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Research Article

The relationship between modular metrics and fuzzy metrics revisited

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ABSTRACT. In a famous article published in 1975, Kramosil and Michálek introduced a notion of fuzzy metric that was the origin of numerous researches and publications in several frameworks and fields. In 2010, Chistyakov introduced and discussed in detail the concept of modular metric. Since then, some authors have investigated the problem of establishing connections between the notions of fuzzy metric and modular metric, obtaining positive partial solutions. In this paper, we are interested in determining the precise relationship between these two concepts. To achieve this goal, we examine a proof, based on the use of uniformities, of the important result that the topology induced by a fuzzy metric is metrizable. As a consequence of that analysis, we introduce the notion of a weak fuzzy metric and show that every weak fuzzy metric, with continuous t-norm the minimum t-norm, generates a modular metric and, conversely, we show that every modular metric generates a weak fuzzy metric, with continuous t-norm the product t-norm. It follows that every modular metric can be generated from a suitable weak fuzzy metric, and that several examples and properties of modular metrics can be directly deduced from those previously obtained in the field of fuzzy metrics.

Keywords: fuzzy metric, weak fuzzy metric, modular metric

2020 Mathematics Subject Classification: 54E35, 54A40.

1. INTRODUCTION

With the aim in offering a fuzzy approach of statistical metric spaces and Menger spaces, Kramosil and Michálek introduced in [13] the fruitful notions of fuzzy metric and fuzzy metric space. Modifications of these concepts were proposed by Grabiec [8], and George and Veeramani [6]. These structures have been extensively explored both from the point of view of their topological and metric properties, as well as the development of a fixed point theory for them, and their application to various fields. There are obviously numerous relevant publications on fuzzy metric spaces and related structures. In order not to make the reference section too long, we will limit ourselves to cite the books [3, 10] and the very recent [7] joint with the references therein.

On the other hand and partly motivated by the studies about modulars on vector spaces ([14, 15, 16, 17]) Chistyakov introduced and discussed in [4] (see also [5]) the concepts of modular metric and modular metric space. Looking at Chistyakov's definition, it can be well intuited that there is a strong connection between modular metrics and fuzzy metrics. In fact, some authors have explored such a connection when working in the construction of a fixed point theory for modular metric spaces, obtaining various positive partial solutions (see, e.g., [11, 20]). In this paper, we are interested in determining the precise relationship between these

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two concepts. To achieve this goal, we examine a proof, based on the use of uniformities, of the important result that the topology induced by a fuzzy metric is metrizable. As a consequence of that analysis, we introduce the notion of a weak fuzzy metric and show that every weak fuzzy metric, with continuous t-norm the minimum t-norm, generates a modular metric such that the induced topologies agree; and, conversely, we show that every modular metric generates a weak fuzzy metric, with continuous t-norm the product t-norm, such that the induced topologies agree. It follows that every modular metric can be generated from a suitable weak fuzzy metric, and that several examples and properties of modular metrics can be directly deduced from those previously obtained in the field of fuzzy metrics.

2. Remarks on the notion of fuzzy metric

First, we emphasize that our notation and terminology will be standard. By $\mathbb{R}^+$, we design the set of non-negative real numbers and by $\mathbb{N}$ the set of natural numbers.

Now, we recall the notions of fuzzy metric and fuzzy metric space in the aforementioned senses. To this end, the following well-known concept will play a fundamental role.

Definition 2.1. *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous triangular norm (continuous t-norm in short) if $([0,1], *)$ is a topological Abelian monoid with neutral 1 such that $a * c \leq b * c$ if $a \leq b$, with $a, b, c \in [0,1]$.*

As distinguished examples of continuous t-norm that will be used throughout this paper, we have the minimum t-norm $\wedge$, the product t-norm $*_P$ and the Łukasiewicz t-norm $*_L$, which are defined as follows: $a \wedge b = \min\{a, b\}$, $a *_P b = ab$, and $a *_L b = \max\{a + b - 1, 0\}$, for all $a, b \in [0,1]$. Recall that $\wedge \geq *_P \geq *_L$. In fact, $\wedge \geq *$ for any continuous t-norm $*$.

