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Pell Leonardo numbers and their matrix representations

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Abstract — In this study, we investigate Pell numbers and Leonardo numbers and describe a new third-order number sequence entitled Pell Leonardo numbers. We then construct some identities, including the Binet formula, generating function, exponential generating function, Catalan, Cassini, and d'Ocagne's identities for Pell Leonardo numbers and obtain a relation between Pell Leonardo and Pell numbers. In addition, we present some summation formulas of Pell Leonardo numbers based on Pell numbers. Finally, we create a matrix formula for Pell Leonardo numbers and obtain the determinant of the matrix.

Subject Classification (2020): 11B39, 11Y55

1. Introduction

There are many famous number sequences whose elements are integers. The most well-known of these is the Fibonacci sequence. Different generalizations of the Fibonacci sequence have been made in the literature. One of these is the Pell sequence. In 1975, Bicknell [1] described the primer of the Pell sequence by the following recurrence relation: For $n \geq 2$ $(n \in \mathbb{N})$,

$$P_n = 2P_{n-1} + P_{n-2} \tag{1.1}$$

and the initial conditions $P_0 = 0$ and $P_1 = 1$. The characteristic equation of (1.1) is

$$\vartheta^2 - 2\vartheta - 1 = 0 \tag{1.2}$$

which has roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Binet's formula for the P_n is

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \tag{1.3}$$

A lot of research has been done on Pell numbers. Some of these are as follows: In [2], Horadam studied modified Pell numbers and their applications. In [3], Melham obtained new identities of Pell numbers, and in [4], Santana and Diaz-Barrero gave new properties about the summation of Pell numbers. In [5], Mushtaq and Hayat presented a matrix with Pell numbers of entries. In [6], Dasdemir built new matrices based on Pell

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numbers and modified Pell and Pell-Lucas numbers. Çelik et al. [7] obtained new recurrences on Pell and Pell-Lucas numbers.

In this study, we are interested in another well-known sequence closely related to the Fibonacci sequence and the Leonardo sequence. Catarino and Borges [8] presented some properties of the Leonardo sequence that they described by the following recurrence relation: For $n \geq 2$,

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

and the boundary conditions $Le_0 = Le_1 = 1$. In addition, there is an equation between the Leonardo numbers as follows: For $n \geq 2$,

$$Le_{n+1} = 2Le_n - Le_{n-2} \tag{1.4}$$

The characteristic equation of (1.4) is

$$\mu^3 - 2\mu^2 + 1 = 0 \tag{1.5}$$

The Binet's formula of the Le_n number is

$$Le_n = \frac{2\delta^{n+1} - 2\theta^{n+1} - \delta + \theta}{\delta - \theta}$$

δ and θ are roots of (1.5). There are many more works on Leonardo's numbers in literature. Some of these are as follows: In [9], Shannon found an inhomogeneous extension of Leonardo's numbers. Alp and Koçer [10] obtained new identities of Leonardo numbers, and in [11], they introduced hybrid Leonardo numbers. Kuhapatanakul and Chobsorn [12] defined generalized Leonardo numbers and obtained matrix representations. In [13], Karatas introduced complex Leonardo numbers and gave some properties and summation formulas. Tan and Leung [14] introduced Leonardo p −numbers and incomplete Leonardo p −numbers. In [15], Soykan described generalized Horadam Leonardo numbers and gave Binet's formulas, generating functions, Simson formulas, and the summation formulas.

Inspired by these studies, in this article, we first aim to define a new third-order sequence we named Pell Leonardo numbers. Thus, we introduce a new type of Leonardo sequence. Additionally, we intend to find some of their equations, including the Binet formula, generating function, exponential generating function, Catalan, Cassini, and d'Ocagne's identities for Pell Leonardo numbers. We obtain some summation formulas for Pell Leonardo numbers based on Pell numbers. Finally, we build a matrix representation for Pell Leonardo numbers and get the determinant of the matrix.

