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Approximation by a new Schurer type Stancu Operators and Associated GBS Operators

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Abstract

This study presents a novel extension of the Schurer type Stancu operators and investigates their properties in terms of approximation. The uniform convergence of these operators is provided using the Korovkin Theorem, and the rates of convergence are expressed in terms of the modulus of continuity. Subsequently, the theorem known as Grüss-Voronovskaja is proven. In addition, the related generalized Boolean sum (GBS) operators are defined, and the rates of approximation for these operators are obtained using the mixed modulus of smoothness and functions from the Lipshitz class. Then, numerical examples and graphical results for both operators are presented.

Keywords: Schurer-Stancu operators; Modulus of continuity; Grüss-Voronovskaja type theorem; GBS operators.

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Yeni bir Schurer tipi Stancu Operatörleri ve ilgili GBS Operatörleri ile Yaklaşım

Öz

Bu çalışma, Schurer tipi Stancu operatörlerinin yeni bir genelleştirmesini sunmakta ve bu operatörlerin yaklaşım özelliklerini incelemektedir. Bu operatörlerin düzgün yakınsaklığı Korovkin Teoremi yardımıyla verilmiş ve yakınsama hızları süreklilik modülü cinsinden ifade edilmiştir. Daha sonra Grüss-Voronovskaja olarak bilinen teorem ispatlanmıştır. Ayrıca, ilgili genelleştirilmiş Boolean toplamı (GBS) operatörleri tanımlanmış ve bu operatörlerin yaklaşım hızları karma düzgünlük modülü ile Lipshitz sınıfından fonksiyonlar kullanılarak elde edilmiştir. Sonrasında, her iki operatör için sayısal örnekler ve grafiksel sonuçlar sunulmuştur.

Anahtar Kelimeler: Schurer-Stancu operatörleri; Süreklilik modülü; Grüss-Voronovskaja tipi teorem; GBS operatörleri.

1. Introduction

Polynomials are considered the fundamental functions that computers can directly compute. The Weierstrass approximation theorem in mathematical analysis states that any continuous function defined on a closed interval may be represented by a polynomial function. The Weierstrass approximation theorem is highly useful in both practical and theoretical contexts, particularly in the domain of polynomial interpolation, due to this characteristic of the property. An exemplary demonstration of this theorem is S.N. Bernstein got it by defining the subsequent polynomials in reference [1]. The Bernstein polynomials are expressed by the following equation:

$$B_\eta(\phi,\tau)=\sum_{\rho=0}^{\eta}\mathbf{p}_{\eta,\rho}(\tau)\phi\left(\frac{\rho}{\eta}\right), \eta\in \mathrm{N},$$

for each function ϕ in the space $C[0,1]$, which includes all real-valued continuous functions defined on the closed interval $[0,1]$ and $\mathbf{p}_{\eta,\rho}(\tau)$ is a Bernstein basis function equal to $\binom{\eta}{\rho}\tau^\rho(1-\tau)^{\eta-\rho}$.

F. Schurer, [2], introduced and investigated the features of the linear positive operators known as Bernstein-Schurer polynomials,

$$B_{\eta,\gamma}(\phi,\tau)=\sum_{\rho=0}^{\eta+\gamma}\boldsymbol{p}_{\eta+\gamma,\rho}(\tau)\phi\left(\frac{\rho}{\eta}\right),\eta\in N,$$

for ϕ be an element of $C[0,1+\gamma]$, which is a set of all real-valued continuous functions defined on the expanded closed interval $[0,1+\gamma]$.

Then, D. Bărbosu as an extension of Bernstein operators, created the linear positive Schurer-Stancu operators, in [3], $\widetilde{S_{\eta,\gamma}^{\alpha,\beta}}: C[0,1+\gamma]\to C[0,1]$ as

$$\widetilde{S_{\eta,\gamma}^{\alpha,\beta}}(\phi,\tau)=\sum_{\rho=0}^{\eta+\gamma}\boldsymbol{p}_{\eta+\gamma,\rho}(\tau)\phi\left(\frac{\rho+\alpha}{\eta+\beta}\right),$$

where γ be a given integer from the set of natural numbers, and α and β be provided parameters from the set of real numbers. These parameters must satisfy the requirements $0\le\alpha\le\beta$. Note that for $\alpha=\beta=0$ these polynomials become Schurer operators, for $\gamma=0$ the operators are the Stancu operators, for $\alpha=\beta=0$ and $\gamma=0$ the operators are the Bernstein operators. Bărbosu's work has inspired many studies in literature. Some of them are as [4-6].

Stancu introduced a new type of linear positive operators named Stancu operators in the literature, as follows,

$$L_{\eta,\mu}(\phi,\tau)=\sum_{\rho=0}^{\eta}w_{\eta,\rho,\mu}(\tau)\phi\left(\frac{\rho}{\eta}\right),\quad \tau\in[0,1] \tag{1}$$

for $\phi\in C[0,1]$, a non-negative integer parameter $\mu,\eta\in N$ such that $\eta>2\mu$, where

$$w_{\eta,\rho,\mu}(\tau)=\begin{cases}(1-\tau)\boldsymbol{p}_{\eta-\mu,\rho}(\tau); & 0\le\rho<\mu\\ (1-\tau)\boldsymbol{p}_{\eta-\mu,\rho}(\tau)+\tau\boldsymbol{p}_{\eta-\mu,\rho-\mu}(\tau); & \mu\le\rho\le\eta-\mu,\\ \tau\boldsymbol{p}_{\eta-\mu,\rho-\mu}(\tau); & \eta-\mu<\rho\le\eta\end{cases} \tag{2}$$

and $\mathbf{p}_{\eta,\rho}(\tau)$ is the well-known Bernstein basis polynomials [7-8]. For the special cases $\mu=0$ and $\mu=1$, Stancu operators defined by Eq.1 give the classical Bernstein operators.

Recently, many academics have focused on Stancu operator approximation. The researchers in [9-11] showed Stancu operator approximation properties in multiple domains, including the complex plane and distinct forms. The complex form of Schurer-type Stancu operators is defined in [13], and their complex approximation properties are investigated by authors. In this study, we introduce and examine the real-valued form of operators whose complex form has been examined in [13].