The books [10, 12] provide suitable sources to the study of continuous t-norms.

Now, consider the following axioms for a set X, a fuzzy set M in $X \times X \times \mathbb{R}^+$, and $x, y, z \in X$:

(KM1) $M(x, y, 0) = 0$;
(GV1) $M(x, y, t) > 0$ for all $t > 0$;
(KM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(GV2) $M(x, x, t) = 1$ for all $t > 0$, and $M(x, y, t) < 1$ whenever $y \neq x$;
(KM3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;
(KM5) the function $t \to M(x, y, t)$ is left continuous on $\mathbb{R}^+$;
(GV5) the function $t \to M(x, y, t)$ is continuous on $\mathbb{R}^+$;
(KM6) $\lim_{t\to\infty} M(x, y, t) = 1$.

The triple $(X, M, *)$ is a fuzzy metric space in the sense of Kramosil and Michálek [13] provided that axioms (KM1), (KM2), (KM3), (KM4), (KM5) and (KM6) are fulfilled. In that case, we will say that the pair $(M, *)$, or simply M, is a KM-fuzzy metric.

In [8], Grabiec removed axiom (KM6) in the definition of fuzzy metric space because it was not necessary for his research about fixed point theory on fuzzy metric spaces. Then, a fuzzy metric $(M, *)$, or simply M, in Grabiec's sense will be called a Gr-fuzzy metric.

Later, George and Veeramani [6] defined a fuzzy metric space as a triple $(X, M, *)$, where X is a set, M is a fuzzy set in $X \times X \times (0, \infty)$ and $*$ is a continuous t-norm such that axioms (GV1), (GV2), (KM3), (KM4) and (GV5) are fulfilled. In that case, we will say that the pair $(M, *)$, or simply M, is a GV-fuzzy metric.

Note that every GV-fuzzy metric $(M, *)$ can be considered as a Gr-fuzzy metric simply defining $M(x, y, 0) = 0$ for all $x, y \in X$.

In [8, Lemma 4], Grabiec stated assertion (1) of Lemma 2.1 below in the framework of Gr-fuzzy metrics.

Lemma 2.1. *Let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2) and (KM4) are fulfilled. Then, for each $x, y \in X$, we get*

(1) *The function $t \to M(x,y,t)$ is nondecreasing on $\mathbb{R}^+$,*
(2) *$x = y$ if and only if $M(x,y,t) \geq 1 - t$ for all $t > 0$.*

Proof.

(1) Fix $x, y \in X$. Let $s, t \in \mathbb{R}^+$ such that $0 \leq s < t$. If $s = 0$, we get $M(x,y,s) = 0$ by (KM1). If $s > 0$ we get $M(x,y,t) \geq M(x,x,t-s) * M(x,y,s) = M(x,y,s)$ by (KM2) and (KM4).
(2) Suppose that $x = y$. Then, $M(x,y,t) \geq 1 - t$ for all $t > 0$ by (KM2). Conversely, suppose that $x \neq y$. By (KM2), there are $t > 0$ and $\delta \in (0,1)$ such that $M(x,y,t) < 1 - \delta$. It follows from hypothesis and assertion (1) that, for any $s \in (0,t)$, $1 - s \leq M(x,y,s) \leq M(x,y,t) < 1 - \delta$, a contradiction.

□

Every Gr-fuzzy metric M on a set X induces in a natural way a topology $\mathfrak{T}_M$ on X. We show that axioms (KM1), (KM2) and (KM4) are sufficient to construct such a topology.

Lemma 2.2. *Let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2) and (KM4) are fulfilled. For each $x \in X, \varepsilon \in (0,1)$ and $t > 0$, set $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1 - \varepsilon\}$. Then, the family*

$$\mathfrak{T}_M = \{A \subseteq X : \text{ for each } x \in A \text{ there is } \varepsilon \in (0,1) \text{ and } t > 0 \text{ such that } B_M(x,\varepsilon,t) \subseteq A\},$$

is a topology on X.

