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Research Article (Araştırma Makalesi)

On a Generalization of Statistical Convergence of Double Sequences

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Abstract

In this study, the concept of delta convergence, which is a generalization of the statistical convergence of double sequences is introduced. In connection with this, $\mathcal{C}_\Delta$ and $\mathcal{W}_p$ function spaces, which consist of some functions of two-variables, are introduced and the relationship between them is examined.

Keywords: *Time scale, statistical convergence, delta convergence, double sequence.*

Çift İndisli Dizilerin İstatistiksel Yakınsaklığının Bir Genelleştirilmesi Üzerine

Özet

Bu çalışmada, çift indisli dizilerin istatistiksel yakınsamasının bir genelleştirmesi olan delta yakınsaklık kavramı tanıtılmaktadır. Bununla bağlantılı olarak, iki değişkenli bazı fonksiyonlardan oluşan $\mathcal{C}_\Delta$ ve $\mathcal{W}_p$ fonksiyon uzayları tanıtılmakta ve aralarındaki ilişki incelenmektedir.

Anahtar Kelimeler: *Zaman skalası, istatistiksel yakınsaklık, delta yakınsaklık, çift indisli dizi.*

1. Introduction and Background

The notion of statistical convergence of real sequences was first introduced by Fast [1]. The interest in this concept increased with the articles of Schoenberg [2], Šălát [3], Fridy [4], and Connor [5]. The statistical convergence of double-sequences and their properties were studied by Mursaleen [6]. The analog of the asymptotic density definition in natural numbers, called delta density, was provided on the time scale by Seyyidoğlu [7]. This has made it possible to extend the concept of statistical convergence to the time scale. In recent years, many articles have been published about the statistical convergence on time scale [8-11].

One can refer to [12,13] for the concepts related to time scale below. A time scale $\mathbb{T}$ is nonempty closed subset of $\mathbb{R}$. The forward jump operator on $\mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

An interval $[a, b)$ on the time scale $\mathbb{T}$ is

$$[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$$

with $a, b \in \mathbb{T}$. Other intervals are defined similarly. The family $\mathcal{F}$, which consists of the intervals $[a, b)$ of $\mathbb{T}$, is a semiring and the function $m: \mathcal{F} \to [0, \infty)$ defined by $m\big([a, b)\big) = b - a$ is a countably additive measure on $\mathcal{F}$. If we

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define $m^*: 2^{\mathbb{T}} \to [0,\infty]$ by

$$m^*(A) := \inf\left\{\sum_{k=1}^{\infty} m(A_k): A \subset \bigcup_{k=1}^{\infty} A_k\,, A_k \subset \mathcal{F}, k \in \mathbb{N}\right\}$$

then m^* is an outer measure. Furthermore, if we restrict this outer measure to the family of m^*-measurable sets, we get a measure called Lebesgue Δ-measure.

Throughout this article, we will agree that the time scale $\mathbb{T}_i$ is unbounded from above; $a_i = \min \mathbb{T}_i$ and σ_i is forward jump operatör on $\mathbb{T}_i$, $(i = 1,2)$. Also, we set $\mathbb{X} := \mathbb{T}_1 \times \mathbb{T}_2$ and μ is product measure on $\mathbb{X}$.

As defined in [6], a real double sequence $x = (x_{i,j})$ is said to be statistical convergent to a real number L if

$$\lim_{m,n\to\infty} \frac{1}{mn}\left|\left\{(i,j): i \le m, j \le n, \left|x_{i,j}\right| \ge \varepsilon\right\}\right| = 0$$

for each $\varepsilon > 0$.

2. Pringsheim Convergence and Δ-Convergence

The definitions below are necessary for the theorems we will construct.

Definition 2.1. (Pringsheim convergence). A function $f: \mathbb{X} \to \mathbb{R}$ is said to be convergent to L in the Pringsheim's sense on a set $A \subset \mathbb{X}$ if for every $\varepsilon > 0$ there exists $(t_1', t_2') \in A$ such that $|f(t_1, t_2) - L| < \varepsilon$ whenever $t_1 > t_1', t_2 > t_2'$ and $(t_1, t_2) \in A$. In this case, we write $\lim_{(t_1,t_2)\in A} f(t_1, t_2) = L$.