Motivated by the beforementioned studies, we introduce the real variable case of the Schurer form of Stancu operators as follows,

$$L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau) = \sum_{\rho=0}^{\eta+\gamma-\mu} \binom{\eta+\gamma-\mu}{\rho} \tau^{\rho}(1-\tau)^{\eta+\gamma-\mu-\rho} \left[(1-\tau)\phi\left(\frac{\rho+\alpha}{\eta+\beta}\right) + \tau\phi\left(\frac{\rho+\mu+\alpha}{\eta+\beta}\right)\right] \quad (3)$$

where α, β are real parameters with $0 \leq \alpha \leq \beta$, μ is a non -negative integer, $\eta \in \mathrm{N}$ such that $\eta + \gamma > 2\mu, \gamma \in \mathrm{N} \cup \{0\}$ and examined the approximation properties of them. Here, for the situation $\gamma = 0$, these operators become the original Stancu operators represented with the equation (1), and for $\mu = 0$, they become Schurer-type Stancu operators.

In this study, we first investigate this new real-valued generalization of the Schurer-type Stancu operators and their properties related to approximation. By applying the renowned Korovkin theorem, we achieve a uniform approximation result and consider the modulus of continuity rate of convergence given. Next, we establish the theorems of Grüss-Voronovskaja. Then, we create related GBS operators of two real variables and examine their approximation properties. Finally, we provide numerical examples to demonstrate the process of approximation using these new operators.

2. Supplementary Findings

To achieve the primary findings, we provide the following information. These findings in this section were written using the results found for the complex variable in [13]. Thus, the results are presented without proof.

Lemma 2.1. Considering the operators (3), we have the following moments,

$$L_{\eta,\mu,\gamma}^{\alpha,\beta}(e_0,\tau) = 1,$$

$$L_{\eta,\mu,\gamma}^{\alpha,\beta}(e_1,\tau) = \frac{(\eta+\gamma)\tau+\alpha}{\eta+\beta},$$

$$L_{\eta,\mu,\gamma}^{\alpha,\beta}(e_2,\tau) = \frac{1}{(\eta+\beta)^2}\{(\eta+\gamma-\mu)(\eta+\gamma+\mu-1)\tau^2 + [(\eta+\gamma-\mu)(2\alpha+1)+\mu(\mu+2\alpha)]\tau + \alpha^2\},$$

$L^{\alpha,\beta}_{\eta,\mu,\gamma}(e_3,\tau) = \frac{1}{(\eta+\beta)^3}\{(\eta+\gamma-\mu)(\eta+\gamma+\mu-1)(\eta+\gamma+2\mu-2)\tau^3 + 3(\eta+\gamma-\mu)\,[(\eta+\gamma-\mu-1)(\alpha+1)+\mu(\mu+2\alpha+1)]\tau^2 + [(\eta+\gamma-\mu)(1+3\alpha+3\alpha^2)+\mu(\mu^2+3\alpha\mu+3\alpha^2)]\tau + \alpha^3\}$,

$L^{\alpha,\beta}_{\eta,\mu,\gamma}(e_4,\tau) = \frac{1}{(\eta+\beta)^4}\{(\eta+\gamma-\mu)(\eta+\gamma+\mu-1)(\eta+\gamma-\mu-2)(\eta+\gamma+3\mu-3)\tau^4 + 2(\eta+\gamma-\mu)(\eta+\gamma-\mu-1)[(\eta+\gamma-\mu-2)(3+2\alpha)+3\mu(2+2\alpha+\mu)]\tau^3 + (\eta+\gamma-\mu)[(\eta+\gamma-\mu-1)(7+12\alpha+6\alpha^2)+4\mu^3+\mu^2(6+12\alpha)+4\mu(1+3\alpha+3\alpha^2)]\tau^2 + [(\eta+\gamma-\mu)(1+2\alpha)(1+2\alpha+2\alpha^2)+\mu^4+4\alpha\mu^3+6\alpha^2\mu^2+4+2\alpha^3\mu]\tau + \alpha^4\}$.

Lemma 2.2. The central moments corresponding to the operators (3) are displayed below:

$$L^{\alpha,\beta}_{\eta,\mu,\gamma}(\xi-\tau,\tau) = \frac{(\gamma-\beta)\tau+\alpha}{\eta+\beta} \coloneqq \Phi^{\alpha,\beta,1}_{\eta,\mu,\gamma},$$

$L^{\alpha,\beta}_{\eta,\mu,\gamma}((\xi-\tau)^2,\tau) = \frac{1}{(\eta+\beta)^2}\{[-(\eta+\gamma)+(\gamma+\beta)^2-\mu(\mu-1)]\tau^2 + [\eta+\gamma+2\alpha(\gamma-\beta)+\mu(\mu-1)]\tau+\alpha^2\} \coloneqq \Phi^{\alpha,\beta,2}_{\eta,\mu,\gamma}$,

$L^{\alpha,\beta}_{\eta,\mu,\gamma}((\xi-\tau)^4,\tau) = \frac{1}{(\eta+\beta)^4}\{[3\eta^2+\eta(12\gamma\beta+6\mu^2-6\mu-6-8\beta-6\beta^2)+\gamma^4+\gamma^3(-6-4\beta)+\gamma^2(-6\mu^2+6\mu+11+12\beta+6\beta^2)+\gamma(8\mu^3+6\mu^2+12\beta\mu^2-14\mu-12\beta\mu-6-8\beta-6\beta^2-4\beta^3)\&-3\mu^4-\mu^3(6+8\beta)+\mu^2(3-6\beta^2)+\mu(6+8\beta+6\beta^2)+\beta^4]\tau^4 + [-6\eta^2+\eta(6\gamma^2-12\gamma(2+\alpha+\beta)-12\mu^2+12\mu+12+8\alpha+12\beta+12\alpha\beta+6\beta^2)+\gamma^3(6+4\alpha)+\gamma^2(6\mu^2-6\mu-18-12\alpha-12\beta-12\alpha\beta)+\gamma(-12\mu^3-12\mu^2(1+\alpha+\beta)+12\mu(2+\alpha+\beta)+12+8\alpha+12\beta+12\alpha\beta+12\alpha\beta^2+6\beta^2)+6\mu^4+4\mu^3(3+2\alpha+3\beta)-6\mu^2(1-2\alpha\beta-\beta^2)-2\mu(6+4\alpha+6\beta+6\alpha\beta+3\beta^2)-4\alpha\beta^3]\tau^3 + [3\eta^2+\eta(\gamma(10+12\alpha)+6\mu^2-6\mu-6\alpha^2-12\alpha-7-12\alpha\beta-4\beta)+\gamma^2(7+12\alpha+6\alpha^2)+\gamma(4\mu^3+\mu^2(6+12\alpha)+\mu(-10-12\alpha)-7-12\alpha-6\alpha^2-4\beta-12\alpha\beta-12\alpha^2\beta)-4\mu^4-2\mu^3(3+6\alpha+2\beta)+3\mu^2(1-4\alpha\beta-2\alpha^2)+\mu(7+12\alpha+6\alpha^2+4\beta+12\alpha\beta)+16\alpha^2\beta^2]\tau^2 + [(\eta-\mu)(1+4\alpha+6\alpha^2)+\gamma(1+2\alpha)(1+2\alpha+2\alpha^2)+\mu^4+4\alpha\mu^3+6\alpha^2\mu^2+4\alpha^3\mu]\tau+\alpha^4\}$.

