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$\mathcal{C}_\alpha-$CURVES AND THEIR $\mathcal{C}_\alpha-$FRAME IN CONFORMABLE DIFFERENTIAL GEOMETRY

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Abstract. The aim of this study is to redesign the space curve and its Frenet framework, which are extremely important in terms of differential geometry, by using conformable derivative arguments. In this context, conformable counterparts of basic geometric concepts such as angle, vector, line, plane and sphere have been obtained. The advantages of the conformable derivative over the classical (Newton) derivative are mentioned. Finally, new concepts produced by conformable derivative are supported with the help of examples and figures.

1. Introduction

Perhaps the most interesting and well representative field of study of differential geometry is the theory of curves. Examination of the local properties of the curves yields different and important results. This theory has very different applications in linear and nonlinear differential equations and physics. Frenet equations are at the forefront of the most widely used and natural structure of the theory of curves. These equations have a very elite status in geometry and have many different uses. These formulas were first used in 1847 and discovered and published by Frenet J.F. Unaware of him, Serret J.A. calculated the same formulas in 1851. For this reason, these formulas are called the Frenet-Serret formulas by giving the names of both today. In this way, many new curve concepts have joined the geometry family with the help of Frenet-Serret vectors. The best examples of this are Bertrand curve pair, Mannheim curve pair and Involute-Evolute curve pair. In addition, Bishop, Darboux and Sabban frames in Euclidean and Minkowski spaces are different approaches to describing the motion of the curve. With the help of these approaches, many studies are carried out for the properties or characterization of curves in 3-dimensional Euclidean and Minkowski spaces according to Frenet, Bishop and Darboux frame [1, 2, 3].

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Classical analysis, a mathematical theory widely used today, was discovered by Leibniz G. and Newton I. in the second half of the 17th century, based on the concepts of derivative and integral, and are also referred to as Newtonian analysis. Over time, alternative analyses to Newton analysis are tried to be produced. Fractional analysis can be considered as the most important of these. Fractional analyses, which is first mentioned in Leibniz's letter to L'Hospital in 1695, aim to expand integer order derivatives to fractional orders. This theory, which is not widely accepted at first, are gained a place in every field today. The most important reason for this is the assumption that fractional analyzes have some advantages over Newtonian analysis. It is a fact that fractional analysis gives more numerical results than Newtonian analysis, especially in the solutions of some special differential equations [4, 5, 6, 7]. In this context, fractional analysis are become extremely popular and as a result, many types of fractional analysis are emerged. In general, fractional derivatives are grouped under two headings: global fractional derivative and local fractional derivative. The most important of the global fractional derivatives are Riemann-Liouville, Caputo, Grünwald-Letnikov, Wely, Riesz [8, 9, 10, 11]. The most important of the local fractional derivatives are proven themselves today as conformable, M-derivative and V-derivative [12, 13, 14, 15]. Global and local fractional derivatives have a big distinction within themselves. The most important difference between them is that global derivatives do not satisfy Leibniz and the chain rule as in the classical derivative, while local fractional derivatives do not have such a disadvantage. In addition, in global fractional derivatives, the derivative of the constant is not zero except for the Caputo derivative, but this is not the case in local derivatives. This situation is made local derivatives more indispensable in some matters.

The theory of curves and surfaces can be defined as the study of the motion of a point in a space with the help of linear algebra and calculus. Moreover, Leibniz and the chain rule are two indispensable elements when making calculations in differential geometry. For this reason, if fractional analysis is to be applied in differential geometry, the most appropriate one is local fractional derivatives. Fractional calculus has been used effectively in the field of differential geometry for the last decade, as it has proven itself in every field. This adventure was first started when Yajima T. and Kamasaki K. examined the Caputo fractional derivative of surfaces [16]. Additionally, Yajima T. et al. succeeded in creating the Frenet frame using fractional calculus [17]. Lazopoulos K.A. and Lazopoulos A.K. are made fractional calculations on manifolds [18]. Evren M.E. explained that local fractional derivatives are more useful and advantageous than global fractional derivatives in differential geometry [19]. Has A. et al obtained some advantages of the conformable derivative in terms of geometry compared to the classical derivative [20]. Gozutok and colleagues created the Frenet frame using conformable derivatives [21]. Following these developments, the use of fractional analysis in differential geometry has increased tremendously and many studies have been carried out on this subject [22, 23, 24, 25, 26, 27, 28, 29, 30].

In this study, the basic geometric properties of the curves were reconstructed using compatible derivative arguments. In the first stage, the main concepts of angle, vector, line, plane and sphere, which are geometric concepts, were redesigned with the help of conformable calculus. In addition, the orthogonal and orthonormal systems, which can be considered the basis of vectors, have been redefined in a similar way. Afterwards, with these of the conformable concepts obtained, the conformable space curve and the conformable Frenet framework at any point of it were created. In the final, examples were given and enriched with figures to make the subject more fluent.