Proof. It is obvious that the union of any family of members of $\mathfrak{T}_M$ belongs to $\mathfrak{T}_M$. Now, let $A_1, ..., A_n$, be a finite family of members of $\mathfrak{T}_M$. Let $x \in \bigcap_{k=1}^n A_k$. For each $k \in \{1,...,n\}$ there is $\varepsilon_k \in (0,1)$ and $t_k > 0$ such that $B_M(x,\varepsilon_k,t_k) \subseteq A_k$. Put $\varepsilon = \min\{\varepsilon_k : k \in \{1,...,n\}\}$ and $t = \min\{t_k : k \in \{1,...,n\}\}$. It follows from Lemma 2.1 that $B_M(x,\varepsilon,t) \subseteq B_M(x,\varepsilon_k,t_k)$ for all $k \in \{1,...,n\}$. Therefore, $B_M(x,\varepsilon,t) \subseteq \bigcap_{k=1}^n A_k$. We conclude that $\mathfrak{T}_M$ is a topology on X. □

Remark 2.1. *It is well known that the topology $\mathfrak{T}_M$ is metrizable, i.e., there is a metric on X such that its induced topology agrees with $\mathfrak{T}_M$. Next, we present an outline of a proof of this fundamental result based on the construction of a suitable uniformity (see, e.g., [9, 19]) and emphasizing about those axioms that are really used. The conclusions derived from this examination will be key later on.*

Indeed, let X be a set, M be a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ be a continuous t-norm for which axioms (KM1), (KM2), (KM3) and (KM4) are fulfilled. For each $n \in \mathbb{N}$ put

$$U_n = \{(x,y) \in X \times X : M(x,y,1/n) > 1 - 1/n\}.$$

Then, we obtain

- *$\{(x,x) : x \in X\} = \bigcap_{n=1}^\infty U_n$ by (KM1), (KM2) and (KM4) (Lemma 2.1),*
- *$U_n = U_n^{-1}$ for all $n \in \mathbb{N}$ by (KM3),*
- *for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $U_m^2 \subseteq U_n$ by continuity of $*$ and (KM4).*

Therefore, the family $\{U_n : n \in \mathbb{N}\}$ is a (countable) base for a uniformity on X whose induced topology agrees with $\mathfrak{T}_M$, which implies that $\mathfrak{T}_M$ is a metrizable topology on X.

Next, we remind some typical and well-known examples of KM, Gr and GV fuzzy metric spaces. In all cases we will assume, without explicit mention, that axiom (KM1) is satisfied.

Example 2.1. *Let (X,d) be a metric space and $*$ be a continuous t-norm. Let $M_d : X \times X \times \mathbb{R}^+ \to [0,1]$ given by*

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $x, y \in X$ and $t > 0$. Then, the pair $(M_d, *)$ is a GV-fuzzy metric on X, whose induced topology coincides with the topology induced by d.

Example 2.2. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{01} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by $M_{01}(x, y, t) = 1$ if $d(x, y) < t$, and $M_{01}(x, y, t) = 0$ if $d(x, y) \geq t$, for all $x, y \in X$ and $t > 0$. Then, the pair $(M_{01}, *)$ is a KM-fuzzy metric on X whose induced topology coincides with the topology induced by d. Clearly, $(M_{01}, *)$ is not a GV-fuzzy metric whenever $|X| \geq 2$ because axiom (GV1) is not satisfied.*

Example 2.3. *Let $X = [0, 1]$ and let $*$ be a continuous t-norm. Then, the pair $(M_*, *)$ is a Gr-fuzzy metric on X, where $M_*(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) = x * y$ for all $x, y \in X$ with $x \neq y$, and $t > 0$. Clearly, $(M_*, *)$ is not a KM-fuzzy metric because axiom (KM6) is not satisfied. Moreover, it is not a GV-fuzzy metric because $M_*(0, y, t) = 0$ whenever $y \neq 0$.*

