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On a Simple Characterization of Conformally Flat 4-Dimensional Spaces of Neutral Signature

4–Boyutlu Nötr Metrik İşaretli Konformal Düz Uzayların Basit Bir Karakterizasyonu Üzerine

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Abstract
In this work, the full characterization of 4-dimensional conformally flat spaces of neutral signature is given by using methods based on holonomy structure. Possible holonomy types are obtained for the spaces in question and several remarks are made. Various examples are presented related to this investigation.
Keywords: Weyl conformal curvature tensor, Conformally flat space, Neutral signature, Holonomy

I. INTRODUCTION

Conformally flat spaces are one of the important research topics not only in differential geometry but also in physics. Such spaces are commonly used in general relativity, for example, describing Friedmann–Lemaître–Robertson–Walker metric. The classification of conformally flat spaces has been a subject of interest for many years and so it is not possible to mention about all these works (among them see, e.g., [1-9]). Our goal in the present study is to provide a systematic examination of holonomy types in $4-$dimensional conformally flat spaces admitting a metric of signature $(+,+,-,-)$ referred to as *neutral signature*. Such studies for Lorentz and positive definite signatures have been carried out by Hall and Lonie in [3] and by Hall in [4], respectively. Adding to these works, the present paper is intended to explore the problem further for neutral signature which is the most complicated metric signature as there are orthogonal null vectors that are independent and there are pairs of timelike $2-$spaces that are orthogonal. The holonomy algebras for $4-$dimensional spaces of neutral signature are known and the labelings given in [10] will be used in the present work.

The rest of the paper is organized as in the following: Some basic notions of the study and the description of holonomy groups of $4-$dimensional spaces of signature $(+,+,-,-)$ are recalled in Section II for further consideration. The main result of the paper is presented in Section III. To illustrate the results of the study, several examples are given in Section IV. Finally, further remarks and conclusion are discussed in Section V.

II. PRELIMINARIES

Let $\mathcal{M}$ be a $4-$dimensional manifold such that it is smooth, connected, simply connected and equipped with a metric g being smooth and having neutral signature $(+,+,-,-)$. The Levi-Civita connection of g will be symbolised by ∇. Let the notations W, $Riem$, $Ricc$ and r respectively represent the Weyl conformal curvature tensor, Riemann curvature tensor, Ricci tensor and scalar curvature of $\mathcal{M}$. We shall assume that $Riem$ is not identically zero on $\mathcal{M}$ by saying that the structure $(\mathcal{M}, g)$ is not flat. The Weyl curvature tensor of type $(0,4)$ is given by

$$W_{ijkh} = R_{ijkh} + \frac{1}{2}(g_{jk}R_{ih} - g_{jh}R_{ik} + g_{ih}R_{jk} - g_{ik}R_{jh}) + \frac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}), \quad \textbf{(1)}$$

where W_{ijkh}, $R_{ijkh} \equiv g_{im}R^{m}{}_{jkh}$ and $R_{ik} \equiv R^{j}{}_{ijk}$ are the components of W, $Riem$ and $Ricc$, respectively. If W vanishes in $\dim\mathcal{M} \geq 4$, then $(\mathcal{M}, g)$ is named as *conformally flat* meaning that there is a neighbourhood $\mathcal{V}$ of each $p \in \mathcal{M}$ on which g, restricted to $\mathcal{V}$ is conformal to a flat metric.

