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Upper Bounds on the Domination and Total Domination Number of Fibonacci Cubes

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Abstract: One of the basic model for interconnection networks is the n-dimensional hypercube graph Q_n and the vertices of Q_n are represented by all binary strings of length n. The Fibonacci cube Γ_n of dimension n is a subgraph of Q_n, where the vertices correspond to those without two consecutive 1s in their string representation. In this paper, we deal with the domination number and the total domination number of Fibonacci cubes. First we obtain upper bounds on the domination number of Γ_n for $n \geq 13$. Then using these result we obtain upper bounds on the total domination number of Γ_n for $n \geq 14$ and we see that these upper bounds improve the bounds given in [1].

Fibonacci Küplerinin Baskınlık ve Toplam Baskınlık Sayıları için Üst Sınırlar

Anahtar Kelimeler
Fibonacci küp,
Baskınlık sayısı,
Toplam baskınlık sayısı

Özet: Bağlantı ağları için en temel modellerden biri n-boyutlu hiperküp grafı Q_n dir ve Q_n nin köşeleri boyu n olan tüm ikilik diziler ile temsil edilir. n-boyutlu Fibonacci küpü Γ_n, Q_n nin bir alt grafıdır ve köşeleri, ikilik dizi gösterimlerinde ardışık 1 içermeyen tüm köşelere karşı gelir. Bu çalışmada, Fibonacci küplerinin baskınlık ve toplam baskınlık sayıları ile ilgilendik. Öncelikle, $n \geq 13$ olmak üzere Γ_n nin baskınlık sayısı için üst sınırlar elde ettik. Sonrasında bu sonuçları kullanarak $n \geq 14$ olmak üzere Γ_n nin toplam baskınlık sayısı için üst sınırlar bulduk ve bu sınırların [1] de verilen üst sınırları geliştirdiğini gördük.

1. Introduction

An interconnection network can be represented by a graph $G = \{V, E\}$ with vertex set V and edge set E. In this representation, V denotes the processors and E denotes the communication links between processors. One of the basic model for interconnection networks is the n-dimensional hypercube graph Q_n. The vertices of Q_n are represented by all binary strings of length n and two vertices are adjacent if and only if they differ in exactly one position. The n-dimensional Fibonacci cube Γ_n is a subgraph of Q_n, where the vertices correspond to those without two consecutive 1s in their string representation. For convenience, Γ_0 is defined as Q_0, the graph with a single vertex and no edges. Γ_n is also used a model of computation for interconnection networks [2].

In literature many interesting properties of Γ_n exist. Their usage in theoretical chemistry and some results on the structure of Fibonacci cubes, including representations, recursive construction, hamiltonicity, the nature of the degree sequence and some enumeration results are presented in [3]. Characterization of induced hypercubes in Γ_n are considered in [4–8] and many additional new properties of Fibonacci cubes are given in the literature, see for example [9–11]. Furthermore, the domination number (see, Section 2) of Γ_n is first considered in [12, 13]. A lower bound for the domination number and its exact values for $n \leq 8$ is presented in [12]. In [13], upper and lower bounds for the domination number of Γ_n are obtained and a comparison with the domination number of Lucas cubes is given. In [14], an integer programming method is used to compute the exact values of the domination number of Γ_n for $n \leq 10$ and then this technique is used in [1] to obtain the exact value of domination for $n = 11$. Also in [1] total domination number of a graph is defined and upper and lower bounds are obtained. In addition, using integer programming method exact values for total domination number is obtained for $n \leq 12$.

In this paper, we obtain some upper bounds on the domination number of Γ_n for $n \geq 13$. Then using the fundamental decomposition of Γ_n we obtain an upper bound on the total domination number of Γ_n for $n \geq 14$. We compare our result with the ones in [1] in Table 2 and see that our bounds are better.

2. Material and Method

Let $G = \{V(G), E(G)\}$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Then $D \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)$ either belongs to D or is adjacent to

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some vertex in D. The *domination number* $\gamma(G)$ is defined as the minimum cardinality of a dominating set of the graph G. Similarly, $D \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to some vertex in D and the *total domination number* $\gamma_t(G)$ is defined as the minimum cardinality of a total dominating set of G.

