-1} M_h\right)$ such that for any $u \in D$ and any $t \in [0,1]$,

$$\int_0^1 H(t,qs) f(s,u(s)) d_q s \leq R_0.$$

Proof: It is obvious that

$$\left|\int_0^1 H(t,qs)f(s,u(s))d_qs\right| \le \int_0^1 |H(t,qs)||f(s,u(s))|d_qs$$

$$\le B.\int_0^1 \left(g(s)+h(s)\big(u(s)\big)^{p-1}\right)d_qs \le B.\left(M_g+\delta^{p-1}M_h\right) := R_0.$$

The proof is completed. □

Theorem 3.1 Suppose that $(H_2)-(H_3)$ hold and

$$\frac{\delta.k.B}{R_0}<1$$

with $1<p<2$. Then the boundary value problem (BVP) (1.1) - (1.2) has a unique solution.

Proof: It is easy to see that $r>2$. Using Lemma 3.1, Lemma 3.2, and the basic properties of the *p*-Laplacian operator, we conclude that for any $u,v\in X$ and any $t\in[0,1]$.

$$|Tu(t)-Tv(t)| = \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\left|\int_0^1 G(t,qs)\left\{\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,u(\eta))d_q\eta\right)\right.\right.$$
$$\left.\left.-\,\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,v(\eta))d_q\eta\right)\right\}d_qs\right|$$

$$\le \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\cdot\frac{A}{\Gamma_q(\alpha)}\left|\int_0^1\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,u(\eta))d_q\eta\right)d_qs-\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,v(\eta))d_q\eta\right)d_qs\right|$$

$$\le \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\cdot\frac{A}{\Gamma_q(\alpha)}.{R_0}^{r-2}(r-1)\int_0^1 H(s,q\eta)\big(|f(\eta,u(\eta))-f(\eta,v(\eta))|d_q\eta\big)d_qs$$

$$\le \underbrace{\frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\cdot\frac{r-1}{\Gamma_q(\alpha)}.A.{R_0}^{r-2}.B.k}_{<1}.\|u-v\|.$$

So, we can calculate

$$\|Tu-Tv\|\le\frac{(r-1).{R_0}^{r-2}.A.B.k}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}\Gamma_q(\alpha)}\|u-v\|.$$

From the definition of R_0, we have

$$\frac{(r-1).{R_0}^{r-2}.A.B.k}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}\Gamma_q(\alpha)}=\frac{(r-1).A.B.k}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}\Gamma_q(\alpha)}\left[B.\left(M_g+\delta^{p-1}M_h\right)\right]^{r-2}$$

$$= \frac{(r-1).A.B^{r-1}.k}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}\Gamma_q(\alpha)}\left(M_g+\delta^{p-1}M_h\right)^{r-2}$$

$$= \left[\frac{\left(M_g+\delta^{p-1}M_h\right)B}{\Omega\Gamma_q(\beta)}\right]^{r-1}.\frac{(r-1).A.k}{\left(M_g+\delta^{p-1}M_h\right)\Gamma_q(\alpha)}$$

$$= \delta\frac{(r-1).k}{\left(M_g+\delta^{p-1}M_h\right)} = \frac{\delta.k.B}{R_0}.$$

Therefore, $T: D \to D$ is a contraction mapping. By the Banach contraction mapping principle, we can see that T has a unique fixed point in D, which means that the boundary value problem (BVP) (1.1) - (1.2) has a unique solution. □

Theorem 3.2 Assume that (H_4): $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous and there exists a positive constant k such that $|f(t,u(t)| \leq k.|u|$ for all $(t,u) \in [0,1] \times \mathbb{R}$.

Then the boundary value problem (BVP) (1.1) - (1.2) has at least one solution on [0,1].

Proof: We consider the set D and the operator $T: D \to D$ defined by (3.1).