Lemma 2.3. From the previous lemma we immediately get the following results.

$$\lim_{\eta\to\infty} L^{\alpha,\beta}_{\eta,\mu,\gamma}(\xi-\tau,\tau) = (\gamma-\beta)\tau+\alpha,$$

$$\lim_{\eta\to\infty} L^{\alpha,\beta}_{\eta,\mu,\gamma}((\xi-\tau)^2,\tau) = \tau(1-\tau),$$

$$\lim_{\eta\to\infty} L^{\alpha,\beta}_{\eta,\mu,\gamma}((\xi-\tau)^4,\tau) = 0.$$

3. Main Results for Univariate Operators

As a result of the previous lemmas, the following Korovkin type theorem is easily reached.

Theorem 3.1. If ϕ belongs to the set of continuous functions defined on the interval $[0,1+\gamma]$, $\lim_{\eta\to\infty} L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau) = \phi(\tau)$, uniformly on $[0,1]$.

Now, we want to provide an upper limit for the approximation error using the K −functional. Let us start by reviewing certain definitions and notations.

The definition of the modulus of continuity for a function ϕ in the space $C[0,1+\gamma]$ with a given value of $\delta > 0$ is as follows:

$$\omega(\phi,\delta) = \sup_{|\xi|<\delta} \sup_{\tau,\tau+\xi\in[0,1+\gamma]} |\phi(\tau+\xi) - \phi(\tau)|$$

The Petree's K −functional is formally defined as follows:

$$\rho(\phi,\delta) = \inf_{\zeta\in C^2[0,1+\gamma]} \{\|\phi-\zeta\| + \delta\|\zeta''\|\}, (\delta > 0),$$

where $C^2[0,1+\gamma] = \{\zeta \in C[0,1+\gamma]: \zeta', \zeta'' \in C[0,1+\gamma]\}$. Then, for a positive constant M, we have the following inequality [12, p.1]

$$\rho(\phi,\delta) \le M\omega_2(\phi,\sqrt{\delta}) \tag{4}$$

where modulus of smoothness of the second order for a function ϕ belonging to the space $C[0,1+\gamma]$ is defined as

$$\omega_2(\phi,\sqrt{\delta}) = \sup_{0<\zeta\le\delta} \sup_{\tau,\tau+2\zeta\in[0,1+\gamma]} |\phi(\tau+2\zeta) - 2\phi(\tau+\zeta) + \phi(\tau)|.$$

The upper bound for the error $\phi(\tau) - L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ can be bounded by the modulus of continuity, as previously described.

Theorem 3.2. If $\phi \in C[0,1+\gamma]$, then

$$\left|L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau) - \phi(\tau)\right| \le 2\omega(\phi,\sqrt{\delta})$$

where ω denotes the usual modulus of continuity.

Proof. By considering the inequality

$$|\phi(t)-\phi(\tau)| \leq \omega(\phi,\delta)\left(1+\frac{(\xi-\tau)^2}{\delta^2}\right) \tag{5}$$

and applying the operators these operators (3) we have

$$\begin{aligned}\left|L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)-\phi(\tau)\right| &\leq L_{\eta,\mu,\gamma}^{\alpha,\beta}(|\phi(t)-\phi(\tau)|,\tau)\\ &\leq \omega(\phi,\delta)\left(1+\frac{1}{\delta^2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2,\tau)\right)\end{aligned}$$

if we choose $\delta=\sqrt{\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,2}}$ we reach the desired result.

Theorem 3.3. Let $\gamma \in N \cup \{0\}$ be fixed. For any value of τ inside the interval $[0,1]$ and every function ϕ belonging to the set of continuous functions defined on the interval $[0,1+\gamma]$,the inequality below,

$$\begin{aligned}\left|L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)-\phi(\tau)\right| &\leq 4\rho\left(\phi,\frac{1}{4}\left(\left(\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,2}\right)+{\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,1}}^2\right)\right)+\omega\left(\phi;\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,1}\right)\\ &\leq M\omega_2\left(\phi,\frac{1}{2}\sqrt{\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,2}+{\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,1}}^2}\,\&\right)+\omega\left(\phi;\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,1}\right)\end{aligned}$$

holds.