An n-dimensional hypercube (or n-cube) Q_n is the simple graph whose vertices are represented by all binary strings of length n and there is an edge between two vertices if and only if they differ in exactly one position. That is,

$$\begin{aligned} V(Q_n) &= \{v_1v_2\cdots v_n \mid v_i \in \{0,1\},\ 1 \le i \le n\} \text{ and} \\ E(Q_n) &= \{(u,v) \mid u,v \in V(Q_n), d_H(u,v) = 1\}, \end{aligned}$$

where $d_H(u,v)$ denotes the Hamming distance between u and v, that is, the number of different positions in u and v. The number of vertices in Q_n is 2^n and each of them has degree n, thus $|E(Q_n)| = n2^{n-1}$. It is known that the number of all binary sequences having length n without two consecutive 1s is enumerated by the Fibonacci numbers. For this reason Fibonacci cube Γ_n can be obtained from Q_n by removing all vertices containing consecutive 1s in its string representation. The number of vertices of the Fibonacci cube Γ_n is F_{n+2}, where F_n is the usual Fibonacci numbers defined as $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Here we remark that the generating function of F_n (see, for example [15]) is

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}. \qquad (1)$$

Note that Fibonacci cubes can also be defined recursively using the recursive relation of Fibonacci numbers. It is called as the fundamental decomposition of Γ_n in [3]. We can describe it as follows:

Γ_n can be decomposed into the subgraphs induced by the vertices that start with 0 and 10 respectively. Let $V(\Gamma_n) = A_n \cup B_n$ where

$$A_n = \{1v \mid v \in B_{n-1}\} \quad \text{and} \quad B_n = \{0v \mid v \in A_{n-1} \cup B_{n-1}\}$$

with $A_0 = \emptyset$ and $B_0 = \{\varepsilon \mid \varepsilon \text{ is the empty string}\}$. Note that for $n \geq 2$ any vertex in A_n must start with 10. Therefore, for $n \geq 2$ the vertices in B_n will constitute a graph isomorphic to Γ_{n-1} and the vertices in A_n will constitute a graph isomorphic to Γ_{n-2}. We will show this fundamental decomposition for $n \geq 2$ as

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}\ . \qquad (2)$$

Here note that $0\Gamma_{n-1}$ has a subgraph isomorphic to $00\Gamma_{n-2}$, and there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. In the following section we will use the decomposition (2) to obtain upper bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

We present first 6 Fibonacci cubes with minimum dominating sets in Figure 1 (those vertices having circles around). Furthermore, we present minimum total dominating sets of $\Gamma_1, \ldots, \Gamma_5$ in Figure 2.

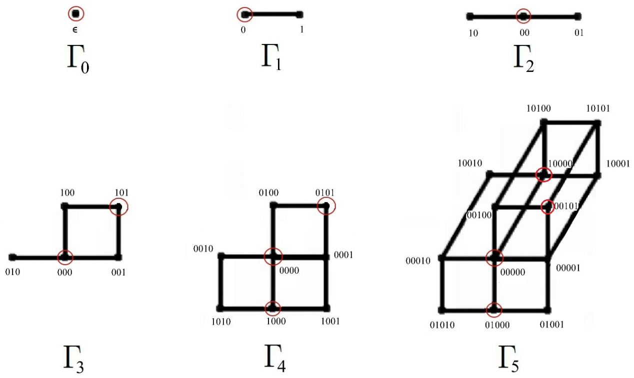

Figure 1. $\Gamma_0, \ldots, \Gamma_5$ and their dominating sets.

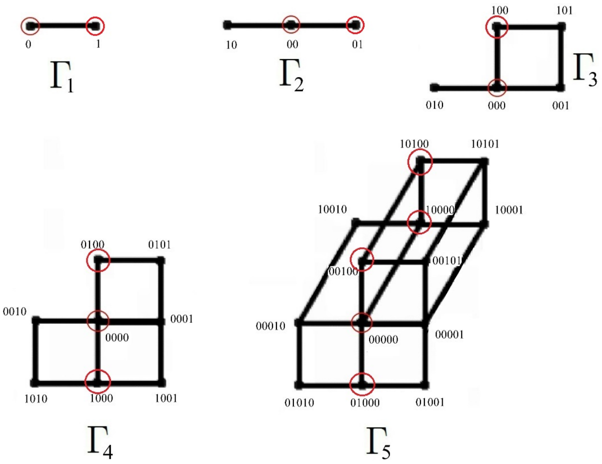

Figure 2. $\Gamma_1, \ldots, \Gamma_5$ and their total dominating sets.

3. Results

In this section, we will obtain upper bounds on the domination number $\gamma(\Gamma_n)$ and total domination number $\gamma_t(\Gamma_n)$ of Fibonacci cube Γ_n. We start by presenting some known values of these numbers in Table 1, which was given in [1, 14].