2. Preliminaries

Khalil R. et al. are introduced a new derivative called the conformable fractional derivative of order α of the function f, which is defined as [12]:

$$D_\alpha(f)(s) = \lim_{\varepsilon \to 0} \frac{f(s+\varepsilon s^{1-\alpha}) - f(s)}{\varepsilon}.$$

where $f : [0,\infty) \to \mathbb{R}$ and $0 < \alpha < 1$. The relationship between the conformable derivative and the classical derivative, where $f'(s) = df(s)/ds$, is obtained as follows:

$$D_\alpha f(s) = s^{1-\alpha} \frac{d}{ds} f(s).$$

We say with the next theorem that the conformable derivative satisfies some properties such as linearity, Leibniz's rule and chain rule, as in the conventional derivative.

Theorem 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ and $0 < \alpha < 1$. The following are provided as functions f, g are α-differentiable functions. For all $a, b, p, \lambda \in \mathbb{R}$ [12],*

(1) $D_\alpha(af + bg)(s) = aD_\alpha f(s) + bD_\alpha g(s)$,
(2) $D_\alpha(s^p) = ps^{p-\alpha}$,
(3) $D_\alpha(\lambda) = 0$,
(4) $D_\alpha(fg)(s) = f(s)D_\alpha g(s) + g(s)D_\alpha f(s)$,
(5) $D_\alpha(\frac{f}{g})(s) = \frac{g(s)D_\alpha f(s) - f(s)D_\alpha g(s)}{g^2(s)}$,
(6) $D_\alpha(g \circ f)(s) = f^{\alpha-1}(s)D_\alpha f(s)D_\alpha g(f(s))$.

The conformable integral was defined by Khalil R. et al. as the inverse operator of the conformable derivative operator. Accordingly, the conformable integral of the $\alpha-$differentiable function f and for $[t, s]$, is as follows [12]

$$I_t^\alpha f(s) = I_t^s(s^{\alpha-1} f) = \int_t^s \frac{f(s)}{s^{1-\alpha}} ds.$$

In addition, f being a conformable differentiable function is given below for $t > 0$

$$D_\alpha I_\alpha[f(s)] = f(s)$$

The derivative limit of vector-valued functions has also been investigated by means of conformal analysis. We give this in the following theorem.

Theorem 2.2. *Let the function f be a function with n variables and each component is conformable differentiable. Then the conformable derivative of the function f is* [32]

$$D_\alpha f(f_1(s), f_n(s), ...f_n(s)) = f(D_\alpha f_1(s), D_\alpha f_m(s), ..., D_\alpha f_n(s)).$$

3. Some concepts of conformable differential geometry

In this section, the most basic concepts of geometry will be reconstructed with conformable arguments.

Notation: Along the study, expressions that are equal to 1 when $\alpha \to 1$ will be denoted as $\mathbf{1}_\alpha$ and expressions that are equal to 0 when $\alpha \to 1$ will be denoted as $\mathbf{0}_\alpha$. In addition, in order to avoid confusion between classical and conformable concepts, $\mathcal{C}_\alpha$ will be left in charge of conformable concepts.

Remark 3.1 (**A geometric approach to conformable derivative**). The geometric interpretation of the conformable derivative is based on the notion of fractal geometry. In fractal geometry, objects exhibit self-similarity at different scales. The conformable derivative captures this self-similar behavior of a function by considering its local fractional variations. Geometrically, it can be understood as analyzing the "zooming in" behavior of the function at that point, similar to the classical derivative capturing the local linear behavior. Overall, the geometric interpretation of the conformable derivative relates to the self-similarity and scaling properties of functions, enabling us to understand their behavior at different levels of detail and resolution. *More specifically, the conformable derivative can be explained as a measure of how much a straight line and plane bend to form a curve and a surface.* Figure 3 shows how a line is curved with the conformable calculus effect.

There are no Euclidean lines in the $\mathcal{C}_\alpha-$(conformable) space, this only happens when $\alpha \to is$ 1. We present this in Figure 3. This situation leads us to define a new angle in $\mathcal{C}_\alpha-$ space. Because we cannot measure the angle between the classical angle and the lines in $\mathcal{C}_\alpha-$ space. This new angle is called the $\mathcal{C}_\alpha-$ angle, and it measures the angle between the $\mathcal{C}_\alpha-$lines.