Example 2.4. *Let (X, d) be a metric space such that $d(x, y) \leq 1$ for all $x, y \in X$. Then, the pair $(M_1, *_L)$ is a Gr-fuzzy metric on X, where $M_1(x, y, t) = 1 - d(x, y)$ for all $x, y \in X$ and $t > 0$. Moreover, the topologies induced by $(M_1, *_L)$ and d coincide. Note also that if there are $x, y \in X$ such that $d(x, y) = 1$, then $(M_1, *_L)$ is not a GV-fuzzy metric because axiom (GV1) is not be satisfied. Moreover, $(M_1, *_L)$ is not a KM-fuzzy metric whenever $|X| \geq 2$ because axiom (KM6) is not be satisfied.*

Remark 2.2. *It is well known (see, e.g.,* [6, Result 3.2]*) that for every Gr-fuzzy metric M on a set X the balls $B_M(x, \varepsilon, t)$ are $\mathfrak{T}_M$-open sets. To show it axiom (KM5) is essential. Fortunately, this result will not be relevant for our targets.*

By virtue of Remark 2.1, we propose the following notion.

Definition 2.2. *A weak fuzzy metric space is a triple $(X, M, *)$, where X is a set, M is a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ is a continuous t-norm for which axioms (KM1), (KM2), (KM3) and (KM4) are fulfilled. In this case, we say that the pair $(M, *)$, or simply M, is a weak fuzzy metric on X.*

Remark 2.3. *Let $(X, M, *)$ be a weak fuzzy metric space. It follows from Remark 2.1 that a sequence $(x_n)_n$ in X is $\mathfrak{T}_M$-convergent to a $x \in X$ if and only if, for each $t > 0$, $\lim_{n\to\infty} M(x, x_n, t) = 1$.*

Remark 2.4. *Recall that Kramosil and Michálek added axioms (KM5) and (KM6) in their definition with the aim of having that, for each $x, y \in X$, the function $t \to M(x, y, t)$ be a (generalized) distribution function, as occurs for statistical metric spaces. Note also that axiom (KM6) is crucial in the realm of fuzzy normed spaces, concretely, to show that every fuzzy normed space is a topological vector space (see* [1, 3, 18]*).*

We conclude this section with three examples of weak fuzzy metrics that are not Gr-fuzzy metric, obtained by suitable modifications in Examples 2.1, 2.2 and 2.3, respectively.

Example 2.5. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{d,w} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by*

$$M_{d,w}(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and $t \in (0, 1)$, and $M_{d,w}(x, y, t) = 1$ for all $x, y \in X$ and $t \geq 1$. Then, the pair $(M_{d,w}, *)$ is a weak fuzzy metric on X whose induced topology coincides with the topology induced by d. Note that $(M_{d,w}, *)$ is not a Gr-fuzzy metric if $|X| \geq 2$ because, for $x \neq y$, the function $t \to M_{d,w}(x, y, t)$ is not left continuous at $t = 1$.

Example 2.6. *Let (X, d) be a metric space and $*$ be a continuous t-norm. Let $M_{01,w} : X \times X \times \mathbb{R}^+ \to [0, 1]$ given by $M_{01,w}(x, y, t) = 1$ if $d(x, y) < t$, $M_{01,w}(x, y, t) = 1/2$ if $d(x, y) = t$, and*

$M_{01}(x,y,t)=0$ if $d(x,y)>t$, for all $x,y\in X$ and $t>0$. Then, the pair $(M_{01,w},*)$ is a weak fuzzy metric on X whose induced topology coincides with the topology induced by d. Note that $(M_{01,w},*)$ is not a Gr-fuzzy metric if $|X|\geq 2$ because, for $x\neq y$, the function $t\to M_{01,w}(x,y,t)$ is not left continuous at $t=d(x,y)$. Note also that the "open" ball $B_{01,w}(x,1,1)$ is not necessarily a $\mathfrak{T}_M$-open set because $B_{01,w}(x,1,1)=\{y\in X: d(x,y)\leq 1\}$.