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If $\mathcal{M}$ is Ricci-flat, in other words, if $Ricc \equiv 0$ on $\mathcal{M}$, one can observe from Equation (1) that $W \equiv Riem$. For this reason, we shall study conformally flat spaces provided that $\mathcal{M}$ is not Ricci-flat. Let $\mathcal{T}_p\mathcal{M}$ be the tangent space of $\mathcal{M}$ at p and $v \cdot \omega$ be the inner product of tangent vectors $v, \omega \in \mathcal{T}_p\mathcal{M}$. A non-zero tangent vector $\omega \in \mathcal{T}_p\mathcal{M}$ is named as *spacelike*, *timelike*, *null* (or *lightlike*) if $\omega \cdot \omega > 0$, $\omega \cdot \omega < 0$, $\omega \cdot \omega = 0$ hold, respectively. For neutral metric, we can set up a (pseudo)-orthonormal basis of $\mathcal{T}_p\mathcal{M}$ shown as $\{x, y, s, t\}$ which satisfies $x \cdot x = y \cdot y = 1$ (x, y being spacelike) and $s \cdot s = t \cdot t = -1$ (s, t being timelike). Besides, an associated null basis $\{l, n, L, N\}$ can be constructed in the way that $l = \frac{1}{\sqrt{2}}(x + t)$, $n = \frac{1}{\sqrt{2}}(x - t)$, $L = \frac{1}{\sqrt{2}}(y + s)$, $N = \frac{1}{\sqrt{2}}(y - s)$ where the relations $l \cdot n = 1$ and $L \cdot N = 1$ are satisfied and the other products between these null vectors are all zero. Moreover, a $2-$dimensional subspace $\mathcal{U}$, named as a $2-$space, of $\mathcal{T}_p\mathcal{M}$ can be classified by the following items: (i) *spacelike*: every non-zero element of $\mathcal{U}$ is timelike or every non-zero element of $\mathcal{U}$ is spacelike (ii) *timelike*: $\mathcal{U}$ involves precisely two, null $1-$dimensional subspaces that are referred to as directions (iii) *null*: $\mathcal{U}$ involves precisely one null direction (iv) *totally null*: every non-zero element of $\mathcal{U}$ must be null and so any two non-zero elements must be orthogonal. Now, let $\Lambda_p\mathcal{M}$ be the space of all bivectors at the point p. A non-zero bivector F with components $F^{ij} (= -F^{ji})$ has even rank which can be either 2 or 4. The case when the rank of F equals 2, it is referred to as a *simple bivector* and in the other case, it is named as a *non-simple bivector*. In the former case it can be expressed as $F^{ij} = 2v^{[i}\omega^{j]}$ for $v, \omega \in \mathcal{T}_p\mathcal{M}$ in which square brackets surrounding indices are used to indicate the ordinary anti-symmetrisation of the indices. In that case, the *blade* of F, which will be written as $v \wedge \omega$, is the $2-$space generated by $v, \omega \in \mathcal{T}_p\mathcal{M}$. If the blade of F is *spacelike* (in order of, *timelike*, *null* or *totally null*) $2-$space at p as defined above, then (the simple bivector) F is named as *spacelike* (in order of, *timelike*, *null* or *totally null*). Note that F is simple including $0 \neq v \in \mathcal{T}_p\mathcal{M}$ in its blade provided that $F_{[ij}v_{h]} = 0$.

It is also remarked that one can define the *curvature map* denoted by $\tilde{f}: \Lambda_p\mathcal{M} \to \Lambda_p\mathcal{M}$ given by $F \to R^i{}_{jkh}F^{kh}$, [11]. On the other hand, it is advantageous to express the decomposition of $Riem$ as follows (see, e. g., [11]):

$$R_{ijkh} = W_{ijkh} + E_{ijkh} + \frac{r}{6}G_{ijkh} \qquad \textbf{(2)}$$

where

$$E_{ijkh} = \tilde{R}_{i[k}g_{h]j} + \tilde{R}_{j[h}g_{k]i}, \quad G_{ijkh} = g_{i[k}g_{h]j},$$
$$\tilde{R}_{ij} = R_{ij} - \frac{r}{4}g_{ij} = E^m{}_{imj}. \qquad \textbf{(3)}$$

It is clear from Equation (3) that the tensor with components are $\tilde{R}_{ij}$ is tracefree and $E = 0 \Leftrightarrow \tilde{R} = 0 \Leftrightarrow R_{ij} = \frac{r}{4}g_{ij}$ meaning that $(\mathcal{M}, g)$ is an *Einstein manifold*.