Table 1. Known values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

n	$\lvert V(\Gamma_n)\rvert$	$\gamma(\Gamma_n)$	$\gamma_t(\Gamma_n)$
1	2	1	2
2	3	1	2
3	5	2	2
4	8	3	3
5	13	4	5
6	21	5	7
7	34	8	10
8	55	12	13
9	89	17	20
10	144	25	30
11	233	39	44
12	377	54-61	65
13	610	?	97-101

For any graph G of minimum degree δ it is known that [16, 17]

$$\gamma(G) \leq \frac{|V(G)|}{\delta+1} \sum_{j=1}^{\delta+1} \frac{1}{j}\ .$$

Then using $\delta(\Gamma_n) = \lfloor \frac{n+2}{3} \rfloor$ (see, [18]) one can write

$$\gamma(\Gamma_n) \le \frac{F_{n+2}}{\lfloor \frac{n+5}{3} \rfloor} \sum_{j=1}^{\lfloor \frac{n+5}{3} \rfloor} \frac{1}{j}, \tag{3}$$

and using $\gamma_t \le 2\gamma$ it is given in [1] that

$$\gamma_t(\Gamma_n) \le \frac{2F_{n+2}}{\lfloor \frac{n+5}{3} \rfloor} \sum_{j=1}^{\lfloor \frac{n+5}{3} \rfloor} \frac{1}{j}. \tag{4}$$

By using the fundamental decomposition (2) of Γ_n, the following upper bound on $\gamma_t(\Gamma_n)$ is obtained in [1].

Theorem 3.1. *[1] If $n \ge 11$, then $\gamma_t(\Gamma_n) \le 2F_{n-10} + 21F_{n-8}$.*

Note that, in [1] by using integer programming it is shown that $54 \le \gamma(\Gamma_{12}) \le 61$ and $97 \le \gamma_t(\Gamma_{13}) \le 101$.

Remark 3.2. By using the fact that $\gamma_t(\Gamma_{13}) \le 101$, it is stated in [1] that for $n \ge 12$ Theorem 3.1 can be further improved to $\gamma_t(\Gamma_n) \le 601F_{n-1} - 371F_n$. But this better result is not presented in [1, Table 2]. We will consider this result in our Table 2.

In the following theorem, using the fundamental decomposition (2) of Γ_n and the best known results in Table 1 we present an upper bound for $\gamma(\Gamma_n)$ which includes Fibonacci numbers.

Theorem 3.3. *If $n \ge 13$, then $\gamma(\Gamma_n) \le 116F_n - 187F_{n-1}$.*

Proof. If we consider the fundamental decomposition (2) of Γ_n we have

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$$

which means that it is enough to find minimum dominating sets for $0\Gamma_{n-1}$ and $10\Gamma_{n-2}$. Thus,

$$\gamma(\Gamma_n) \le \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2}). \tag{5}$$

We know that $\gamma(\Gamma_{11}) = 39$ and $\gamma(\Gamma_{12}) \le 61$. Now set $b_{11} = 39$, $b_{12} = 61$ and $b_n = b_{n-1} + b_{n-2}$ for $n \ge 13$. Then using (5) one can easily see that $\gamma(\Gamma_n) \le b_n$ for $n \ge 11$.

Now, let $S = \sum_{n\ge 0} b_{n+11}x^n$ be the generating function of the sequence b_n. We know that $b_{11} = 39$, $b_{12} = 61$ and $b_n = b_{n-1} + b_{n-2}$ for $n \ge 13$. Therefore S satisfies

$$S - 39 - 61x = x(S - 39) + x^2 S$$

which gives

$$S = \frac{39 + 22x}{1 - x - x^2}.$$

Then using (1) we obtain that $b_{n+11} = 39F_{n+1} + 22F_n$ for $n \ge 0$, which is equivalent to

$$b_n = 116F_n - 187F_{n-1}$$

for all $n \ge 11$. □

Using the fundamental decomposition of Γ_n we obtain the following result.