Example 2.7. *Let $X=[0,1]$. Then, the pair $(M_\wedge,\wedge)$ is a weak fuzzy metric on X, where $M_\wedge(x,x,t)=1$ for all $x\in X$ and $t>0$, $M(x,y,t)=x\wedge y$ for all $x,y\in X$ with $x\neq y$ and $t\in(0,1)$, and $M(x,y,t)=1$ for all $x,y\in X$ and $t\geq 1$. Clearly, $(M_\wedge,\wedge)$ is not a Gr-fuzzy metric because the function $t\to M(x,y,t)$ is not left continuous at $t=1$ for $x\neq y$.*

3. Modular metrics and their relation with fuzzy metrics

We start this section recalling the notion of modular metric as given by Chistyakov ([4, 5]).

Definition 3.3. *A modular metric on a set X is a function $w:(0,\infty)\times X\times X\to[0,\infty]$ that fulfills the following axioms for all $x,y,z\in X$:*

(MM1) $x=y$ *if and only if* $w(t,x,y)=0$ *for all* $t>0$;
(MM2) $w(t,x,y)=w(t,y,x)$ *for all* $t>0$;
(MM3) $w(t+s,x,y)\leq w(t,x,z)+w(s,z,y)$ *for all* $t,s>0$.

A modular metric space is a pair (X,w) such that X is a set and w is a modular metric on X.

Let w be a modular metric on a set X. For each $x\in X$, and $\varepsilon,t>0$, set $B_w(x,\varepsilon,t)=\{y\in X: w(t,x,y)<\varepsilon\}$. Since, for each $x,y\in X$, the function $t\to w(t,x,y)$ is nonincreasing on $(0,\infty)$ ([4, p. 3]), we deduce, similarly to the proof of Lemma 2.2, that the family

$$\mathfrak{T}_w=\{A\subseteq X: \text{ for each } x\in A \text{ there exist } \varepsilon,t>0 \text{ such that } B_w(x,\varepsilon,t)\subseteq A\},$$

is a topology on X.

Remark 3.5. *If, for each $x,y\in X$, the function $t\to w(t,x,y)$ is left continuous on $(0,\infty)$, then each ball $B_w(x,\varepsilon,t)$ is a $\mathfrak{T}_w$-open set. Indeed, suppose that $y\in B_w(x,\varepsilon,t)$ for some $x\in X$ and $\varepsilon,t>0$. Put $\delta=\varepsilon-w(t,x,y)$ and choose $r\in(0,t)$ such that $w(t-r,x,y)<w(t,x,y)+\delta/2$. Thus, for each $z\in B_w(y,\delta/2,r)$, we have*

$$w(t,x,z)\leq w(t-r,x,y)+w(r,y,z)<w(t,x,y)+\delta/2+\delta/2=\varepsilon.$$

We conclude that $B_w(y,\delta/2,r)\subseteq B_w(x,\varepsilon,t)$, and hence, $B_w(x,\varepsilon,t)$ is a $\mathfrak{T}_w$-open set.

In the light of the preceding remark, and by analogy with the fuzzy metric case, by a Gr-modular metric on a set X we will mean a modular metric w on X satisfying that, for each $x,y\in X$, the function $t\to w(t,x,y)$ is left continuous on $(0,\infty)$. If, in addition, w satisfies that $\lim_{t\to\infty}w(t,x,y)=0$ for all $x,y\in X$, we will say that w is a KM-modular metric on X.

Proposition 3.1. *Let $(M,*)$ be a weak fuzzy metric on a set X such that $*=\wedge$. Then, the function $w_M: X\times X\times(0,\infty)\to[0,\infty]$ defined by*

$$w_M(t,x,y)=\frac{1-M(x,y,t)}{M(x,y,t)} \tag{3.1}$$

for all $x,y\in X$ and $t>0$, is a modular metric on X such that the topologies $\mathfrak{T}_M$ and $\mathfrak{T}_{w_M}$ coincide on X. Furthermore, if $(M,\wedge)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X, then w_M is a Gr-modular metric (resp. a KM-modular metric) on X.