Let $\|\mathbf{u}\| = \mathbf{1}_\alpha$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$unit vector that is, they are vectors of the form $\|\mathbf{u}\| = \mathbf{1}_\alpha$ and $\|\mathbf{v}\| = \mathbf{1}_\alpha$. Then, the *$\alpha-$conformable radian measure* of *$\mathcal{C}_\alpha-$angle* between $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\theta_\alpha = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

It is also said that $\mathbf{u}$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$orthogonal when the following condition is proved,

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{0}_\alpha.$$

When x=1, $\mathcal{C}_\alpha-$ space has a different structure than Euclidean space, so the concept of $\mathcal{C}_\alpha-$ orthogonality will differ from. For example, let's consider the vectors $\mathbf{u} = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $\mathbf{v} = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$ in $\mathcal{C}_\alpha-$ space. Since $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{0}_\alpha$, vectors $\mathbf{u}$ and $\mathbf{v}$ are $\mathcal{C}_\alpha-$orthogonal. We showed this in Figure 1.

As in Euclidean space, in $\mathcal{C}_\alpha-$space the vector $u \times v$ is $\mathcal{C}_\alpha-$orthogonal to the vectors $\mathbf{u}$ and $\mathbf{v}$. For example, if $\mathbf{u} = (s^{1-\alpha}, 1-\alpha, \frac{1}{s^{1-\alpha}})$ and $\mathbf{v} = (\frac{1-\alpha}{s^\alpha}, s^\alpha, 2-2\alpha)$, $\mathbf{u}\times\mathbf{v} = (2\alpha^2 - 4\alpha - s^{2\alpha-1} + 2, 2\alpha s^{1-\alpha} - 2s^{1-\alpha} - \frac{\alpha}{s} + \frac{1}{s}, -\alpha^2 s^{-\alpha} + 2\alpha s^{-\alpha} - s^{-\alpha} + s)$ is obtained. It is also seen that $\langle \mathbf{u}\times\mathbf{v}, \mathbf{u}\rangle = 0_\alpha$ and $\langle \mathbf{u}\times\mathbf{v}, \mathbf{v}\rangle = 0_\alpha$. We showed this in Figure 2.

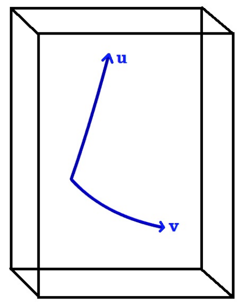

FIGURE 1. $\mathcal{C}_\alpha-$orthogonal vectors.

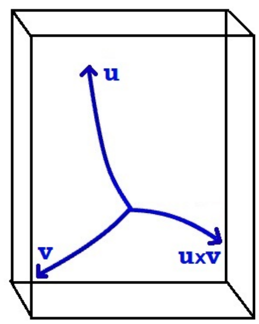

FIGURE 2. $\mathcal{C}_\alpha-$orthogonal system.

Definition 3.2. $\mathcal{C}_\alpha-$*line* l with a direction $\mathbf{v}$ through the point $P_\alpha = ((p_1)_\alpha, (p_2)_\alpha)$ is a subset of $\mathbb{E}^2$ is defined as

$$l = \{X \in \mathbb{E}^2 : X = P_\alpha + \mathbf{v}_\alpha f(t)\}$$

where $f(t) = \int t^{1-\alpha} dt$, $\mathbf{v}_\alpha = ((v_1)_\alpha, (v_2)_\alpha)$ and P_α is the point whose coordinates contain α.

Example 3.3. Let consider the $s \mapsto \mathbf{x}(s) = (s, \int s^{1-\alpha} ds)$, $\mathcal{C}_\alpha-$line passing through the point $P = (0, 0)$ and whose direction is $v = (s^{1-\alpha}, s^{1-\alpha})$.
In Fig. (3) we present the graph of the $\mathcal{C}_\alpha-$line for different α values.

Definition 3.4. $\mathcal{C}_\alpha-$*plane* Γ passing through a point $P_\alpha = ((p_1)_\alpha, (p_2)_\alpha, (p_3)_\alpha)$ and $\mathcal{C}_\alpha-$orthogonal to $\mathbf{v}$ is a subset of $\mathbb{E}^3$ defined by (see Fig. 4)

$$\Gamma = \{X \in \mathbb{E}^3 : \langle X - P_\alpha, \mathbf{v}_\alpha \rangle = 0_\alpha\}$$

where $X = (I^a_\alpha x_1, I^a_\alpha x_2, I^a_\alpha x_3)$, $\mathbf{v}_\alpha = ((v_1)_\alpha, (v_2)_\alpha, (v_3)_\alpha)$ and P_α is the point whose coordinates contain α.