Let Φ be the holonomy group of $(\mathcal{M}, g)$ which is a Lie group formed by the collection of all linear isomorphisms on $\mathcal{T}_p\mathcal{M}$ arising from the parallel transfer of each tangent vector of $\mathcal{T}_p\mathcal{M}$ around a smooth, closed curve c at p (for details on holonomy group, see [12]).

When the metric is of neutral signature, the Lie algebra of Φ is subalgebra of $o(2,2)$, which will be denoted by ϕ. By using the matrix characterization of this algebra, one achieves a bivector representation of ϕ for this signature. The labelings that were tabulated in [10] are utilized in Table 1 containing exactly 23 types. All these holonomy types are shown in columns 1 and 3 together with the generators in bivector representation respectively indicated in columns 2 and 4. The dimension of each holonomy type can easily be seen from its label. Note that these holonomy algebras are the ones arising for a metric connection and they are not all of the subalgebras of $o(2,2)$.

It is noted that one can define $3-$dimensional subspaces of $\Lambda_p\mathcal{M}$ denoted by $\overset{+}{S} = \{F \in \Lambda_p\mathcal{M}: \overset{*}{F} = F\}$ and $\overset{-}{S} = \{F \in \Lambda_p\mathcal{M}: \overset{*}{F} = -F\}$ where $*$ is the Hodge duality operator. The dual of $F \in \Lambda_p\mathcal{M}$, shown as $\overset{*}{F}$, is described by $\overset{*}{F}_{ij} = \frac{1}{2}\epsilon_{ijkh}F^{kh}$ with $\epsilon_{ijkh} = \sqrt{\det g}\,\delta_{ijkh}$ being the classical pseudo-tensor, δ being the standard alternating symbol. One has $\overset{**}{F} = F$ for neutral signature. In Table 1, $\overset{+}{B} = \langle l \wedge n - L \wedge N, l \wedge N \rangle$, $\overset{-}{B} = \langle l \wedge n + L \wedge N, l \wedge L \rangle$ where the symbol $\langle\ \rangle$ denotes a spanning set. Furthermore, $\eta, \zeta \in \mathbb{R}$ and $\eta \neq \pm\zeta$ are valid for types $2(h)$ and $3(d)$ whilst for type $2(j)$, both of them are non-zero. A basis of $\overset{+}{S}$ is $\{l \wedge N, l \wedge n - L \wedge N, n \wedge L\}$.

An essential concept in the theory of holonomy is the Ambrose-Singer theorem [13] which states that if one fixes $p \in \mathcal{M}$ and for arbitrary $p' \in \mathcal{M}$ calculates the range space of the curvature map, $Rg(\tilde{f})$, and parallel transports the range space to p throughout a curve $\alpha: p' \to p$ and carry on doing this for every p' and α, the collection of bivectors acquired at p generates ϕ.

