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RESEARCH ARTICLE

A COUNTEREXAMPLE TO ELAYDI'S CONJECTURE

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Abstract

In this work, we define a chaotic map that contradicts Elaydi's conjecture. Firstly, we present some important concepts used in this paper and define a continuous map f on $[0,2]$, which is connected according to the usual topology on $\mathbb{R}$. Moreover, we show that f is chaotic on $[0,2]$ by using topological conjugacy with the 'tent map'. Finally, we conclude that $f^2 = f \circ f$ is not chaotic on $[0,2]$. In addition, this example also shows that topological transitivity does not imply total transitivity.

Keywords

Chaos,
Topologically Transitive,
Totally Transitive,
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1. INTRODUCTION

Chaotic dynamical systems are often used in image encryption, cryptology, fractal geometry, etc. [1-3]. Devaney presented the definition of a chaotic map, which is widely used in mathematics [4]. The definition of a chaotic map consists of three conditions: topological transitivity, the density of the set of periodic points, and sensitive dependence on initial conditions. Topological transitivity indicates that the system exhibits complex behavior, while the density of the set of periodic points suggests that the system exhibits regular behavior. Additionally, sensitive dependence on initial conditions indicates that the system is unpredictable.

Many researchers examined the relations between chaos conditions [5-8]. Banks and his colleagues showed that topological transitivity and the density of the set of periodic points imply sensitive dependence on initial conditions in a non-finite metric space with a continuous map [5]. Vellekoop and Berglund showed that topological transitivity is sufficient for chaos on intervals [6]. Değirmenci and Koçak investigated the relationship between topological transitivity and dense orbit [7]. Chaos conditions were adapted to product spaces by Değirmenci and Koçak [8]. The question may arise whether, for a metric space X and a chaotic map $f: X \to X$, the map f^m, which is the composition of f with itself m times for all $m \in \mathbb{Z}^+$, is also chaotic. In [9] (p. 143), Elaydi put forward the following claim: "Let $f: X \to X$ be a continuous map on a metric space X (an interval I) which is chaotic. Show that if X is connected, then f^m is chaotic for all $m \in \mathbb{Z}^+$."

The aim of this paper is to define a chaotic map that contradicts Elaydi's conjecture. Firstly, we present some important concepts used in this paper and define a continuous map f on $[0,2]$, which is connected

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according to the usual topology on $\mathbb{R}$. Moreover, we show that f is chaotic on $[0,2]$ by using topological conjugacy with the 'tent map'. Finally, we conclude that $f^2 = f \circ f$ is not chaotic on $[0,2]$. In addition, this example also shows that topological transitivity does not imply total transitivity.

2. PRELIMINARIES

Let X be a topological space and $f: X \to X$, then (X, f) is called *discrete dynamical system* [4]. For any $a \in X$, the set $Orb_f(a) = \{a, f(a), f^2(a), f^3(a), \dots, f^m(a), \dots\}$ is called *orbit* of a under f [4]. A point $p \in X$ is called a *periodic point* of f if there is a positive integer m such that $f^m(p) = p$ [4]. The smallest such m is called *prime period* of p [4]. The set $Per(f) = \{p \in X : f^m(p) = p,\ m \in \mathbb{Z}^+\}$ is called the *set of periodic points of* f [4].

Definition 2.1. ([4]) Let (X, f) be a discrete dynamical system. f is called *topologically transitive* if for every non-empty open subsets $U, V \subset X$, there exists a $m \in \mathbb{Z}^+$ such that $f^m(U) \cap V \neq \emptyset$.

Definition 2.2. ([10]) Let (X, f) be a discrete dynamical system. f is called *totally transitive* if $f^m: X \to X$ is topologically transitive for all $m \in \mathbb{Z}^+$.

From the definition of total transitivity, it is easy to see that every totally transitive map is topologically transitive. However, as we will see, the reverse is not always true.

Definition 2.3. ([4]) Let (X, f) be a discrete dynamical system where X is a metric space. f is called s*ensitive dependent on initial conditions* if for $r > 0$ following hold, for all $x \in X$ and for every open neighborhood U of x, there exists a $m \in \mathbb{Z}^+$ and $y \in U$ such that, $d\big(f^m(x), f^m(y)\big) \geq r$.

Definition 2.4. ([4]**)** Let (X, f) be a discrete dynamical system, where X is a metric space. The map f is called *chaotic* (sense of Devaney) if f is topologically transitive, $Per(f)$ is dense in X, and f is sensitive dependent on initial conditions.

Theorem 2.5. ([5]**)** Let (X, f) be a discrete dynamical system, where X is a non-finite metric space and f is continuous. If f is topologically transitive and $Per(f)$ is dense in X, then f is sensitive dependent on initial conditions, i.e., f is a chaotic map.