Example 3.5. Let X be a representation point of the $\mathcal{C}_\alpha-$plane that contains the point $P = (0, 0, 0)$ and whose normal is $v = (2^{1-\alpha}, -3^{1-\alpha}, 0)$. If X representative

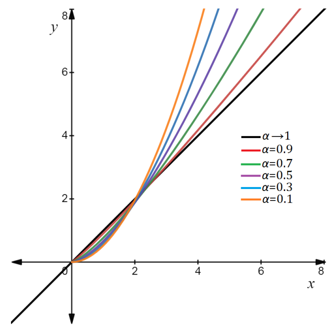

FIGURE 3. $\mathcal{C}_\alpha-$lines.

point is chosen as follows

$$\begin{aligned}\mathbf{x}_1(s) &= \int x^{1-\alpha}dx,\\ \mathbf{x}_2(s) &= \int y^{1-\alpha}dy,\\ \mathbf{x}_3(s) &= 0\end{aligned}$$

we get the $\mathcal{C}_\alpha-$plane. In Fig. (4) we present the graph of the $\mathcal{C}_\alpha-$plane for different α values.

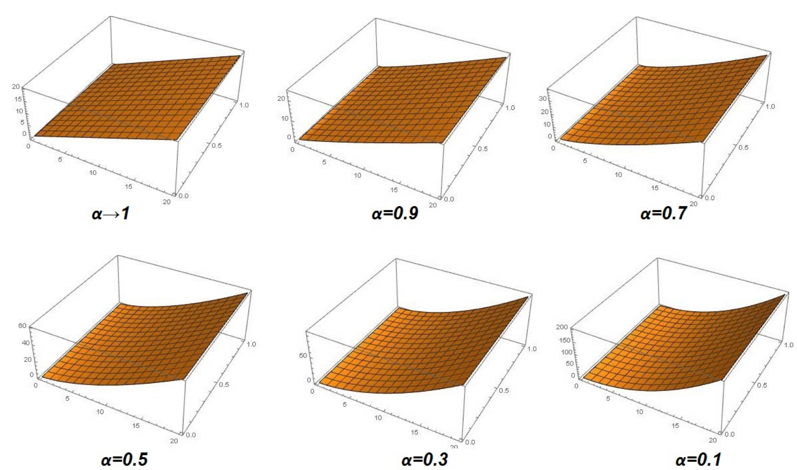

FIGURE 4. $\mathcal{C}_\alpha-$plane for different α values.

Definition 3.6. $\mathcal{C}_\alpha-$*sphere* with radius r_α and centered $C_\alpha = ((x_0)_\alpha, (y_0)_\alpha, (z_0)_\alpha)$ is a subset of $\mathbb{E}^3$ defined by (see Fig. 5)

$$\mathbb{S}^2_\alpha(C_\alpha, r_\alpha) = \{X \in \mathbb{E}^3 : \|X - C_\alpha\| = r_\alpha\}$$

where $X = (I^a_\alpha x_1, I^a_\alpha x_2, I^a_\alpha x_3)$. It should be noted here that the r_α and C_α values are not constant values. That is, the center and radius of the $\mathcal{C}_\alpha-$sphere change for each value of α. We shall denote by $\mathbb{S}^2_\alpha$ the $\mathcal{C}_\alpha-$sphere with radius 1_α and centered at $C_\alpha = (0_\alpha, 0_\alpha, 0_\alpha)$.

Example 3.7. Let $\mathbb{S}^2_\alpha(C_\alpha, r_\alpha)$ be a $\mathcal{C}_\alpha-$sphere in $\mathbb{R}^3$ parameterized by φ. If the coordinate functions of φ is chosen as follows,

$$\begin{aligned} f_1(u,v) &= -\int\int v^{\alpha-1}u^{\alpha-1}\sin u\cos v du dv \\ f_2(u,v) &= \int\int v^{\alpha-1}u^{\alpha-1}\sin u\sin v du dv \\ f_3(u,v) &= \int u^{\alpha-1}\cos u du \end{aligned}$$

we get the $\mathcal{C}_\alpha-$sphere as $\varphi(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$. In Fig. (5), we present the graph of the $\mathcal{C}_\alpha-$sphere for different α values.