Proof. We first note that w_M satisfies axioms (MM1) and (MM2) as a direct consequence of (KM2) and (KM3), respectively.

In order to verify that w_M satisfies (MM3) we shall distinguish two cases.

Case 1. $\min\{M(x,z,t), M(z,y,s)\} = 0$. Then, $\max\{w_M(t,x,z), w_M(s,z,y)\} = \infty$.
Case 2. $\min\{M(x,z,t), M(z,y,s)\} > 0$. Then, $M(x,y,t+s) > 0$ by (KM4).
Put $a = M(x,z,t)$, $b = M(z,y,s)$ and $c = M(x,y,t+s)$. Then, $a+b-ab \geq a \vee b$, so

$$c(a+b-ab) \geq (a \wedge b)(a+b-ab) \geq ab.$$

Hence, $c(a+b-2ab) \geq (1-c)ab$, and, thus

$$w_M(t+s,x,y) = \frac{1-c}{c} \leq \frac{1-a}{a} + \frac{1-b}{b} = w_M(t,x,z) + w_M(s,z,y).$$

We conclude that w_M is a modular metric on X.

The fact that the topologies $\mathfrak{T}_M$ and $\mathfrak{T}_{w_M}$ coincide on X is a consequence from the following easy relations:

$$B_M(x,\varepsilon/L,t) \subseteq B_{w_M}(x,\varepsilon,t) \subseteq B_M(x,\varepsilon,t)$$

for all $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$, where $L > 1+\varepsilon$.

Finally, it is obvious that if $(M, \wedge)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X, then w_M is a Gr-modular metric (resp. a KM-modular metric) on X. □

Conversely, we have:

Proposition 3.2. *Let w be a modular metric on a set X and let M_w the fuzzy set in $X \times X \times \mathbb{R}^+$ defined by $M_w(x,y,0) = 0$ for all $x,y \in X$ and*

$$M_w(x,y,t) = \frac{1}{1+w(t,x,y)} \tag{3.2}$$

for all $x,y \in X$ and $t > 0$. Then, the pair $(M_w, *_P)$ is a weak fuzzy metric on X such that the topologies $\mathfrak{T}_w$ and $\mathfrak{T}_{M_w}$ coincide on X. Furthermore, if w is a Gr-modular metric (resp. a KM-modular metric) on X, then $(M_w, *_P)$ is a Gr-fuzzy metric (resp. a KM-fuzzy metric) on X.

Proof. We first note that M_w satisfies axioms (KM2) and (KM3) as a direct consequence of (MM1) and (MM2), respectively.

To show that $(M_w, *_P)$ is a weak fuzzy metric on X it remains to verify that w_M satisfies (KM4). To this end, let $x,y,z \in X$ and $t,s \geq 0$. If $\min\{t,s\} = 0$, we have $M_w(x,y,t+s) \geq 0 = M_w(x,z,t) *_P M(z,y,s) = 0$ by the definition of M_w. So, we will assume that $t > 0$ and $s > 0$. Then, we obtain

$$\begin{aligned} M_w(x,y,t+s) = \frac{1}{1+w(t+s,x,y)} &\geq \frac{1}{1+w(t,x,z)+w(s,z,y)} \\ &\geq \frac{1}{1+w(t,x,z)}\frac{1}{1+w(s,z,y)} = M_w(x,z,t) *_P M_w(z,y,s). \end{aligned}$$

Similarly to the proof of Proposition 3.1, the fact that the topologies $\mathfrak{T}_w$ and $\mathfrak{T}_{M_w}$ coincide on X is a consequence from the following easy inclusions:

$$B_{M_w}(x,\varepsilon/2,t) \subseteq B_w(x,\varepsilon/2,t) \subseteq B_{M_w}(x,\varepsilon,t) \tag{3.3}$$

for all $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$. □

Remark 3.6. *Remark 2.1 and relations (3.3) imply that the topology $\mathfrak{T}_w$ is metrizable. Moreover, by Remark 2.3, we get that a sequence $(x_n)_n$ in X is $\mathfrak{T}_w$-convergent to a $x \in X$ if and only if, for each $t > 0$, $\lim_{n\to\infty} w(t,x,x_n) = 0$, thus recovering an important result by Chystiakov (compare [4,* Theorem 2.13]*).*

With the help of Proposition 3.1 and examples given in Section 2, it is easy to obtain several instances of modular metrics (compare [4, Examples 2.4]).