2. Pell Leonardo Numbers

In this section, we describe nth Pell Leonardo number by the following recurrence relation for $n \geq 2$,

$$PLe_n = 2PLe_{n-1} + PLe_{n-2} + 1 \tag{2.1}$$

and the initial conditions $PLe_0 = 0$ and $PLe_1 = 1$. The first fifteen Pell Leonardo numbers are

$$0, 1, 3, 8, 20, 49, 119, 288, 696, 1681, 4059, 9800, 23660, 57121, 137903$$

In addition, we can mention the terms with negative subscripts of Pell Leonardo number sequences. By using (2.1), the first five terms with negative subscripts of Pell Leonardo numbers are as follows:

$$PLe_{-1} = 0, PLe_{-2} = -1, PLe_{-3} = 1, PLe_{-4} = -4, \text{and } PLe_{-5} = 8$$

And then, according to (2.1), it is observed that

$$PLe_{n+1} = 3PLe_n - PLe_{n-1} - PLe_{n-2} \tag{2.2}$$

where PLe_n is nth Pell Leonardo number.

The characteristic equation of (2.2) is

$$\gamma^3 - 3\gamma^2 + \gamma + 1 = 0$$

The relation between Pell Leonardo and Pell numbers is expressed in the following proposition.

Proposition 2.1. For $n \geq 0$,

$$PLe_n - PLe_{n-1} = P_n \tag{2.3}$$

Proof. We prove by induction on n. We know that from the definition of Pell Leonardo numbers, $PLe_{-1} = 0$. It is easily seen that (2.3) is held for $n = 0$ and $n = 1$. We suppose that (2.3) is true for all $1 < l \leq n$, and we prove that (2.3) holds for $l = n + 1$. In fact, by using the induction hypothesis and the recurrence relation (2.2), we can write

$$\begin{aligned} PLe_{n+1} &= 3PLe_n - PLe_{n-1} - PLe_{n-2} \\ &= 2PLe_n - PLe_{n-1} - PLe_{n-2} + PLe_n \\ &= 2(PLe_n - PLe_{n-1}) + PLe_{n-1} - PLe_{n-2} + PLe_n \\ &= 2P_n + P_{n-1} + PLe_n \end{aligned}$$

and thus,

$$PLe_{n+1} - PLe_n = P_{n+1}$$

□

Theorem 2.2. The Binet formula for Pell Leonardo numbers is given by

$$PLe_n = \frac{1}{4}(\rho^{n+1} + \sigma^{n+1}) - \frac{1}{2}$$

where $\rho = 1 + \sqrt{2}$ and $\sigma = 1 - \sqrt{2}$.

Proof. Because the roots of (1.2) are equal to ρ and σ, Binet's formula of the Pell Leonardo numbers can be easily calculated by using (1.3) and (2.3). □

Theorem 2.3. The generating function of PLe_n is determined by

$$GF_{PLe}(t) = \frac{t}{(1 - 3t + t^2 + t^3)}$$

Proof. For the generating function of $\{PLe_n\}_{n=0}^{\infty}$. Firstly, we will consider the following power series:

$$GF_{PLe}(t) = \sum_{n=0}^{\infty} PLe_n t^n$$

Therefore,

$$GF_{PLe}(t) = PLe_0 + PLe_1 t + PLe_2 t^2 + \cdots + PLe_k t^k + \cdots$$

Thus,

$$-3tGF_{PLe}(t) = -3PLe_0t - 3PLe_1t^2 - 3PLe_2t^3 - \cdots - 3PLe_kt^{k+1} + \cdots$$

$$+t^2GF_{PLe}(t) = PLe_0t^2 + PLe_1t^3 + PLe_2t^4 + \cdots + PLe_kt^{k+2} + \cdots$$

$$+t^3GF_{PLe}(t) = PLe_0t^3 + PLe_1t^4 + PLe_2t^5 + \cdots + PLe_kt^{k+3} + \cdots$$

If the three equations above are considered together, we obtain

$$\begin{aligned}(1-3t+t^2+t^3)GF_{PLe}(t) &= PLe_0 + (PLe_1 - 3PLe_0)t + (PLe_2 - 3PLe_1 + PLe_0)t^2 \\ &+(PLe_3 - 3PLe_2 + PLe_1 + PLe_0)\, t^3 \\ &+ \cdots + (PLe_{k+1} - 3PLe_k + PLe_{k-1} + PLe_{k-2})t^{k+1} + \cdots\end{aligned}$$

By using (2.2), we have

$$\begin{aligned}GF_{PLe}(t) &= \frac{PLe_0 + (PLe_1 - 3PLe_0)t + (PLe_2 - 3PLe_1 + PLe_0)t^2}{(1-3t+t^2+t^3)} \\ &= \frac{t}{(1-3t+t^2+t^3)}\end{aligned}$$

□

Theorem 2.4. The exponential generating function of PLe_n is

$$EGF_{PLe}\,(t) = \frac{1}{4}(e^{\rho t} + e^{\sigma t}) - \frac{1}{2}$$

where $\rho = 1 + \sqrt{2}$ and $\sigma = 1 - \sqrt{2}$.