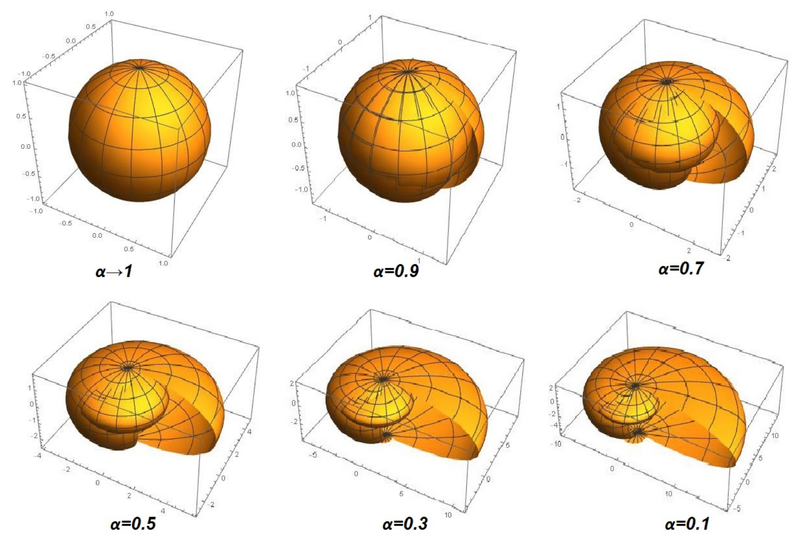

FIGURE 5. $\mathcal{C}_\alpha-$sphere for different α values.

4. $\mathcal{C}_\alpha-$PARAMETRIZED CURVES AND THEIR $\mathcal{C}_\alpha-$FRAME

Given that a 3-dimensional vector valued function to the $\mathcal{C}_\alpha-$ space as follow

$$\mathbf{x} \; : \; I \subset \mathbb{R} \to \mathbb{E}^3 \tag{4.1}$$

$$s \to \mathbf{x}(s) = (\mathbf{x}_1(s), \mathbf{x}_2(s), \mathbf{x}_3(s)) \tag{4.2}$$

We call $\mathbf{x}$ that satisfies the following equation $\mathcal{C}_\alpha-$naturally parameterized curve.

$$\|D_\alpha \mathbf{x}(s)\| = s^{1-\alpha}$$

where $D_\alpha \mathbf{x}(s) = (D_\alpha \mathbf{x}_1(s), D_\alpha \mathbf{x}_2(s), D_\alpha \mathbf{x}_3(s))$. Also, when $D_\alpha \mathbf{x}(s) \neq 0_\alpha$, $\mathbf{x}$ is called a $\mathcal{C}_\alpha-$regular curve and $D_\alpha \mathbf{x}(s) \times D^2_\alpha \mathbf{x}(s) \neq 0_\alpha$, $\mathbf{x}$ is called a $\mathcal{C}_\alpha-$biregular curve in $\mathcal{C}_\alpha-$space for each $s \in I$.

We named the triple apparatus $\{E_1, E_2, E_3\}$, defined as follows, as the $\mathcal{C}_\alpha-$frame vectors at point $s \in I$ of $\mathcal{C}_\alpha-$naturally parameterized curve $\mathbf{x}$:

$$E_1(s) = D_\alpha \mathbf{x}(s), \quad E_2(s) = \frac{D_\alpha E_1(s)}{\|D_\alpha E_1(s)\|}, \quad E_3(s) = E_1(s) \times E_2(s). \tag{4.3}$$

The E_1, E_2 and E_3 trio are called the $\mathcal{C}_\alpha$ tangent, principle normal and binormal at the point $s \in I$ of $\mathbf{x}$, respectively. Moreover, the vectors of the $\mathcal{C}_\alpha-$frame E_1, E_2 and E_3 are $\mathcal{C}_\alpha-$orthogonal and $\mathcal{C}_\alpha-$orthonormal.

Theorem 4.1. *The conformable derivative change of the $\mathcal{C}_\alpha-$frame at point $s \in I$ of the $\mathcal{C}_\alpha-$naturally parameterized $\mathbf{x}$ curve is as follows*

$$\begin{bmatrix} D_\alpha E_1 \\ D_\alpha E_2 \\ D_\alpha E_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \tag{4.4}$$

Proof. Considering Eq. (4.3), as follows

$$E_1 = s^{1-\alpha} T, \tag{4.5}$$

$$E_2 = \frac{(1-\alpha)s^{1-2\alpha}}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} T + \frac{s^{2-2\alpha}\kappa}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} N, \tag{4.6}$$

$$E_3 = \frac{s^{3-3\alpha}\kappa}{\sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}} B. \tag{4.7}$$

Differentiating of both sides of the above first equation α-th order conformable derivative as for s, we obtain

$$D_\alpha E_1 = (1-\alpha)s^{1-2\alpha} T + s^{2-2\alpha}\kappa N. \tag{4.8}$$

and

$$\|D_\alpha E_1\| = \sqrt{(1-\alpha)^2 s^{2-4\alpha} + s^{4-4\alpha}\kappa^2}. \tag{4.9}$$