Proof. We denote $\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)=L_{\eta,\mu,\gamma}^{\alpha,\beta}(t;\tau)=\frac{(\eta+\beta)\tau+\alpha}{\eta+\beta}$ and define the auxiliary operator,

$$\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\phi;\tau)=L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)+\phi(\tau)-\phi\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right). \tag{6}$$

By using Lemma 2.1. we can easily reach that,

$$\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(1;\tau)=1$$
$$\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\xi;\tau)=\tau$$

Consider a function ϕ that belongs to the class $C^2[0,1+\gamma]$. For any values of τ and ξ that are inside the interval $[0,1+\gamma]$ we have,

and applying $\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}$ we get

$$\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\zeta(\xi)-\zeta(\tau);\tau)=\zeta'(\tau)\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\xi-\tau;\tau)+\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}\left(\int_\tau^\xi(\xi-\upsilon)\zeta''(\upsilon)d\upsilon\,;\tau\right)$$
$$=L_{\eta,\mu,\gamma}^{\alpha,\beta}\left(\int_\tau^\xi(\xi-\upsilon)\zeta''(\upsilon)d\upsilon\,;\tau\right)-\int_\tau^{\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)}\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)-\upsilon\right)\zeta''(\upsilon)d\upsilon.$$

If we continue with absolute values of both sides of last equation

$$\begin{aligned}\left|\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\zeta(\xi)-\zeta(\tau);\tau)\right|&\\ &\le L_{\eta,\mu,\gamma}^{\alpha,\beta}\left(\left|\int_\tau^\xi(\xi-\upsilon)\zeta''(\upsilon)d\upsilon\right|;\tau\right)-\left|\int_\tau^{\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)}\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)-\upsilon\right)\zeta''(\upsilon)d\upsilon\right|\\ &\le\frac{\|\zeta''\|}{2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2;\tau)+\frac{\|\zeta''\|}{2}\left(\tau-\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right)^2\\ &=\frac{\|\zeta''\|}{2}\left[\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,2}(\tau)+\left(\frac{(\beta-\gamma)\tau+\alpha}{\eta+\beta}\right)^2\right].\end{aligned}\tag{7}$$

In the view of Eq. (6) we obtain

$$\left|\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\phi;\tau)\right|=\left|L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)\right|+|\phi(\tau)|+\left|\phi\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right)\right|\le 3\|\phi\|\tag{8}$$

Now, for $\phi\in C[0,1+\gamma]$ and $\zeta\in C^2[0,1+\gamma]$, using (7) and (8) we have,

$$\begin{aligned}\left|L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)-\phi(\tau)\right|&=\left|\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\phi;\tau)-\phi(\tau)+\phi\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right)-\phi(\tau)\right|\\ &\le\left|\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\phi-\zeta;\tau)\right|+\left|\widetilde{L_{\eta,\mu,\gamma}^{\alpha,\beta}}(\zeta;\tau)-\zeta(\tau)\right|+|\zeta(\tau)-\phi(\tau)|\\ &+\left|\phi\left(\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right)-\phi(\tau)\right|\\ &\le 4|\phi-\zeta|+\frac{\|\zeta''\|}{2}\left[\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,2}(\tau)+\left(\frac{(\beta-\gamma)\tau+\alpha}{\eta+\beta}\right)^2\right]+\omega\left(\phi,\tau-\epsilon_{\eta,\mu,\gamma}^{\alpha,\beta}(\tau)\right).\end{aligned}$$

Choosing $\delta=\sqrt{\Phi_{\eta,\mu,\gamma}^{\alpha,\beta,1}}$, we obtain the desired inequality.

The Voronovskaja's-type theorem is given in the study [13] as Theorem 2.1. Therefore, the theorem is presented without proof.

Theorem 3.4. ([13]) Let ϕ be a function in the set of continuous functions defined on the interval $[0,1+\gamma]$. Suppose that function ϕ has a second-order derivative at a point τ on the interval

$[0,1+\gamma]$. Then, by considering the behavior of ϕ around τ and ensuring that the expression accounts for limits approaching from both sides of τ, we have the following conclusion:

$$\lim_{\eta\to\infty}\eta\left[L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)-\phi(\tau)\right]=[(\gamma-\beta)\tau+\alpha]\phi'(\tau)+\frac{1}{2}\tau(1-\tau)\phi''(\tau),$$

where $0\leq\alpha\leq\beta$.

In the subsequent discussion, we will provide a Grüss-Voronovskaya type theorem using the methodology outlined in (*cf*, [14]).

Theorem 3.5. Let $\phi,\zeta\in C[0,1+\gamma]$, for each $\tau\in[0,1]$ we have,

$$\lim_{\eta\to\infty}(\eta+\beta)\left[L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi\zeta,\tau)-L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\zeta,\tau)\right]=\tau(1-\tau)\phi'(\tau)\zeta'(\tau).$$

Proof. Since

$$(\phi\zeta)(\tau)=\phi(\tau)\zeta(\tau)$$

$$(\phi\zeta)'(\tau)=\phi'(\tau)\zeta(\tau)+\phi(\tau)\zeta'(\tau)$$

and

$$(\phi\zeta)''(\tau)=\phi''(\tau)\zeta(\tau)+2\phi'(\tau)\zeta'(\tau)+\phi(\tau)\zeta''(\tau).$$

$$\begin{aligned}
&L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi\zeta;\tau)-L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\zeta;\tau)\\
&=\left[L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi\zeta;\tau)-\phi(\tau)\zeta(\tau)+(\phi\zeta)'(\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\xi-\tau;\tau)+\frac{(\phi\zeta)''(\tau)}{2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2;\tau)\right]\\
&\quad-\zeta(\tau)\left[L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)-\phi(\tau)+\phi'(\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\xi-\tau;\tau)+\frac{\phi''(\tau)}{2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2;\tau)\right]\\
&\quad-L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)\left[L_{\eta,\mu,\gamma}^{\alpha,\beta}(\zeta;\tau)-\zeta(\tau)+\zeta'(\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\xi-\tau;\tau)+\frac{\zeta''(\tau)}{2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2;\tau)\right]\\
&\quad+\frac{1}{2}L_{\eta,\mu,\gamma}^{\alpha,\beta}((\xi-\tau)^2;\tau)\left[\phi(\tau)\zeta''(\tau)+2\phi'(\tau)\zeta'(\tau)+\zeta''(\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)\right]\\
&\quad+\zeta'(\tau)L_{\eta,\mu,\gamma}^{\alpha,\beta}(\xi-\tau;\tau)\left[\phi(\tau)-L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi;\tau)\right]
\end{aligned}$$