Definition 2.2. (Δ-Density). Let A be a μ-measurable subset of $\mathbb{X}$. Δ-density of A is defined by

$$\delta(A) := \lim_{(t_1,t_2)\in A} \frac{\mu(A \cap I(t_1, t_2))}{\alpha(t_1, t_2)}$$

provided that this limit exists where $I(t_1, t_2) = [a_1, t_1] \times [a_2, t_2]$ and $\alpha(t_1, t_2) = (\sigma_1(t_1) - a_1)(\sigma_2(t_2) - a_2)$.

Definition 2.3. (Δ-Convergence). A μ-measurable function $f: \mathbb{X} \to \mathbb{R}$ is Δ-convergent to the number L provided that for each $\varepsilon > 0$ there exists $K_\varepsilon \subset \mathbb{X}$ such that $\delta(K_\varepsilon) = 1$ and $|f(t_1, t_2) - L| < \varepsilon$ holds for all $(t_1, t_2) \in K_\varepsilon$ and we write $\Delta\text{-}\lim_{(t_1,t_2)} f(t_1, t_2) = L$.

Theorem 2.4. If $f: \mathbb{X} \to \mathbb{R}$ a μ-measurable function then $\Delta\text{-}\lim_{(t_1,t_2)} f(t_1, t_2) = L$ if and only if there exists a subset $K \subset \mathbb{T}$ such that $\delta(K) = 1$ and $\lim_{(t_1,t_2)\in K} f(t_1, t_2) = L$.

Proof. If there exists a set with the mentioned properties and $\varepsilon > 0$ is an arbitrary number, then we can choose a point $(t_1', t_2') \in K$ that for all $t_1 > t_1', t_2 > t_2'$ and $(t_1, t_2) \in K$ we have

$$|f(t_1, t_2) - L| < \varepsilon. \tag{2.1}$$

Δ-density of $K^* := K \cap ([t_1', \infty) \times [t_2', \infty))$ is equal to 1 and the inequality (2.1) holds for all $(t_1, t_2) \in K^*$, it follows that f satisfies $\Delta\text{-}\lim_{(t_1,t_2)} f(t_1, t_2) = L$.

Conversely, suppose that f is Δ-convergent to L. Put

$$K_0 := \mathbb{X}$$

$$K_n := \left\{(t_1, t_2) \in \mathbb{X}: |f(t_1, t_2) - L| < \frac{1}{n}\right\} \ (n = 1,2, \dots)$$

Obviously,

$$K_0 \supset K_1 \supset K_2 \supset \cdots \tag{2.2}$$

and

$$\delta(K_n) = 1 \quad (n = 0,1,\dots) \tag{2.3}$$

hold. According to (2.3) there exist such points $\left(t_1^{(n)}, t_2^{(n)}\right) \in K_n$,

$$t_1^{(n)} > t_1^{(n-1)} > \cdots > t_1^{(1)}$$

$$t_2^{(n)} > t_2^{(n-1)} > \cdots > t_2^{(1)}$$

that for all $t_1 > t_1^{(n)}$ and $t_2 > t_2^{(n)}$ we have

$$\frac{\mu(K_n \cap I(t_1,t_2))}{\alpha(t_1,t_2)} > \frac{n-1}{n}. \tag{2.4}$$

If we define the sets D_n, S_n and K as follows

$$D_n := \left[a_1, t_1^{(n+1)}\right] \times \left[a_2, t_2^{(n+1)}\right] \ (n = 0,1,\dots),$$

$$S_n := D_n - D_{n-1} \ (n = 0,1,\dots) \text{ where } D_{-1} := \emptyset,$$

$$K := \bigcup_{i=0}^{\infty} (S_i \cap K_i),$$

we obtain

$$(K_n \cap I(t_1,t_2) \subset \left(\bigcup_{i=0}^{n-1} (S_i \cap K_i)\right) \cup \left(\left((S_n \cap K_n) \cap I(t_1,t_2)\right)\right)$$

$$= K \cap I(t_1,t_2)$$

when $t_1^{(n)} < t_1 \leq t_1^{(n+1)}$, $t_2^{(n)} < t_2 \leq t_2^{(n+1)}$. From the last inclusion and (2.4) we get

$$\frac{\mu(K \cap I(t_1,t_2))}{\alpha(t_1,t_2)} \geq \frac{\mu(K_n \cap I(t_1,t_2))}{\alpha(t_1,t_2)} > \frac{n-1}{n}$$