Step 1. We show that T is a continuous operator. Let $\{u_n\}_{n\in N}$ be a sequence in D such that $u_n \to u$ in D as $n \to \infty$. We obtain

$$|Tu_n(t) - Tu(t)| = \left|\frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\int_0^1 G(t,qs)\left\{\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,u_n(\eta))d_q\eta\right) - \varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,u(\eta))d_q\eta\right)\right\}d_qs\right|$$

$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\int_0^1 |G(t,qs)|\left|\varphi_r\left(\int_0^1 H(s,q\eta)f\big(\eta,u_n(\eta)\big)d_q\eta\right) - \varphi_r\left(\int_0^1 H(s,q\eta)f\big(\eta,u(\eta)\big)d_q\eta\right)\right|.$$

Thus, we get

$$\|Tu_n - Tu\| \leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\frac{A}{\Gamma_q(\alpha)}(r-1){R_0}^{r-2}.B.\left\|f\big(\eta,u_n(\eta)\big) - f\big(\eta,u(\eta)\big)\right\|.$$

Using the continuity of the function f, it follows that $\|Tu_n - Tu\| \to 0$ as $n \to \infty$, which implies that T is a continuous operator.

Step 2. We see that T maps bounded sets in to uniformly bounded sets in D. For this reason, we need to show that for all $r_1 > 0$ there exists some $r_2 > 0$ such that for all $u \in B_{r_1} := \{u \in D: \|u\| < r_1\}$, $\|Tu\| \leq r_2$ is satisfied.
Indeed, let $u \in B_{r_1}$, for all $t \in [0,1]$, we have

$$|Tu(t)| = \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}}\left|\int_0^1 G(t,qs)\varphi_r\left(\int_0^1 H(s,q\eta)f(\eta,u(\eta))d_q\eta\right)d_qs\right|$$

$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \int_0^1 |G(t,qs)| \left|\varphi_r\left(\int_0^1 H(s,q\eta) f(\eta,u(\eta)) d_q\eta\right)\right| d_q s$$
$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \int_0^1 \frac{A}{\Gamma_q(\alpha)} \varphi_r\left(\int_0^1 B.k.|u(\eta)| d_q\eta\right) d_q s$$
$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \frac{A}{\Gamma_q(\alpha)} B^{r-1} k^{r-1} \|u\|^{r-1}$$
$$\leq \frac{A}{\left(\Omega\Gamma_q(\beta)\right)^{r-1} \Gamma_q(\alpha)} (B.k.r_1)^{r-1}.$$

which is a constant. Hence, there exists

$$r_2 := \frac{A}{\Gamma_q(\alpha)} \left(\frac{B.k.r_1}{\Omega\Gamma_q(\beta)}\right)^{r-1},$$

such that $\|Tu\| \leq r_2$. Thus, $\{Tu\}$ is uniformly bounded set.

<u>Step 3</u>. We show that T maps bounded sets into equi-continuous sets of D. Let B_{r_1} be a bounded set of D as in Step 2, and $u \in B_{r_1}$.
Consequently, for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$|Tu(t_1) - Tu(t_2)| = \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} \left|\int_0^1 \left(G(t_1,qs) - G(t_2,qs)\right) \varphi_r\left(\int_0^1 H(s,q\eta) f(\eta,u(\eta)) d_q\eta\right) d_q s\right|$$
$$\leq \frac{1}{\left(\Omega\Gamma_q(\beta)\right)^{r-1}} (B.k.r_1)^{r-1} \int_0^1 |G(t_1,qs) - G(t_2,qs)|\, d_q s.$$

Using the continuity of the function $G(t,s)$, as $t_1 \to t_2$, the right side of the above inequality tends to zero. Therefore, we can conclude that $T: D \to D$ is a completely continuous operator by the Arzela-Ascoli theorem.

<u>Step 4</u>. We show that the set $S = \{u \in D: u = \lambda Tu$ for some $\lambda \in (0,1)\}$ is bounded.