Proof. For the exponential generating function of $\{PLe_n\}_{n=0}^{\infty}$, we will deal with the following series representation:

$$EGF_{PLe}(t) = \sum_{n=0}^{\infty} PLe_n \frac{t^n}{n!}$$

Thus, using the Binet formula of PLe_n and $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, it is obtained

$$EGF_{PLe}\,(t) = \frac{1}{4}(e^{\rho t} + e^{\sigma t}) - \frac{1}{2}$$

□

Theorem 2.5. For $m \geq n$, the Catalan identity for Pell Leonardo numbers is

$$PLe_{m-n}PLe_{m+n} - {PLe_m}^2 = \frac{(-1)^{m-n+1}}{2}{P_n}^2 + PLe_m - \frac{1}{2}(PLe_{m+n} + PLe_{m-n})$$

where m and n are nonnegative numbers, P_n is nth Pell number, $\rho = 1 + \sqrt{2}$, and $\sigma = 1 - \sqrt{2}$.

Proof. To find Catalan identity for Pell Leonardo numbers, we first calculate the following:

$$PLe_{m-n}PLe_{m+n} - PLe_m{}^2 = \left(\frac{\rho^{m-n+1}+\sigma^{m-n+1}}{4}-\frac{1}{2}\right)\left(\frac{\rho^{m+n+1}+\sigma^{m+n+1}}{4}-\frac{1}{2}\right)$$
$$-\left(\frac{\rho^{m+1}+\sigma^{m+1}}{4}-\frac{1}{2}\right)\left(\frac{\rho^{m+1}+\sigma^{m+1}}{4}-\frac{1}{2}\right)$$
$$=\left(\frac{\rho^{2m+2}+\sigma^{2m+2}}{16}+\frac{(\rho\sigma)^{m-n+1}(\sigma^{2n}+\rho^{2n})}{16}\right)$$
$$-\left(\frac{\rho^{m-n+1}+\sigma^{m-n+1}}{8}\right)-\left(\frac{\rho^{m+n+1}+\sigma^{m+n+1}}{8}\right)+\frac{1}{4}$$
$$-\left(\frac{\rho^{2m+2}+\sigma^{2m+2}+2(\rho\sigma)^{m+1}}{16}\right)+\left(\frac{\rho^{m+1}+\sigma^{m+1}}{8}\right)$$
$$+\left(\frac{\rho^{m+1}+\sigma^{m+1}}{8}\right)-\frac{1}{4}$$

By using $\rho\sigma = -1$,

$$PLe_{m-n}PLe_{m+n} - PLe_m{}^2 = \frac{(-1)^{m-n+1}}{2}\left(\frac{\sigma^n-\rho^n}{2\sqrt{2}}\right)^2-\frac{\rho^{m-n+1}}{4}\left(\frac{\rho^n-1}{\rho-1}\right)^2-\frac{\sigma^{m-n+1}}{4}\left(\frac{\sigma^n-1}{\sigma-1}\right)^2$$
$$=\frac{(-1)^{m-n+1}}{2}P_n{}^2-\frac{\rho^{m-n+1}}{4}\left(\frac{\rho^n-1}{\rho-1}\right)^2-\frac{\sigma^{m-n+1}}{4}\left(\frac{\sigma^n-1}{\sigma-1}\right)^2$$
$$=\frac{(-1)^{m-n+1}}{2}P_n{}^2+PLe_m-\frac{1}{2}(PLe_{m+n}+PLe_{m-n})$$

□

If $n = 1$ in Catalan identity, Cassini identity is obtained as follows:

Corollary 2.6. For $m \geq 1$,

$$PLe_{m-1}PLe_{m+1} - PLe_m{}^2 = \frac{(-1)^m}{2} - PLe_{m-1} - \frac{1}{2}$$

Theorem 2.7. For $m > n, n \geq 1$, and $m - n > 1$, d'Ocagne's identity for Pell Leonardo numbers is

$$PLe_mPLe_{n+1} - PLe_{m+1}PLe_n = \frac{(-1)^{n+2}}{4}(PLe_{m-n-2}+PLe_{m-n}+1)+\frac{1}{2}(P_{m+1}-P_{n+1})$$

where m and n are nonnegative numbers and P_n is nth Pell number.