By utilizing Lemma 2.2, we obtain

$$\lim_{\eta\to\infty}(\eta+\beta)\left[L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi\zeta;\tau)-L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi;\tau)L^{\alpha,\beta}_{\eta,\mu,\gamma}(\zeta;\tau)\right]$$

$$=\lim_{\eta\to\infty}(\eta+\beta)\left[L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi\zeta;\tau)-\phi(\tau)\zeta(\tau)\right]-(\phi\zeta)'(\tau)\gamma\tau-\frac{(\phi\zeta)''(\tau)}{2}\tau(1-\tau)$$

$$-\zeta(\tau)\lim_{\eta\to\infty}(\eta+\beta)\left[L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi;\tau)-\phi(\tau)\right]-\phi'(\tau)\gamma\tau-\frac{\phi''(\tau)}{2}\tau(1-\tau)$$

$$-\lim_{\eta\to\infty}L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi;\tau)\left[\lim_{\eta\to\infty}(\eta+\beta)\left[L^{\alpha,\beta}_{\eta,\mu,\gamma}(\zeta;\tau)-\zeta(\tau)\right]-\zeta'(\tau)\gamma\tau-\frac{\zeta''(\tau)}{2}\tau(1-\tau)\right]$$

$$+\frac{\tau(1-\tau)}{2}\zeta''(\tau)\lim_{\eta\to\infty}\eta\left[\zeta(\tau)-L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi;\tau)\right]+2\phi'(\tau)\zeta'(\tau)$$

$$+\zeta'(\tau)\gamma\tau\lim_{\eta\to\infty}(\eta+\beta)\left[\phi(\tau)-L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi;\tau)\right].$$

Considering Theorems 3.1 and 3.4 completes the proof.

4. Approximation Properties in the Space of Bögel Continuous Functions

Yang et. al. examined the multivariate context of the Stancu operators on a simplex and demonstrated that these operators preserve the Lipschitz property of the original function [15]. The operators in question are examined in their two-variable form in reference [16]. In their work [16], the authors presented Voronovskaja-type and Grüss-Voronovskaja-type theorems using the standard modulus of continuity for bivariate Stancu operators, within the context of quantitative mean.

In the bivariate scenario, the ideas that are commonly used in the univariate case are naturally extended. Let $I_{\mu q}:=I_\mu\times I_q$ and $I_\mu=[0,1+\mu], I_\nu=[0,1+\nu]$ and $I^2=I\times I, I=[0,1]$. The bivariate Stancu-Schurer operators define as follows,

$$L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi)(\tau,y)=\sum_{\rho=0}^{\eta+\mu-\gamma}\sum_{j=0}^{m+\nu-s}w_{\eta+\mu,\rho,\gamma}(\tau)w_{m+\nu,j,s}(y)\,\phi\left(\frac{\rho}{\eta},\frac{j}{m}\right)$$

where $\phi\in C(I_{\mu q}), w_{\eta+\mu,\rho,\gamma}(\tau)$ and $w_{m+\nu,j,s}(y)$ are similar in the definition (2), $p_{\eta-\gamma,\rho}(\tau), p_{m-s,l}(y)$ are similar to the previous Equation (2), γ, s are non-negative integers, $\eta, m\in N$ such that $\eta+\mu>2\gamma, m+\nu>2s$, and $\mu,\nu\in N_0$. Throughout this paper, $e_{\rho,j}(\tau,y)=\tau^\rho y^j, j\in N_0$ and $\rho+j\le 2$.

In this part, we will provide a generalization of operator (3) specifically for the B-continuous functions. To accomplish this, we will create a GBS operator that is linked to the bivariate type operators and examine some of its features related to smoothness.

Karl Bögel developed the ideas of B-continuous and B-differentiable functions in his work, as referenced in sources [17-19]. The Korovkin theorem, which is widely recognized in approximation theory, was established by Badea et al. in their works [20] and [21]. This theorem specifically focuses on B-continuous functions. The authors in [20] established a Korovkin-type theorem for approximating B-continuous functions by employing the Boolean sum technique.

The investigation in [22, 23] focused on the approximation features of the bivariate Bernstein-type operators and their associated generalized Boolean sum operators. In recent years, numerous researchers have made substantial contributions to this area. Please consult the mentioned articles for further information [24-29].

Now, we shall provide fundamental concepts and notations that will be utilized in this research. For more information, please refer to references [17, 18].

Let I and J represent closed intervals in the real numbers and let D be the Cartesian product of I and J. A function $\phi\colon \mathrm{D} \rightarrow \mathbb{R}$ is called a B-continuous (Bögel continuous) at a point $(\tau_0, y_0) \in$ D if $\lim_{(\tau,y)\rightarrow(\tau_0,y_0)} \Delta_{\tau y}\phi[\tau_0, y_0; \tau, y] = 0$, for any $(\tau, y) \in D$, with $\Delta_{\tau y}\phi[\tau_0, y_0; \tau, y] = \phi(\tau, y) - \phi(\tau, y_0) - \phi(\tau_0, y) + \phi(\tau_0, y_0)$.

The notation $C_b(Q)$ represents the set of all B-continuous functions on $Q = I \times J$ while $C_B(Q)$ represents the set of all ordinary continuous and bounded functions on Q. The GBS operator associated with the operator $L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\phi; \tau, y)$ is defined as follows:

$$GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\phi; \tau, y) \coloneqq \sum_{\rho=0}^{\eta+\mu-\gamma} \sum_{j=0}^{m+q-s} w_{n+\mu,\rho,\gamma}(\tau) w_{m+\nu,j,s}(y) \times \left[\phi\left(\frac{\rho+\alpha}{\eta+\beta}, y\right) + \phi\left(\tau, \frac{j+\vartheta}{m+\varsigma}\right) - \phi\left(\frac{\rho+\alpha}{\eta+\beta}, \frac{j+\vartheta}{m+\varsigma}\right)\right]. \tag{9}$$

The operator $GL_{n,m,\mu,q,\gamma,s}^{\alpha,\beta,\vartheta,\delta}$ is well-defined on the space $C_b(Q)$ and acts on itself. Additionally, $\phi \in C_b(Q)$. $L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\phi; \tau, y)$ is an evident linear positive operator that reproduces linear functions.