Theorem 2.6. ([6]**)** Let (I, f) be a discrete dynamical system, where $I \subset \mathbb{R}$ is an interval and f is continuous. If f is topologically transitive, then $Per(f)$ is dense in I, i.e., f is a chaotic map.

The notion of topological conjugacy is used for the equivalence of the dynamics of the maps.

Definition 2.7. ([4]) Let (X, f) and (Y, g) be two discrete dynamical systems. If a homeomorphism $h: X \to Y$ exists such that $h \circ f = g \circ h$, then f and g are said to be *topologically conjugate* maps, and (X, f) and (Y, g) are said to be *topologically equivalent dynamical systems.*

Theorem 2.8. ([4,10]) Let (X, f) and (Y, g) be topologically equivalent dynamical systems, where X, Y are non-finite metric spaces and f, g are continuous maps. Then,
(i) $Per(f)$ is dense in X iff $Per(g)$ is dense in Y.
(ii) f is topologically transitive iff g is topologically transitive.
(iii) f is totally transitive iff g is totally transitive.
(iv) f is chaotic on X iff g is chaotic on Y.

3. COUNTEREXAMPLE

We define $f: [0,2] \to [0,2]$ by

$$f(x) = \begin{cases} 2x+1, & 0 \le x \le \frac{1}{2} \\ -2x+3, & \frac{1}{2} \le x \le \frac{3}{2} \\ 2x-3, & \frac{3}{2} \le x \le 2. \end{cases} \tag{1}$$

To show f is a chaotic map, it is sufficient to show that f is topologically transitive by Theorem 2.6, because f is defined on an interval and f is continuous. Graphs of f and f^2 are shown in Figure 1.

(a) **(b)**

Figure 1. (a) Graph of f; (b) Graph of f^2.

We use the famous 'tent map' and a map which is topologically conjugate to it to achieve our aim. The tent map is defined by $T: [0,1] \to [0,1]$,

$$T(x) = \begin{cases} 2x \quad , & 0 \le x \le \frac{1}{2} \\ 2-2x\, , & \frac{1}{2} \le x \le 1. \end{cases} \tag{2}$$

T is a well-known chaotic map in the theory of chaotic dynamical systems [4,10]. In addition, T is a totally transitive map, i.e., T^m is topologically transitive for all $m \in \mathbb{Z}^+$ [10]. We will use transitivity of T and T^2 (see Figure 2).

(a) **(b)**

Figure 2. (a) Graph of T; (b) Graph of T^2.

Define $R: [0,1] \to [0,1]$ by

$$R(x) = \begin{cases} -2x+1\, , & 0 \le x \le \frac{1}{2} \\ 2x-1\, , & \frac{1}{2} \le x \le 1. \end{cases} \tag{3}$$

Example 3.1. The tent map T defined in (2) is topologically conjugate to the map R defined in (3) via homeomorphism $h: [0,1] \to [0,1]$, $h(x) = 1 - x$. Let $x \in \left[0, \frac{1}{2}\right]$, then $(h \circ R)(x) = 2x = (T \circ h)(x)$. If $x \in \left[\frac{1}{2}, 1\right]$, then $(h \circ R)(x) = 2 - 2x = (T \circ h)(x)$. Hence, $(h \circ R)(x) = (T \circ h)(x)$ for all $x \in [0,1]$. Finally, T and R are topologically conjugate maps.

Result 3.2 According to Theorem 2.8, the dynamical behavior of T and R are same. Since T is chaotic and totally transitive, R is also chaotic and totally transitive.

(a) **(b)**

Figure 3. (a) Graph of R; (b) Graph of R^2.

Example 3.3. The map f defined in (1) is chaotic on $[0,2]$. By Theorem 2.6, it is sufficient to show that f is topologically transitive. Consider the restricted maps of f^2 as $g_1 = f^2|_{[0,1]}$ and $g_2 = f^2|_{[1,2]}$. Note that, the map g_1 is equal to R^2 (see Figure 1 (b) and Figure 3 (b)). Since R is totally transitive, by Result 3.2, R^2 is topologically transitive. Hence, g_1 is topologically transitive. We show that g_2 and T^2 are topologically conjugate via homeomorphism $k: [0,1] \to [1,2]$, $k(x) = x + 1$. Explicit forms of $g_2: [1,2] \to [1,2]$ and $T^2: [0,1] \to [0,1]$ are

$$g_2(x) = \begin{cases} 4x - 3, & 1 \le x \le \frac{5}{4} \\ -4x + 7, & \frac{5}{4} \le x \le \frac{3}{2} \\ 4x - 5, & \frac{3}{2} \le x \le \frac{7}{4} \\ -4x + 9, & \frac{7}{4} \le x \le 2 \end{cases}$$

and

$$T^2(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ -4x + 2, & \frac{1}{4} \le x \le \frac{1}{2} \\ 4x - 2, & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x + 4, & \frac{3}{4} \le x \le 1 \end{cases}$$

respectively. We will show that $(k \circ T^2)(x) = (g_2 \circ k)(x)$ for all $x \in [0,1]$. Let $x \in [0,1]$.