for all $(t_1,t_2) \in \mathbb{X}$ such that $t_1^{(n)} < t_1 \leq t_1^{(n+1)}$, $t_2^{(n)} < t_2 \leq t_2^{(n+1)}$. From this we get $\delta(K) = 1$. Finally let us Show that $\lim_{(t_1,t_2)\in K} f(t_1,t_2) = L$. Let ε be an arbitrary positive number. Choose an n such that $\frac{1}{n} < \varepsilon$ and next choose a point $(t_1', t_2') \in K$ such that $t_1' > t_1^{(n)}$, $t_2' > t_2^{(n)}$. According to construction of K there exists a set K_m and $(t_1', t_2') \in K_m$ $(n \leq m)$. Thus

$$|f(t_1,t_2) - L| < \frac{1}{m} \leq \frac{1}{n} < \varepsilon$$

for all $(t_1,t_2) \in K$, $t_1 > t_1'$, $t_2 > t_2'$ which shows that $\lim_{(t_1,t_2)\in K} f(t_1,t_2) = L$. ■

3. $\mathcal{C}_\Delta$ and $\mathcal{W}_p$ Spaces

Let's define the sets of bounded, Δ-convergent and strongly p-Cesàro summable functions $(p > 0)$ consisting of some μ-measurable functions, as follows:

$$\mathcal{B} := \left\{ f : \mathbb{X} \to \mathbb{R} \mid \sup_{(t_1,t_2)\in\mathbb{X}} |f(t_1,t_2)| < \infty \right\},$$

$$\mathcal{C}_\Delta := \left\{ f : \mathbb{X} \to \mathbb{R} \mid \Delta - \lim_{(t_1,t_2)} f(t_1,t_2) = L, \text{for some } L \in \mathbb{R} \right\},$$

$$\mathcal{W}_p := \left\{ f: \mathbb{X} \to \mathbb{R} \mid \lim_{(t_1,t_2)\in\mathbb{X}} \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) = 0, \text{for some } L \in \mathbb{R} \right\}$$

respectively. It is clear that the sets above real vector spaces according to the usual addition and scalar multiplication operations. Obviously, the vector space $\mathcal{B}$ is a Banach space with the respect to the supremum norm. Also, one can obtain the relation $\mathcal{W}_q \subset \mathcal{W}_p$ using the Hölder inequality under the condition $0 < p < q$.

Theorem 3.1. $\mathcal{B} \cap \mathcal{C}_\Delta \subset \mathcal{W}_p$.

Proof. Let ε be an arbitrary positive number. There is a set $K \subset \mathbb{X}$ such that $\delta(K) = 1$ and

$$|f(t_1,t_2)| < \left(\frac{\varepsilon}{2}\right)^{1/p} \tag{3.1}$$

holds at each $(t_1,t_2) \in K$. If we say

$$M := |L| + \sup_{(t_1,t_2)\in\mathbb{X}} |f(t_1,t_2)| , \tag{3.2}$$

then there exists a point (t_1', t_2') in $\mathbb{X}$ such that

$$\frac{\mu(K^c \cap I(t_1,t_2))}{\alpha(t_1,t_2)} < \frac{\varepsilon}{2M^p} \tag{3.3}$$

satisfy for all $t_1 > t_1'$ and $t_2 > t_2'$; thus by (3.1), (3.2) and (3.3)

$$\begin{aligned} \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) &= \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)\cap K} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) \\ &+ \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)\cap K^c} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) \\ &< \frac{\varepsilon}{2\alpha(t_1,t_2)} \mu(I(t_1,t_2)) + M^p \frac{\mu(K^c \cap I(t_1,t_2))}{\alpha(t_1,t_2)} \\ &< \varepsilon. \end{aligned}$$

holds for all $t_1 > t_1'$ and $t_2 > t_2'$. Since $\varepsilon > 0$ arbitrary, the proof is complete. ∎

Theorem 3.2. $\mathcal{W}_p \subset \mathcal{C}_\Delta$.

Proof. Suppose that $\varepsilon > 0$. We define the set

$$K := \{(t_1,t_2) \in \mathbb{X}: |f(t_1,t_2) - L|^p \geq \varepsilon\}.$$

If the limit is taken in the Pringsheim's sense on $\mathbb{X}$ in the following inequality, the desired result is obtained.

$$\begin{aligned} \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) &\geq \frac{1}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)\cap K} |f(\xi,\eta) - L|^p d\mu(\xi,\eta) \\ &\geq \frac{\varepsilon}{\alpha(t_1,t_2)} \int_{I(t_1,t_2)\cap K} d\mu(\xi,\eta) \\ &= \frac{\varepsilon}{\alpha(t_1,t_2)} \mu(K \cap I(t_1,t_2)). \ \blacksquare \end{aligned}$$

Corollary 3.3. $\mathcal{B} \cap \mathcal{W}_p = \mathcal{B} \cap \mathcal{C}_\Delta$.