Let's consider the $\kappa_\alpha = \|D_\alpha E_1\|$ equation here and use this equation in Eqs. (4.5), (4.6) and (4.7), we get

$$E_1 = s^{1-\alpha} T, \tag{4.10}$$

$$E_2 = \frac{(1-\alpha)s^{1-2\alpha}}{\kappa_\alpha} T + \frac{s^{2-2\alpha}\kappa}{\kappa_\alpha} N, \tag{4.11}$$

$$E_3 = \frac{s^{3-3\alpha}\kappa}{\kappa_\alpha} B. \tag{4.12}$$

Since the triple $\{E_1, E_2, E_3\}$ is $\mathcal{C}_\alpha-$orthogonal basis in $\mathbb{E}^3$,the following equation exist

$$D_\alpha E_1 = a_{11}E_1 + a_{12}E_2 + a_{13}E_3. \tag{4.13}$$

On the other hand, let's consider the definition of κ_α and Eq. (4.3)

$$D_\alpha E_1 = \kappa_\alpha E_2$$

is obtained. Considering this equation in Eq. (4.13), we can be write as

$$a_{12} = \kappa_\alpha.$$

Now, considering Eqs. (4.9), (4.10) and (4.11) by taking advantage of the scalar product of Eq. (4.13) with E_1, we get

$$\langle D_\alpha E_1, E_1 \rangle = a_{11}\langle E_1, E_1 \rangle + a_{12}\langle E_1, E_2 \rangle + a_{13}\langle E_1, E_3 \rangle,$$

$$(1-\alpha)s^{2-3\alpha} = a_{11}s^{2-2\alpha} + \kappa_\alpha \frac{(1-\alpha)s^{2-3\alpha}}{\kappa_\alpha}.$$

So we get the following result as

$$a_{11} = 0.$$

Anologously, considering Eqs. (4.7) and (4.8), by taking advantage of the scalar product of Eq. (4.13) with E_3, we get

$$a_{13} = 0.$$

The other part of the theorem is proved similarly. □

Conclusion 1. There is a relationship between the conformable derivative and the classical (Newton) derivative. When $\alpha \to 1$ is selected in the Conformable derivative, it is possible to return to the classical derivative results. Similarly, since the $\mathcal{C}_\alpha-$frame is obtained by conformable derivative, when $\alpha \to 1$ is selected, the $\mathcal{C}_\alpha-$frame turns into a classical Frenet frame. For this reason, a curve can be examined and compared both in $\mathcal{C}_\alpha-$space and Euclidean space.

Conclusion 2. The conformable derivative has some advantages over the classical derivative in terms of geometric meaning. The most important of these advantages is that conformable derivatives can be defined at points where the classical derivative is not defined. Thus, at points where tangents cannot be created with the classical derivative, alternative tangents can be created with the help of conformable derivative. For example, the derivative of the function $f(x) = 2\sqrt{x}$ is not defined at $x = 0$. Then it is impossible to create a tangent at $x = 0$. However, if the conformable derivative is applied by selecting $\alpha = \frac{1}{2}$ in the function $f(x)$, $D_\alpha f(x) = 0$. In other words, while a classical tangent cannot be mentioned at the point $x = 0$, a conformable tangent can be mentioned. The most important element of the Frenet frame is the tangent vector. Because other Frenet vectors can be obtained depending on the tangent vector. Thus, at points where the Frenet frame cannot be created, the curve can be examined by creating a $\mathcal{C}_\alpha-$frame.

Example 4.2. Let $\mathbf{x} : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$\mathbf{x}(s) = \left(2s^{\frac{1}{2}},\ s^{\frac{3}{2}},\ s^{\frac{5}{2}}\right).$$

The classical derivative of x and the conformable derivative for $\alpha = \frac{1}{2}$ are as follows

$$\mathbf{x}'(s) = \left(s^{\frac{-1}{2}}, \frac{3}{2}s^{\frac{1}{2}}, \frac{5}{2}s^{\frac{3}{2}}\right), \tag{4.14}$$

$$D_{\frac{1}{2}}\mathbf{x}(s) = \left(1, \frac{3}{2}s, \frac{5}{2}s^2\right). \tag{4.15}$$

where $x'(s)$ and $D_{\frac{1}{2}}\mathbf{x}(s)$ are the classical tangent and $\mathcal{C}_\alpha-$tangent of $\mathbf{x}(s)$, respectively. Considering Eqs. (4.14) and (4.14), while $\mathbf{x}'(0)$ is undefined for s=0, $D_{\frac{1}{2}}\mathbf{x}(0))$ is defined. We show this situation in Figure 6.

Moreover, as mentioned in Conclusion 1, while a Frenet frame cannot be obtained at points where there is no derivative of a curve, a $\mathcal{C}_\alpha-$frame can be established at the same point.