If $x \in \left[0, \frac{1}{4}\right]$, then $(k \circ T^2)(x) = 4x + 1$. Since $x + 1 \in [1, \frac{5}{4}]$, $(g_2 \circ k)(x) = g_2\,(x + 1) = 4x + 1$.

If $x \in [\frac{1}{4}, \frac{1}{2}]$, then $(k \circ T^2)(x) = -4x + 3$. Since $x + 1 \in [\frac{5}{4}, \frac{3}{2}]$, $(g_2 \circ k)(x) = g_2(x + 1) = -4x + 3$.

If $x \in [\frac{1}{2}, \frac{3}{4}]$, then $(k \circ T^2)(x) = 4x - 1$. Since $x + 1 \in [\frac{3}{2}, \frac{7}{4}]$, $(g_2 \circ k)(x) = g_2(x + 1) = 4x - 1$.

If $x \in [\frac{3}{4}, 1]$, then $(k \circ T^2)(x) = -4x + 5$. Since $+1 \in \left[\frac{7}{4}, 2\right]$, $(g_2 \circ k)(x) = g_2(x + 1) = -4x + 5$.

Hence, $(k \circ T^2)(x) = (g_2 \circ k)(x)$ for all $x \in [0,1]$, i.e., T^2 and g_2 topologically conjugate maps. Since T is totally transitive, T^2 is topologically transitive. By Theorem 2.8, g_2 is topologically transitive. Let U and V be non-empty open subsets of $[0,2]$. We will show that f is topologically transitive, i.e., we obtain a $m \in \mathbb{Z}^+$ such that $f^m(U) \cap V \neq \emptyset$. We investigate five cases.

Case 1: If $U, V \subset [0,1]$, since g_1 is topologically transitive, there exists a $n \in \mathbb{Z}^+$ such that $g_1^n(U) \cap V \neq \emptyset$. Hence, $f^{2n}(U) \cap V \neq \emptyset$.

Case 2: If $U, V \subset [1,2]$, since g_2 is topologically transitive, there exists a $k \in \mathbb{Z}^+$ such that $g_2^k(U) \cap V \neq \emptyset$. Hence, $f^{2k}(U) \cap V \neq \emptyset$.

Case 3: Let $\subset [0,1], V \subset [1,2]$. Since f is continuous, $f^{-1}(V) \subset [0,1]$ is an open set. By Case 1, there exists a $n \in \mathbb{Z}^+$ such that $f^{2n}(U) \cap f^{-1}(V) \neq \emptyset$. If $x \in f^{2n}(U) \cap f^{-1}(V)$, then $x \in f^{2n}(U)$ and $x \in f^{-1}(V)$. Therefore, $f(x) \in f^{2n+1}(U)$ and $f(x) \in V$ (since f is onto, $f\left(f^{-1}(V)\right) = V$). Hence, $f^{2n+1}(U) \cap V \neq \emptyset$.

Case 4: Let $\subset [1,2], V \subset [0,1]$. Since f is continuous, $f^{-1}(V) \subset [1,2]$ is an open set. By Case 2, there exists a $k \in \mathbb{Z}^+$ such that $f^{2k}(U) \cap f^{-1}(V) \neq \emptyset$. If $x \in f^{2k}(U) \cap f^{-1}(V)$, then $x \in f^{2k}(U)$ and $x \in f^{-1}(V)$. Therefore, $f(x) \in f^{2k+1}(U)$ and $f(x) \in V$ (since f is onto, $f\left(f^{-1}(V)\right) = V$). Hence, $f^{2k+1}(U) \cap V \neq \emptyset$.

Case 5: If U or V are open sets containing 1, the desired result can be similarly obtained from Case 1,2.

Therefore, f is topologically transitive. Consequently, by Theorem 2.6, f is chaotic on $[0,2]$.

Result 3.4. f^2 is not a chaotic map. Let $U \subset [0,1]$ and $V \subset [1,2]$ be non-empty open subsets. Since $(f^2)^m(U) \subset [0,1]$ for all $m \in \mathbb{Z}^+$ (see Figure 1 (b)), $(f^2)^m(U) \cap V = \emptyset$. Therefore, f^2 is not topologically transitive, i.e., by Definition 2.4, f^2 is not chaotic. Although we construct a chaotic and continuous map f on $[0,2]$, which is connected, f^m is not chaotic on $[0,2]$ for $m = 2$. Hence, this situation contradicts Elaydi's conjecture in [9] (p. 143). Moreover, f is an example of a map that is topologically transitive but not totally transitive.

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CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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