Example 3.8.

(A) *Let (X,d) be a metric space. By applying Example 2.1 and Proposition 3.1 (formula (3.1)), we immediately deduce that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,y)=d(x,y)/t$ for all $x,y\in X$ and $t>0$, is a KM-modular metric on X. If we apply Example 2.5 instead of Example 2.1, we obtain that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,y)=d(x,y)/t$ for all $x,y\in X$ and $t\in(0,1)$, and $w(t,x,y)=0$ for all $x,y\in X$ and $t\geq 1$, is a modular metric on X that is not a Gr-modular metric whenever $|X|\geq 2$.*

(B) *Let (X,d) be a metric space. By applying Example 2.2 and Proposition 3.1 (formula (3.1)), we immediately deduce that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,y)=0$ if $d(x,y)<t$, and $w(t,x,y)=\infty$ if $d(x,y)\geq t$, is a KM-modular metric on X. If we apply Example 2.6 instead of Example 2.2, we obtain that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,y)=0$ if $d(x,y)<t$, $w(t,x,y)=1$ if $d(x,y)=t$, and $w(t,x,y)=\infty$ if $d(x,y)>t$, is a modular metric on X that is not a Gr-modular metric whenever $|X|\geq 2$.*

(C) *Let $X=[0,1]$. By applying Example 2.3, with $*=\wedge$, and Proposition 3.1 (formula (3.1)) we immediately deduce that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,x)=0$ for all $x\in X$ and $t>0$, and $w(t,x,y)=(1/(x\wedge y))-1$ whenever $x\neq y$ and $t>0$, is a Gr-modular metric on X that is not a KM-modular metric on X. If we apply Example 2.7 instead of Example 2.3, we obtain that the function $w:(0,\infty)\times X\times X\to[0,\infty]$ given by $w(t,x,x)=0$ for all $x\in X$ and $t>0$, $w(t,x,y)=(1/(x\wedge y))-1$ whenever $x\neq y$ and $t\in(0,1)$, and $w(t,x,y)=0$ whenever $x\neq y$ and $t\geq 1$, is a modular metric on X that is not a Gr-modular metric on X.*

Note that Propositions 3.1 and 3.2 imply the following statements:

(s1) If $(M,\wedge)$ is a weak fuzzy metric on a set X, then $M=M_{w_M}$,

(s2) If w is a modular metric on a set X, then $w=w_{M_w}$.

In turn, statements (s1) and (s2) suggest the next notions.

Definition 3.4.

(i) *A weak fuzzy metric $(M,*)$ on a set X is called modulable if there is a modular metric w on X such that $M=M_w$.*

(ii) *A modular metric w on a set X is called fuzziable if there is a weak fuzzy metric $(M,*)$ on X such that $w=w_M$.*

Therefore, we obtain:

Proposition 3.3.

(a) *Every fuzzy metric $(M,*)$ on a set X such that $*=\wedge$ is modulable.*

(b) *Every modular metric on a set X is fuzziable.*

We finish the paper by recalling that precedents of Propositions 3.1 and 3.2 can be found in [11] and [20], respectively. Thus, in [11] it was proved Proposition 3.1 for the case that $(M,*)$ is a triangular GV-fuzzy metric in the sense of [2], while that in [20] is was proved that under the assumption that w is a non-Archimedean modular metric on a set X such that, for each $x,y\in X$, the function $t\to w(t,x,y)$ is continuous on $(0,\infty)$, and condition $w(t,x,y)>0$ whenever $x\neq y$ is also satisfied, then equality (3.2) defines a non-Archimedean triangular GV-fuzzy metric on X for the product t-norm.

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