Proof. It is clear from theorem 3.1 and 3.2. ∎

Theorem 3.4. The set $\mathcal{B} \cap \mathcal{C}_\Delta$ is closed subspace of normed space $\mathcal{B}$ which equipped with the supremum norm.

Proof. Suppose that $f_n \in \mathcal{B} \cap \mathcal{C}_\Delta$, $f_n \to f \in \mathcal{B}$ and $\Delta\text{-}\lim_{(t_1,t_2)} f_n(t_1,t_2) = L_n$. For a given $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ we have

$$\|f_n - f_m\| < \frac{\varepsilon}{3}. \tag{3.4}$$

For each n there is a set $K_n \subset \mathbb{X}$ such that $\delta(K_n) = 1$ and

$$|f_n(t_1,t_2) - L_n| < \frac{\varepsilon}{3} \tag{3.5}$$

holds for all $(t_1,t_2) \in K_n$. From (3.4) and (3.5), we get

$$|L_m - L_n| \le |L_m - f_m(t_1,t_2)| + |f_m(t_1,t_2) - f_n(t_1,t_2)| + |f_n(t_1,t_2) - L_n| < \varepsilon$$

for $m, n > n_0$ and $(t_1,t_2) \in K_n \cap K_m$. This shows that the sequence (L_n) is convergent , say, $L_n \to L$. Hence there is a natural number N such that

$$|L_n - L| \le \frac{\varepsilon}{3} \tag{3.6}$$

and

$$|f_N(t_1,t_2) - f(t_1,t_2)| < \frac{\varepsilon}{3} \quad \text{for } (t_1,t_2) \in \mathbb{X}. \tag{3.7}$$

Thus from (3.5), (3.6) and (3.7) we obtain

$$|f(t_1,t_2) - L| \le |f(t_1,t_2) - f_N(t_1,t_2)| + |f_N(t_1,t_2) - L_n| + |L_n - L| < \varepsilon$$

for $(t_1,t_2) \in K_n$. Therefore, $f \in \mathcal{B} \cap \mathcal{C}_\Delta$. This shows that the set $\mathcal{B} \cap \mathcal{C}_\Delta$ is closed subspace of $\mathcal{B}$. ■

Mursaleen's Theorem [6], which states that the space of bounded and statistically convergent double sequences is not dense in the space of bounded double sequences, is a special case of the following theorem:

Theorem 3.5. The set $\mathcal{B} \cap \mathcal{C}_\Delta$ is nowhere dense in $\mathcal{B}$.

Proof. The proof is based on the fact that a closed proper subspace of a normed space is nowhere dense in that space. Therefore, considering Theorem 3.4, it is sufficient to show that $\mathcal{B} \cap \mathcal{C}_\Delta \ne \mathcal{B}$. We can choose the points $t_1^{(1)} < t_1^{(2)} < \cdots$ and $t_2^{(1)} < t_2^{(2)} < \cdots$ such that

$$\frac{\mu\left(\left(\bigcup_{k=1}^{n} A_k\right) \cap I\left(t_1^{(2n-1)}, t_2^{(2n-1)}\right)\right)}{\alpha\left(t_1^{(2n-1)}, t_2^{(2n-1)}\right)} > \frac{3}{4}$$

$$\frac{\mu\left(\left(\bigcup_{k=1}^{n} A_k\right) \cap I\left(t_1^{(2n)}, t_2^{(2n)}\right)\right)}{\alpha\left(t_1^{(2n)}, t_2^{(2n)}\right)} < \frac{1}{4}$$

are satisfied, where

$$A_1 \coloneqq I\left(t_1^{(1)}, t_2^{(1)}\right)$$

$$A_k \coloneqq I\left(t_1^{(2k-1)}, t_2^{(2k-1)}\right) - I\left(t_1^{(2k-2)}, t_2^{(2k-2)}\right) \quad (k \ge 2).$$

Thus, Δ-density of the set $A \coloneqq \bigcup_{k=1}^{\infty} A_k$ does not exist. So, the function $f: \mathbb{X} \to \mathbb{R}$ defined as

$$f(t_1,t_2) \coloneqq \begin{cases} 1 & (t_1,t_2) \in A \\ 0 & (t_1,t_2) \notin \mathbb{X} - A \end{cases}$$

is not Δ-convergent. ■

Ethics Consent

Ethics committee approval is not required for this study.

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