We calculate the rate at which the sequences of operators (10) converge to ϕ in the space of B-continuous functions on Q, using the modulus of continuity in the Bögel sense. Let us start by reviewing the definition of the Bögel (mixed) modulus of smoothness for a function $\phi \in C_b(Q)$. The Bögel modulus of smoothness of a function f belonging to the space $C_b(Q)$ is defined as

$$\omega_B(\phi;\delta_1,\delta_2)=\sup\left\{\left|\Delta_{(\tau,y)}\phi[t,s;\tau,y]\right|:|\tau-t|<\delta_1,|y-s|<\delta_2\right\},$$

for all $(\tau,y),(t,s)\in Q$ and for any $(\delta_{1,}\delta_2)\in(0,\infty)\times(0,\infty)$ with $\omega_B:[0,\infty)\times[0,\infty)\to R$ **,** [30]. This modulus has similar properties to the usual modulus of continuity. For example, if $\phi \in C_b(Q)$ then ϕ is uniform B-continuous on Q and

$$\lim_{\eta,m\to\infty}\omega_B(\phi;\delta_\eta,\delta_m)=0.$$

Theorem 4.1. For every $\phi\in C_b(Q)$, in each $(\tau,y)\in Q$, the operators (9) satisfy the following inequality,

$$\left|GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)-\phi(\tau,y)\right|\le 4\omega_B(\phi;\delta_\eta,\delta_m) \tag{10}$$

where $\delta_\eta=\sqrt{3\frac{b_\eta}{\eta}\tau+\left(\frac{b_\eta}{\eta}\right)^2}$, $\delta m=\sqrt{3\frac{c_m}{m}y+\left(\frac{c_m}{m}\right)^2}$.

Proof. From the definition of $\omega_B(f;\delta_\eta,\delta_m)$ and by the elementary inequality

$$\omega_B(\phi;\lambda_\eta\delta_\eta,\lambda_m\delta_m)\le(1+\lambda_\eta)(1+\lambda_m)\omega_B(\phi;\delta_\eta,\delta_m);\lambda_\eta,\lambda_m>0$$

we get,

$$\begin{aligned}\left|\Delta_{(\tau,y)}\phi[t,s;\tau,y]\right|&\le\omega_B(\phi;|\tau-t|,|y-s|)\\&\le\left(1+\frac{|\tau-t|}{\delta_\eta}\right)\left(1+\frac{|y-s|}{\delta_m}\right)\omega_B(\phi;\delta_\eta,\delta_m)\end{aligned} \tag{11}$$

for every $(\tau,y),(t,s)\in Q$ and for any $\delta_\eta,\delta_m>0$. Using the definition of $\Delta_{(\tau,y)}f[t,s;\tau,y]$, we may write,

$$\phi(\tau,s)+\phi(t,y)-\phi(t;s)=\phi(\tau;y)-\Delta_{(\tau,y)}\phi[t;s;\tau;y].$$

When we apply the $GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)$ operator to this equality we get

$$GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\phi;\tau,y)=\phi(\tau;y)L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(e_{0,0};\tau,y)-$$
$$L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\Delta_{(\tau,y)}\phi[t;s;\tau;y];\tau,y).$$

Since $L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(e_{0,0};\tau,y)=1$, considering inequality (11), using the linearity of the $D_{n,m}$ operator and using Cauchy-Schwarz inequality we have,

$$\left|GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(\phi;\tau,y)-\phi(\tau,y)\right|\le L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}\left(\left|\Delta_{(\tau,y)}\phi[t;s;\tau;y]\right|;\tau,y\right)$$
$$\le\left(L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}(e_{0,0};\tau,y)+\delta_\eta^{-1}\sqrt{L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}\left(\left(e_{1,0}-\tau\right)^2;\tau,y\right)}\right.$$
$$+\delta_m^{-1}\sqrt{L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}\left(\left(e_{0,1}-y\right)^2;\tau,y\right)}$$
$$\left.+\delta_\eta^{-1}\delta_m^{-1}\sqrt{L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}\left(\left(e_{1,0}-\tau\right)^2;\tau,y\right)L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}\left(\left(e_{0,1}-y\right)^2;\tau,y\right)}\right)\omega_B(\phi;\delta_\eta,\delta_m).$$

From Lemma 2, we can write,

$$L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}((t-\tau)^2;\tau,y)=\frac{1}{(\eta+\beta)^2}\{[-(\eta+\mu)+(\mu+\beta)^2-\gamma(\gamma-1)]\tau^2$$
$$+[\eta+\mu+2\alpha(\mu-\beta)+\gamma(\gamma-1)]\tau+\alpha^2\}\le\frac{\mu(\mu+4\beta)}{\eta+\beta}=\Phi_{\eta,\gamma,\mu}^{\alpha,\beta,*},$$

and

$$L_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}((l-y)^2;\tau,y)\le\frac{\nu(\nu+4\varsigma)}{m+\varsigma}=\Phi_{m,s,\nu}^{\gamma,\varsigma,*}.$$

We obtain,

$$\left|GL_{\eta,m,\mu,\nu,r,s}^{\alpha,\beta,\vartheta,\delta}(\phi;\tau,y)-\phi(\tau,y)\right|\le\left(1+\delta_\eta^{-1}\sqrt{\frac{\mu(\mu+4\beta)}{\eta+\beta}}\right.$$
$$\left.+\delta_m^{-1}\sqrt{\frac{\nu(\nu+4\varsigma)}{m+\varsigma}}+\delta_\eta^{-1}\delta_m^{-1}\sqrt{\frac{\mu(\mu+4\beta)}{\eta+\beta}\frac{\nu(\nu+4\varsigma)}{m+\varsigma}}\right)\omega_B(\phi;\delta_\eta,\delta_m)$$

which gives the assertion of Theorem the desired inequality (9) by choosing $\delta_\eta=\sqrt{\Phi_{\eta,\gamma,\mu}^{\alpha,\beta,*}}$ and $\delta_m=\sqrt{\Phi_{m,s,\nu}^{\vartheta,\varsigma,*}}$.

Now, we study the degree of approximation for the operators $GL^{\alpha,\beta,\gamma,\delta}_{\eta,m,\mu,q,r,s}(\phi;\tau,y)$ by means of the Lipschitz class for B-continuous functions. For $f\in C_b(Q)$, we define the Lipschitz class $Lip_M(\lambda,\upsilon)$ with $\lambda,\upsilon\in(0,1]$ as follows:

$$Lip_M(\lambda,\mu)=\left\{\phi\in C_b(Q):\left|\Delta_{(\tau,y)}\phi[t,s;\tau,y]\right|\le M|t-\tau|^{\lambda}|s-y|^{\upsilon}\right\}$$

for $(t,s),(\tau,y)\in Q$, $M>0$.