Proof. To find d'Ocagne's identity for Pell Leonardo numbers, we first calculate the following:

$$PLe_mPLe_{n+1} - PLe_{m+1}PLe_n = \left(\frac{\rho^{m+1}+\sigma^{m+1}}{4}-\frac{1}{2}\right)\left(\frac{\rho^{n+2}+\sigma^{n+2}}{4}-\frac{1}{2}\right)$$
$$-\left(\frac{\rho^{m+2}+\sigma^{m+2}}{4}-\frac{1}{2}\right)\left(\frac{\rho^{n+1}+\sigma^{n+1}}{4}-\frac{1}{2}\right)$$
$$=\left(\frac{\rho^{m+n+3}+\sigma^{m+n+3}}{16}+\frac{(\rho\sigma)^{n+2}(\sigma^{m-n-1}+\rho^{m-n-1})}{16}\right)$$

$$-\left(\frac{\rho^{m+1}+\sigma^{m+1}}{8}\right)-\left(\frac{\rho^{n+2}+\sigma^{n+2}}{8}\right)+\frac{1}{4}$$
$$-\left(\frac{\rho^{m+n+3}+\sigma^{m+n+3}}{16}+\frac{(\rho\sigma)^{n+1}\,(\sigma^{m-n+1}+\rho^{m-n+1})}{16}\right)$$
$$+\left(\frac{\rho^{m+2}+\sigma^{m+2}}{8}\right)+\left(\frac{\rho^{n+1}+\sigma^{n+1}}{8}\right)-\frac{1}{4}$$

By using $\rho\sigma=-1$,

$$PLe_mPLe_{n+1}-PLe_{m+1}PLe_n=\frac{(-1)^{n+2}}{16}\rho^{m-n-1}(1+\rho^2)+\sigma^{m-n-1}(1+\sigma^2)$$
$$+\frac{\rho^{m+1}\,(\rho-1)-\rho^{n+1}\,(\rho-1)+\sigma^{m+1}\,(\sigma-1)-\sigma^{n+1}\,(\sigma-1)}{8}$$

By using $\rho=1+\sqrt{2}$ and $\sigma=1-\sqrt{2}$,

$$PLe_mPLe_{n+1}-PLe_{m+1}PLe_n=\frac{(-1)^{n+2}}{4}(PLe_{m-n-2}+PLe_{m-n}+1)$$
$$+\frac{1}{2}\left(\frac{\rho^{m+1}-\sigma^{m+1}}{2\sqrt{2}}\right)-\frac{1}{2}\left(\frac{\rho^{n+1}-\sigma^{n+1}}{2\sqrt{2}}\right)$$
$$=\frac{(-1)^{n+2}}{4}(PLe_{m-n-2}+PLe_{m-n}+1)+\frac{1}{2}(P_{m+1}-P_{n+1})$$

□

Theorem 2.8. Some summation formulas of Pell Leonardo numbers based on Pell numbers are as follows:

i. $\sum_{k=0}^{n}PLe_k=4(\sum_{k=1}^{n-1}P_k)$

ii. $\sum_{k=0}^{n}PLe_{3k}=(\sum_{k=3n-2}^{3n}P_k)+2(\sum_{k=3n-5}^{3n-3}P_k)+\cdots+(n-1)(\sum_{k=4}^{6}P_k)+n(\sum_{k=1}^{3}P_k)$

where $k\geq 0$ and P_k is kth Pell number.