Table 1. Holonomy types relevant for neutral signature are indicated

Type	Generators	Type	Generators
$1(a)$	$l \wedge n$	$2(j)$	$l \wedge N,\ \eta(l \wedge n - L \wedge N) + \zeta(l \wedge L)$
$1(b)$	$x \wedge y$	$2(k)$	$l \wedge y, l \wedge n$ (or $l \wedge s, l \wedge n$)
$1(c)$	$l \wedge y$ (or $l \wedge s$)	$3(a)$	$l \wedge N, l \wedge n, L \wedge N$
$1(d)$	$l \wedge N$	$3(b)$	$l \wedge N, l \wedge n - L \wedge N, l \wedge L$
$2(a)$	$\overset{+}{B}$	$3(c)$	$x \wedge y, x \wedge t, y \wedge t$ (or $x \wedge s, x \wedge t, s \wedge t$)
$2(b)$	$l \wedge n, L \wedge N$	$3(d)$	$l \wedge N, l \wedge L, \eta(l \wedge n) + \zeta(L \wedge N)$
$2(c)$	$l \wedge n - L \wedge N, l \wedge L + n \wedge N$	$4(a)$	$\overset{+}{S}, l \wedge n + L \wedge N$
$2(d)$	$l \wedge n - L \wedge N, l \wedge L$	$4(b)$	$\overset{+}{S}, l \wedge L + n \wedge N$
$2(e)$	$x \wedge y, s \wedge t$	$4(c)$	$\overset{+}{B}, \overset{-}{B}$
$2(f)$	$l \wedge N + n \wedge L, l \wedge L$	5	$\overset{+}{S}, \overset{-}{B}$
$2(g)$	$l \wedge N, l \wedge L$	6	$o(2,2)$
$2(h)$	$l \wedge N,\ \eta(l \wedge n) + \zeta(L \wedge N)$		

Then, $Rg(\tilde{f})$ is a subspace of ϕ and the Riemann curvature tensor may always be expressed as a symmetrized sum of products of bivectors of ϕ (see, [11]).

Finally, it will be useful to give a remark on parallel and recurrent vector fields. A vector field v is said to be *recurrent* on an open and connected subset $\mathcal{U} \neq \emptyset$ of $\mathcal{M}$ if $\nabla v = q \otimes v$ for some $1-$form q. If q vanishes on $\mathcal{U}$, in other words, the case when $\nabla v = 0$ on $\mathcal{U}$, v is named as *parallel* on $\mathcal{U}$. If v is parallel, it is either non-null everywhere or null everywhere.

In the sense of holonomy theory, if $0 \neq v \in \mathcal{T}_p\mathcal{M}$ is an eigenvector of *all* bivectors of ϕ, then on some neighbourhood of $p \in \mathcal{M}$, there exists a smooth vector field which is recurrent and whose value at p is v (for details, see, e.g., [11]). Further, if every eigenvalue of v is zero for all $F \in \phi$, then it is eligible as a parallel vector field. Therefore, as $\mathcal{M}$ is simply connected, all recurrent and parallel vector fields (if any) can be detected by considering Table 1 for every holonomy type (see [14]).

For instance, for holonomy type $3(c)$ with its one of the generators presented in Table 1, it can be checked that $\nabla s = 0$ (or $\nabla y = 0$), in other words, s (or y) causes a parallel vector field whilst for type $4(c)$, l turns out a recurrent vector field on $\mathcal{U}$.

III. THE MAIN RESULT

Let us now look for the potential holonomy types for conformally flat spaces with a metric of neutral signature. Assume that $(\mathcal{M}, g)$ is conformally flat (but it is neither flat nor Ricci-flat). For this case, one gets from Equation (1) that

$$R_{ijkh} = \tfrac{1}{2}(g_{jh}R_{ik} - g_{ih}R_{jk} + g_{ik}R_{jh} - g_{jk}R_{ih}) - \tfrac{r}{6}(g_{ik}g_{jh} - g_{ih}g_{jk}). \quad \textbf{(4)}$$

First of all, assume that $\mathcal{M}$ contains a non-zero parallel vector field v. In this case, the Ricci identity implies that $R_{ijkh}v^h = 0$ and thus, $R_{jh}v^h = 0$. Contracting Equation (4) by $v^j v^h$ and using the Ricci identity, we obtain

$$\tfrac{1}{2}R_{ik}v_h v^h + \tfrac{r}{6}(v_i v_k - g_{ik}v_h v^h) = 0. \quad \textbf{(5)}$$