Theorem 4.3. *Let $\mathbf{x} = \mathbf{x}(s)$ be $\mathcal{C}_\alpha-$naturally parametrized curve in the Euclidean $3-$space where s measures its $\mathcal{C}_\alpha-$arc length. When $\alpha \to 1$, its curvature and torsion are $\kappa_\alpha \to \kappa$ and $\tau_\alpha \to \tau$, respectively.*

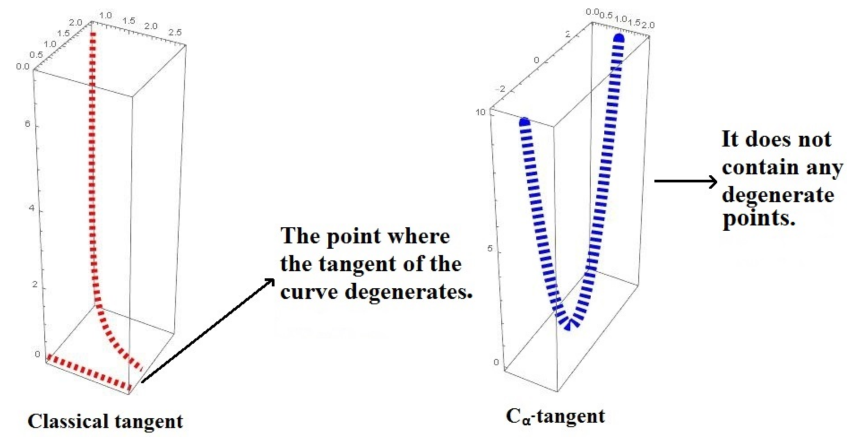

FIGURE 6. Classical tangent and $\mathcal{C}_\alpha-$tangent of the curve $\mathbf{x}(s)$.

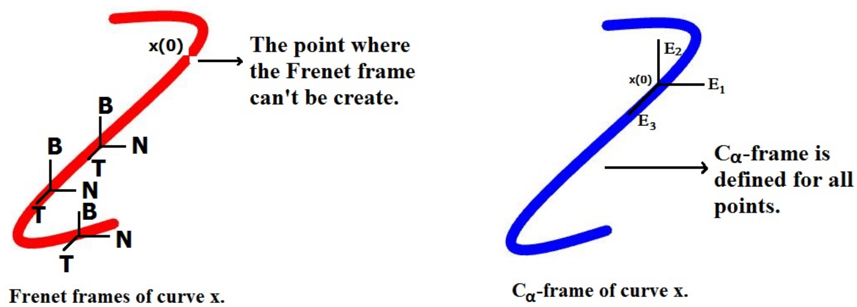

FIGURE 7. Frenet frame and $\mathcal{C}_\alpha-$frame of the curve $\mathbf{x}(s)$ at $s = 0$.

Proof. Let $\mathbf{x} = \mathbf{x}(s)$ be a regular $\mathcal{C}_\alpha-$curve $\mathbf{x}$. Here let's consider the definition of κ_α and Eq. (4.9), we have

$$\kappa_\alpha = s^{1-\alpha}\sqrt{(1-\alpha)^2 s^{-2\alpha} + s^{2-2\alpha}\kappa^2}. \tag{4.16}$$

Also considering the definition of τ_α and Eqs. (4.11), (4.12) we get

$$\tau_\alpha = \frac{s^{5-5\alpha}\kappa^2}{\kappa_\alpha^2}\tau. \tag{4.17}$$

Here is seen, while $\alpha \to 1$, $\kappa_\alpha \to \kappa$ and $\tau_\alpha \to \tau$. □

Example 4.4. Let $x : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{R}^3$ parameterized by

$$x(s) = \left(\frac{3}{5}\cos s,\ \frac{3}{5}\sin s,\ \frac{4}{5}s\right).$$

From Eqs. (4.5), (4.6) and (4.7), we get

$E_1 = \left(\frac{-3s^{1-\alpha}}{5}\sin s, \frac{3s^{1-\alpha}}{5}\cos s, \frac{4s^{1-\alpha}}{5}\right),$

$E_2 = \left(\frac{-3(1-\alpha)s^{1-2\alpha}\sin s - 3s^{2-2\alpha}\cos s}{5\kappa_\alpha}, \frac{3(1-\alpha)s^{1-2\alpha}\cos s - 3s^{2-2\alpha}\sin s}{5\kappa_\alpha}, \frac{4(1-\alpha)s^{1-2\alpha}}{5\kappa_\alpha}\right),$