Proof. By using (2.3), it is easily obtained that the above equations are satisfied. □

3. Matrix Formula of Pell Leonardo Numbers

This section introduces a new matrix formulation for Pell Leonardo numbers. By using (2.2), we have

$$\begin{bmatrix}PLe_{n+2}\\PLe_{n+1}\\PLe_n\end{bmatrix}=\begin{bmatrix}3&-1&-1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}PLe_{n+1}\\PLe_n\\PLe_{n-1}\end{bmatrix}$$

Then, the new matrix formulation for Pell Leonardo numbers is as follows:

$$\begin{bmatrix}PLe_{n+2}\\PLe_{n+1}\\PLe_n\end{bmatrix}=\begin{bmatrix}3&-1&-1\\1&0&0\\0&1&0\end{bmatrix}^n\begin{bmatrix}PLe_2\\PLe_1\\PLe_0\end{bmatrix}$$

Here, we provide a matrix formulation for Pell Leonardo numbers, called the PL-matrix, and denote the PL-matrix by $PLeM$. If $PLeM$ is

$$\begin{bmatrix}3&-1&-1\\1&0&0\\0&1&0\end{bmatrix}$$

then $\det PLeM=-1$.

Theorem 3.1. The PL-matrix for nth Pell Leonardo number is given as

$$PLeM_n = \begin{bmatrix} PLe_{n+1} & -PLe_n - PLe_{n-1} & -PLe_n \\ PLe_n & -PLe_{n-1} - PLe_{n-2} & -PLe_{n-1} \\ PLe_{n-1} & -PLe_{n-2} - PLe_{n-3} & -PLe_{n-2} \end{bmatrix}$$

Proof. To find the PL-matrix, we calculate the following:

$$PLeM_n = (PL)^n = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n$$

By using (2.2), $PLeM_n$ is obtained as follows:

$$PLeM_n = \begin{bmatrix} PLe_{n+1} & -PLe_n - PLe_{n-1} & -PLe_n \\ PLe_n & -PLe_{n-1} - PLe_{n-2} & -PLe_{n-1} \\ PLe_{n-1} & -PLe_{n-2} - PLe_{n-3} & -PLe_{n-2} \end{bmatrix}$$

□

Corollary 3.2. For $n \geq 0$, the determinant of the PL-matrix for nth Pell Leonardo number can be found as follows:

$$|PLeM_n| = \begin{vmatrix} PLe_{n+1} & -PLe_n - PLe_{n-1} & -PLe_n \\ PLe_n & -PLe_{n-1} - PLe_{n-2} & -PLe_{n-1} \\ PLe_{n-1} & -PLe_{n-2} - PLe_{n-3} & -PLe_{n-2} \end{vmatrix} = (-1)^n \quad (3.1)$$

Proof. To ensure (3.1), we use induction on n. For $n = 0$, $PLeM_0 = I_3$, where I is the identity matrix. Therefore, it is easily seen that (3.1) is held. We suppose that (3.1) is held for $n = u$, that is,

$$|PLeM_u| = \begin{vmatrix} PLe_{u+2} & -PLe_{u+1} - PLe_u & -PLe_{u+1} \\ PLe_{u+1} & -PLe_u - PLe_{u-1} & -PLe_u \\ PLe_u & -PLe_{u-1} - PLe_{u-2} & -PLe_{u-1} \end{vmatrix} = |PL^u| = |PL|^u = (-1)^u$$

Then, by induction, for $n = u + 1$,

$$|PLeM_{u+1}| = \begin{vmatrix} PLe_{u+2} & -PLe_{u+1} - PLe_u & -PLe_{u+1} \\ PLe_{u+1} & -PLe_u - PLe_{u-1} & -PLe_u \\ PLe_u & -PLe_{u-1} - PLe_{u-2} & -PLe_{u-1} \end{vmatrix} = |PL^{u+1}| = |PL|^u|PL| = (-1)^{u+1}$$

Thus, (3.1) holds for all $n \geq 0$. □

4. Conclusion

In this study, we first introduce a new third-order Leonardo sequence and thus add a new third-order number sequence to the literature. We named that sequence Pell Leonardo numbers. Moreover, we obtain some equations for that sequence, including the Binet formula, generating function, exponential generating function, Catalan, Cassini, and d'Ocagne's identities, and some summation formulas based on the Pell sequence. Finally, we describe a PL-matrix for Pell Leonardo numbers and obtain the determinant of PL-matrix. Different sequences such as complex, bicomplex, gaussian, polynomial, and quaternion sequences can be defined using this sequence. Again, this number sequence and matrix representation can be used in coding theory, cryptography, and other engineering and physics applications.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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