Lemma 3.4. *If $n \ge 14$, then $\gamma_t(\Gamma_n) \le 2\gamma(\Gamma_{n-2}) + \gamma_t(\Gamma_{n-3})$.*

Proof. We know that $\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$ and $\Gamma_{n-1} = 0\Gamma_{n-2} + 10\Gamma_{n-3}$. Then we can write

$$\begin{aligned} \Gamma_n &= 0(0\Gamma_{n-2} + 10\Gamma_{n-3}) + 10\Gamma_{n-2} \\ &= 00\Gamma_{n-2} + 010\Gamma_{n-3} + 10\Gamma_{n-2} \end{aligned} \tag{6}$$

where there is a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. This matching guarantees that union of the dominating sets of $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ becomes a total dominating set for $00\Gamma_{n-2} \cup 10\Gamma_{n-2}$. Therefore, to find a total dominating set for Γ_n, we only need to find a total dominating set for $010\Gamma_{n-3}$ by (6). Hence we have

$$\gamma_t(\Gamma_n) \le 2\gamma(\Gamma_{n-2}) + \gamma_t(\Gamma_{n-3}).$$

□

Here we note that if one will obtain better upper bounds for $\gamma_t(\Gamma_n)$ then using Lemma 3.4 the bounds can be further improved for some cases. Now, using Theorem 3.3, Lemma 3.4 and the fundamental decomposition of Γ_n more than once we obtain the following result. We note that Theorem 3.5 presents an upper bound for the total domination number of Γ_n, using the values of domination number of Γ_{n-3} and Γ_{n-4}.

Theorem 3.5. *If $n \ge 15$, then $\gamma_t(\Gamma_n) \le 3\gamma(\Gamma_{n-3}) + 2\gamma(\Gamma_{n-4}) \le 116F_n - 187F_{n-1}$.*

Proof. We first apply the fundamental decomposition to (6) one more time and obtain that

$$\begin{aligned} \Gamma_n &= 000\Gamma_{n-3} + 010\Gamma_{n-3} + 100\Gamma_{n-3} \\ &\quad + 0010\Gamma_{n-4} + 1010\Gamma_{n-4} \end{aligned}$$

where there are perfect matchings between $000\Gamma_{n-3}$ and $010\Gamma_{n-3}$; $000\Gamma_{n-3}$ and $100\Gamma_{n-3}$; and $0010\Gamma_{n-4}$ and $1010\Gamma_{n-4}$. These matchings guarantees that the union of the dominating sets becomes a total dominating set for Γ_n. Then we get

$$\gamma_t(\Gamma_n) \le 3\gamma(\Gamma_{n-3}) + 2\gamma(\Gamma_{n-4}). \tag{7}$$

Then by using the sequence b_n defined in Theorem 3.3 and (7) for $n \ge 15$ we have

$$\gamma_t(\Gamma_n) \le 3b_{n-3} + 2b_{n-4} = b_n$$

which completes the proof. □

Remark 3.6. It is noted in [1] that the bound obtained in Theorem 3.1 is better than the bound in (4) for $n \le 33$. By using the properties of Fibonacci numbers, it is clear that our upper bound in Theorem 3.3 is better than (3) and moreover the upper bound in Theorem 3.5 is better than the bounds in (4), Theorem 3.1 and Remark 3.2.

We collect our results in Table 2 in which we present the known upper bounds of $\gamma_t(\Gamma_n)$ for $n \ge 13$. We include the upper bounds given in [1] (see Theorem 3.1) and Remark 3.2 in the second and third column respectively, and our results obtained in Lemma 3.4 and Theorem 3.5 in the last column of Table 2. One can easily see that our bounds are better.

Table 2. Known upper bounds on $\gamma_t(\Gamma_n)$ for $13 \leq n \leq 33$.

	Theorem 3.1[1]	Remark 3.2	Our results
n	$\gamma_t(\Gamma_n) \leq$	$\gamma_t(\Gamma_n) \leq$	$\gamma_t(\Gamma_n) \leq$
13	101	101	101
14	174	166	166
15	283	267	261
16	457	433	422
17	740	700	683
18	1197	1133	1105
19	1937	1833	1788
20	3134	2966	2893
21	5071	4799	4681
22	8205	7765	7574
23	13276	12564	12255
24	21481	20329	19829
25	34757	32893	32084
26	56238	53222	51913
27	90995	86115	83997
28	147233	139337	135910
29	238228	225452	219907
30	385461	364789	355817
31	623689	590241	575724
32	1009150	955030	931541
33	1632839	1545271	1507265

4. Discussion and Conclusion

In this paper we deal with the domination number and total domination number of Fibonacci cubes. We obtain some upper bounds on the domination number of Γ_n for $n \geq 13$. Then using the fundamental decomposition of Γ_n we obtain the best known upper bounds on the total domination number of Γ_n for $n \geq 14$ and observe that these upper bounds for the domination number and total domination number of Γ_n coincides for $n \geq 15$. We compare our result with the ones in [1] in Table 2 and see that our bounds are better.

Finally we remark that if one obtains better upper bounds for the domination number or total domination number of Γ_n, then by combining these bounds with our results many improved bounds can be obtained.

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