$E_3 = \left(\frac{4s^{3-3\alpha}}{5\kappa_\alpha}\sin s, \frac{-4s^{3-3\alpha}}{5\kappa_\alpha}\cos s, \frac{3s^{3-3\alpha}}{5\kappa_\alpha}\right).$

In addition, the $\mathcal{C}_\alpha-$curvature and torsion of the $\mathcal{C}_\alpha-$curve $\mathbf{x}$ is calculated as in Eqs. (4.16) and (4.17) as follows

$$\begin{aligned} \kappa_\alpha &= \frac{s^{1-\alpha}}{5}\sqrt{25(1-\alpha)^2 s^{-2\alpha} + 9s^{2-2\alpha}}, \\ \tau_\alpha &= \frac{36s^{3-3\alpha}}{125(1-\alpha)^2 s^{-2\alpha} + 45s^{2-2\alpha}}. \end{aligned}$$

For different values of α the graphs of the curvature κ_α and torsion τ_α with fractional-order as in following Fig. 8 and Fig. 9.

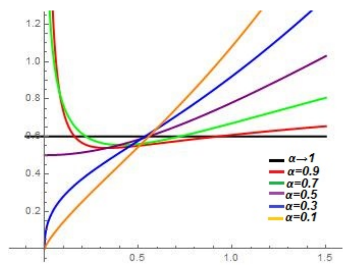

Figure 8. $\mathcal{C}_\alpha-$curvatures, κ_α.

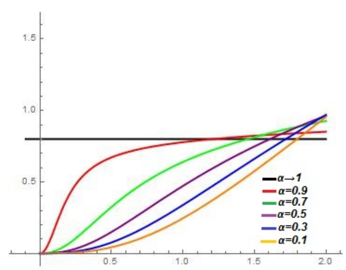

Figure 9. $\mathcal{C}_\alpha-$torsion, τ_α.

Example 4.5. Let $\mathbf{x} : I \subset \mathbb{R} \to E^3$ be a $\mathcal{C}_\alpha-$naturally parametrized curve in $\mathbb{E}^3$ parameterized by

$\mathbf{x}(s) = \left(-\frac{225}{16}\int s^{\alpha-1}(\sin 25s + \sin 9s)ds,\ \frac{225}{16}\int s^{\alpha-1}(\cos 25s - \cos 9s)ds,\ -\frac{225}{8}\int s^{\alpha-1}\sin 17s ds\right).$

In Fig. (10) we present the graph of the $\mathcal{C}_\alpha-$naturally parametrized curve for different α values

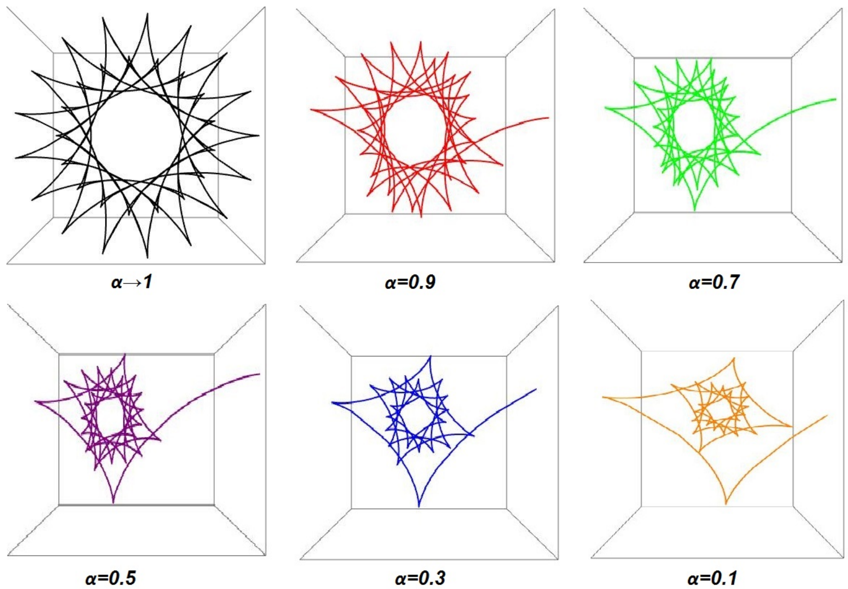

FIGURE 10. $\mathcal{C}_\alpha-$naturally parameterized curve for different α values.

5. Conclusion

In this study, we want to bring a new perspective to some problems that cannot be solved in Euclidean space, with the help of conformable derivatives. Alternatives have been created for some concepts that cannot be defined in Euclidean space with the help of conformable derivatives and have now become examinabla. In addition, one of the most attractive features of this situation is that new concepts can be compared with their classical forms since the return to Euclidean space is achieved at $\alpha \to 1$.

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No potential confict of interest was reported by the author(s).

Data availability
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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