Let $u \in S$ and $\lambda \in (0,1)$ be such that $u = \lambda Tu$. By Step 2, for all $t \in [0,1]$, we have

$$|Tu(t)| \leq \frac{A}{\Gamma_q(\alpha)} \left(\frac{B.k}{\Omega\Gamma_q(\beta)}\right)^{r-1} \|u\|^{r-1}.$$

Since $\lambda \in (0,1)$, $u \leq Tu$, and hence,

$$\|u\| \leq \|Tu\| \leq \frac{A}{\Gamma_q(\alpha)} \left(\frac{B.k}{\Omega\Gamma_q(\beta)}\right)^{r-1} \|u\|^{r-1}$$
$$\|u\|^{2-r} \leq \frac{A}{\Gamma_q(\alpha)} \left(\frac{B.k}{\Omega\Gamma_q(\beta)}\right)^{r-1},$$

and so

$$\|u\| \leq \left[\frac{A}{\Gamma_q(\alpha)}\left(\frac{B.k}{\Omega\Gamma_q(\beta)}\right)^{r-1}\right]^{\frac{1}{2-r}}.$$

Thus, confirming the boundedness of S, we then employ Schaefer's fixed point theorem to establish the existence of least one fixed u of the operator T within the set D. Therefore, we conclude that the boundary value problem (BVP) (1.1) - (1.2) has at least one solution in D and the proof is finished. □

4. An illustrative example

Example 4.1 Consider the following fractional q-difference equation BVP with φ-Laplacian:

$$D_q^{\frac{3}{2}}\left(\varphi_3\left(D_q^{\frac{5}{4}}u(t)\right)\right)+\frac{1}{10}\sin u(t) = 0, \tag{4.1}$$

$$u(0) = 0, \qquad u(1) = \frac{1}{4}\int_0^1 t^{\frac{3}{4}}.u(t)d_qt,$$

$$D_q^{\frac{5}{4}}u(0) = \frac{1}{5}D_q^{\frac{5}{4}}u\left(\frac{1}{2}\right), \quad D_q^{\frac{5}{4}}u(1) = \frac{1}{3}D_q^{\frac{5}{4}}u\left(\frac{1}{4}\right), \tag{4.2}$$

Here $\alpha = \frac{5}{4},\ \beta = \frac{3}{2},\ a = \frac{1}{5},\ b = \frac{1}{3},\ c = \frac{1}{4},\ \xi = \frac{1}{2},\ \eta = \frac{1}{4},\ h(t) = t^{\frac{3}{4}}$.

$$k = 1 - \frac{1}{4}\int_0^1 t^{\frac{3}{4}}\ .\ t^{\frac{5}{4}-1}d_qt = 1 - \frac{1}{4}\int_0^1 t^{\frac{3}{4}}\ .\ t^{\frac{1}{4}}\ d_qt = 1 - \frac{1}{4}\int_0^1 t\, d_qt = 1 - \frac{1}{4}.\frac{t^2}{[2]}\Big|_0^1 = \frac{7}{8} > 0.$$

So, (H_1) is satisfied.

$$f(t,u(t)) = \frac{1}{10}\sin u(t)$$

is continuous and also we get $|f(t,u(t))| \leq \left|\frac{1}{10}\sin u(t)\right| \leq \frac{1}{10}|u(t)|$.

So the condition (H_4) is satisfied for $k = \frac{1}{10}$.
Thus, the problem (4.1) - (4.2) has at least one solution by Theorem 3.2.

5. Conclusions

In this paper, we have studied the existence and uniqueness of solutions for fractional q-difference equations with the p-Laplacian operator. Our investigation focused on the intricate dynamics introduced by the fractional q-difference and the nonlinearity of the p-Laplacian, which together create a rich structure for mathematical analysis.

We employed a combination of fixed point theorems and variational methods to establish the existence results. Specifically, we utilized the Schaefer's and Banach fixed point theorems and found that the solution to the fractional q-difference equation given.

Our results contribute to the broader understanding of fractional q-difference equations by extending existing theories and providing new insights into their behavior with p-Laplacian operators. The fractional q-difference operator, with its ability to generalize classical difference equations, offers a versatile tool for modeling diverse phenomena in science and engineering.

Future work can explore the uniqueness and stability of these solutions, as well as their numerical approximations. Additionally, the methods developed in this study can be applied to other types of fractional difference equations, further enriching the field of discrete fractional calculus.

In summary, this paper has provided a foundational framework for analyzing the existence of solutions to fractional q-difference equations with p-Laplacian, paving the way for further research and applications in this promising area of mathematical study.

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