Theorem 4.2. Let $\phi\in Lip_M(\lambda,\mu)$, then we have,

$$\left|GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)-\phi(\tau,y)\right|\le M\delta_\eta^{\lambda/2}\delta_m^{\upsilon/2},$$

where δ_η,δ_m are as given in the proof of the previous theorem and $\lambda,\upsilon\in(0,1]$, $(\tau,y)\in Q$.

Proof. By the definition of $GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}$ operator and by linearity of the $L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}$ operator, we may write,

$$GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)=L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi(\tau,s)+\phi(t,y)-\phi(t;s);\tau,y)$$

$$=L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(\phi(\tau;y)-\Delta_{(\tau,y)}\phi[t;s;\tau;y];\tau,y\right)=\phi(\tau;y)L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(e_{0,0};\tau,y\right)$$

$$-L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(\Delta_{(\tau,y)}\phi[t;s;\tau;y];\tau,y\right).$$

And, by using the hypothesis, we have,

$$\begin{aligned}\left|GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)-\phi(\tau,y)\right|&\le L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(\left|\Delta_{(\tau,y)}\phi[t;s;\tau;y]\right|;x,y\right)\\&\le ML^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(|t-\tau|^{\lambda}|s-y|^{\upsilon};\tau,y\right)\\&=ML^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}\left(|t-\tau|^{\lambda};\tau,y\right)L^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(|s-y|^{\upsilon};\tau,y).\end{aligned}$$

Applying the Hölder's inequality with $p_1=2/\lambda, q_1=2/(2-\lambda)$ and $p_2=2/\upsilon, q_1=2/(2-\upsilon)$, we get

$$\left|GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}(\phi;\tau,y)-\phi(\tau,y)\right|\le MD_n(|t-\tau|^2;\tau)^{\frac{\lambda}{2}}D_n(|s-y|^2;y)^{\frac{\upsilon}{2}}\le M\delta_\eta^{\frac{\lambda}{2}}\delta_m^{\frac{\upsilon}{2}}.$$

5. Graphical Examples

Here, we examine the theoretical findings from the preceding parts via the use of graphical illustrations and a numerical demonstration.

Example 5.1. For the changing values of η as $\eta = 20$(red),$\eta = 50$(green), $\eta = 100$(blue), the convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \sin(2\pi\tau)$ (black) is illustrated as the first example in the Figure 1.

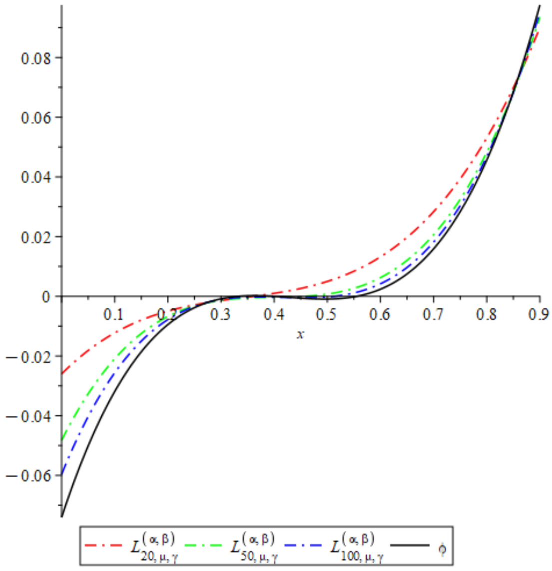

Figure 1: The convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \sin(2\pi\tau)$ for $\eta = 20,50,100$.

Example 5.2. For the changing values of η as $\eta = 20$(red), $\eta = 50$(green), $\eta = 100$(blue), the convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \left(\tau - \frac{1}{3}\right)\sin(\pi\tau)$ (black) is illustrated in the coming figure.

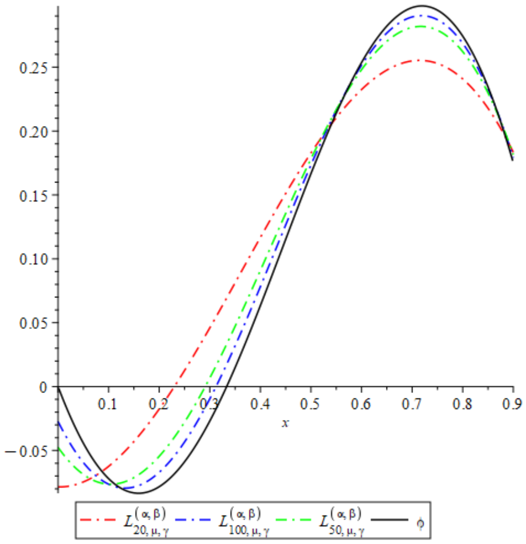

Figure 2: The convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \left(\tau - \frac{1}{3}\right)\sin(\pi\tau)$ (black) is illustrated for $\eta = 20,50,100$.

Example 5.3. For the changing values of η as $\eta = 20$(red), $\eta = 50$(green), $\eta = 100$(blue), the convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \cos(2\pi\tau) + 2\sin(\pi\tau)$ (black) is illustrated in the figure showing that the expression leans toward both sides as η increases.

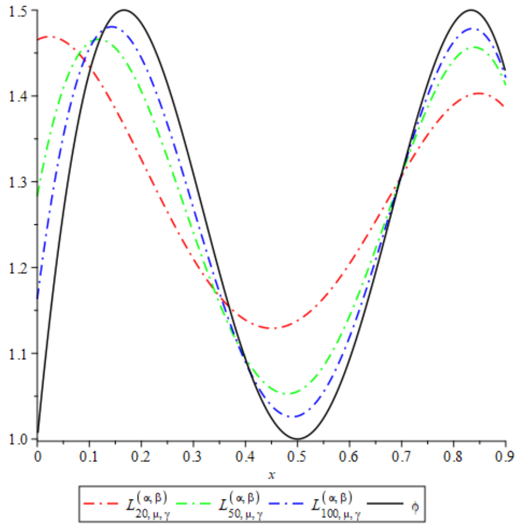

Figure 3: The convergence of $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ to $\phi(\tau) = \cos(2\pi\tau) + 2\sin(\pi\tau)$ (black) for $\eta = 20,50,100$.

In the next table, the error estimation of the $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ operators are given in comparison with the $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ operators. The absolute value of the difference of both operators with the $\phi(\tau) = \tan(\tau^2 + 2\tau + 10)$ function is given according to the increasing η values and the value of τ. Accordingly, as η values increase in multiples of 10, the approximation of the $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ operators to the $\phi(\tau)$ function is better than the $L_{\eta,\mu,\gamma}^{\alpha,\beta}(\phi,\tau)$ operators.

Table 1: Error estimation for operators $L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)$.

	$\boldsymbol{\tau = 0.5}$	
$\boldsymbol{\eta}$	$\boldsymbol{\left\lvert L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)-\phi(\tau)\right\rvert}$	$\boldsymbol{\left\lvert L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)-\phi(\tau)\right\rvert}$
10	4.918308847	4.030402557
10^2	0.022928159	0.091543553
10^3	0.120741290	0.095097202
10^4	0.010993630	0.008609165
10^5	0.001090463	0.000853504

Finally, here we visualized the approach of the new type operators $L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)$ for the changing values of τ where $\alpha = 3, \beta = 5, \gamma = 1$ values are taken.

Example 5.4. Visualization of the convergence of $L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)$ with varying values of τ for the value $\eta = 80$, to $\phi(\tau) = \frac{\tau^2-1}{(\tau-1)^2}$ (black) is illustrated in the figure. The μ values here are chosen as $1,5,10,15$ respectively and the colors in which each value is represented can be seen in the figures' legend.

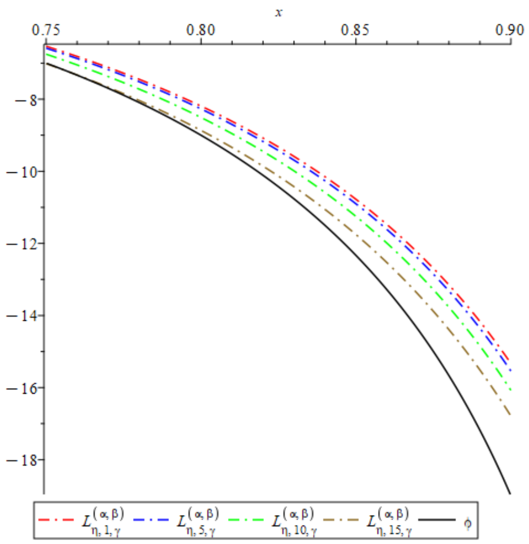

Figure 4: The convergence of $L^{\alpha,\beta}_{\eta,\mu,\gamma}(\phi,\tau)$operators to $\phi(\tau) = (\tau^2-1)/(\tau-1)^2$ (black) for changing μ values.

Now we give some graphics comparing the approximation of the bivariate and GBS operators of corresponding operators.

Example 5.5. Visualization of the convergence of $GL^{\alpha,\beta,\vartheta,\varsigma}_{\eta,m,\mu,\nu,\gamma,s}$ to function $\phi(\tau,y) = sin(4\pi\tau) + sin(\pi y)$. Here ϕ function is blue colored and GBS operators are yellow $for\ \eta, m =$

2 and red for changing $\eta, m = 5$ values. A better approximation is observed as η, m values increase.

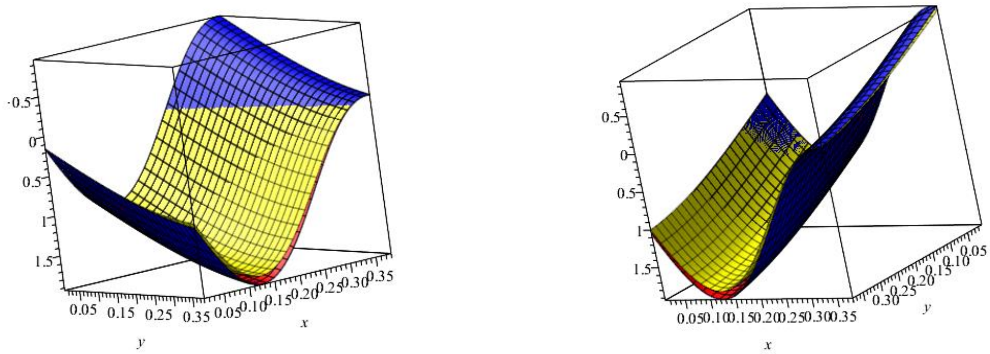

Figure 5: The convergence of $GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}$ operators to $\phi(\tau, y) = 2\tau^2 + 3y^2$ (blue) for changing η, m values.

Example 5.6. Visualization of the convergence of $GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}$ to function $\phi(\tau, y) = 2\tau^2 + 3y^2$. Here function is blue colored and GBS operators are yellow one, so as red is present the bivariate operators. It is clear that GBS operators have a better approach.

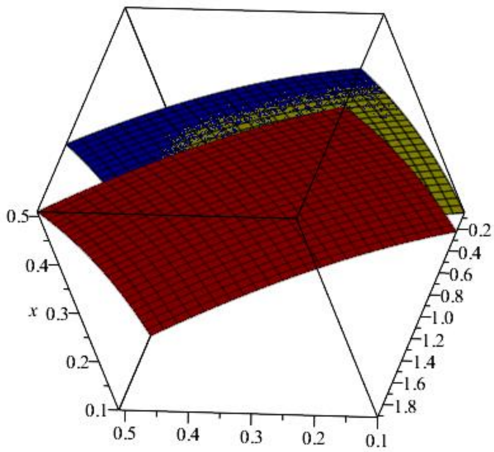

Figure 6: The convergence of $GL_{\eta,m,\mu,\nu,\gamma,s}^{\alpha,\beta,\vartheta,\varsigma}$ operators and bivariate operators to $\phi(\tau, y) = 2\tau^2 + 3y^2$ (blue) for GBS and bivariate operators.

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