Case 1: If v is null, then we get from Equation (5) that $r = 0$ and vice versa. Contracting Equation (4) by v^h, one can get the following equation:

$$R_{ik}v_j = R_{jk}v_i. \quad \textbf{(6)}$$

The condition (6) is equivalent to $Ricc = \lambda(v \otimes v)$ for some nowhere zero function $\lambda: \mathcal{M} \to \mathbb{R}$, where the Segre type of $Ricc$ is $\{(211)\}$ *with zero eigenvalue* (for the Segre classification, see, e. g., [11, 14]). As $r = 0$, we also obtain from Equation (2) that $Riem = E$. Moreover, as the tensor E has the duality properties ${}^{*}E_{ijkh} = -E^{*}_{ijkh}$, we have ${}^{*}R_{ijkh} = -R^{*}_{ijkh}$. It then follows that if a bivector F is in the range of the curvature map, so is its dual F^{*}. Plugging $Ricc$ into Equation (4), a direct computation shows that $Riem$ can be written in terms of a pair of totally null bivectors whose blades contain v. More explicitly, for that case the causal character of the bivectors in $Riem$ is preserved by parallel transporting them from each $p' \in \mathcal{M}$ to a fixed point $p \in \mathcal{M}$ and considering the Ambrose-Singer theorem (see Section II). Moreover, since v is parallel, it remains as null under parallel translation and stays in the blades of these bivectors. By the aid of Table 1, it can be seen that Φ must be of type $2(g)$ admitting l as a parallel vector field as this type is generated by the bivectors $l \wedge N$ and $l \wedge L$ for both of which l is an eigenvector corresponding to the zero eigenvalue.

Case 2: Assume that v is non-null. Then $v_k v^k \neq 0$ and it can be deduced from Equation (5) that $Ricc$ takes the following form:

$$R_{ij} = \frac{r}{3}\left(g_{ij} - \frac{1}{v_k v^k} v_i v_j\right) \qquad \textbf{(7)}$$

where the Segre type of $Ricc$ is $\{1(111)\}$ and $r \neq 0$. Let F be an arbitrary bivector at p satisfying the condition $F_{ij} v^j = 0$. In this case, putting Equation (7) into (4) and multiplying the resulting equation by F^{kh}, we get that $R_{ijkh} F^{kh} = \frac{r}{3} F_{ij}$. It then follows that F is an eigenbivector of the Riemann curvature tensor corresponding to eigenvalue $r/3$. Moreover, such bivectors must be simple and they form a $3-$dimensional subspace of $\Lambda_p \mathcal{M}$ such that the dual subspace to it is generated by three bivectors each of which annihilates $Riem$ and that they are independent, simple, and v is in their blades. This yields that the rank of $Riem$ is 3 at p. Moreover, as the holonomy admits a parallel vector field v which is non-null, the dimension of the holonomy algebra is at most 3. Combining these findings, it is achieved that $\dim\phi = 3$. Hence, it can be observed from Table 1 that Φ must be of type $3(c)$ as for this type the basis members, as being simple bivectors, are $F_1 \equiv x \wedge y$, $F_2 \equiv x \wedge t$, $F_3 \equiv y \wedge t$ (or $G_1 \equiv x \wedge s$, $G_2 \equiv x \wedge t$, $G_3 \equiv s \wedge t$) and they have a common annihilator s (or y) which is timelike (or spacelike) and parallel. Note that if $Ricc$ takes the form (7), then such a manifold is known as *quasi-Einstein manifold* in the literature.

Next, let $\mathcal{M}$ contains a null vector field v which is recurrent, more explicitly, for some $1-$form q, one has $\nabla v = q \otimes v$. From the Ricci identity, we get the following:

$$\begin{aligned}\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i &= v^h R_{hijk} \\ &= v_i (\nabla_k q_j - \nabla_j q_k). \end{aligned} \qquad \textbf{(8)}$$

By using the identity $R_{h[ijk]} = 0$ and Equation (8), one gets that the bivector $F_{jk} \equiv \nabla_k q_j - \nabla_j q_k$ is simple whose blade contains v. In this case, a contraction of Equation (4) with $v^i v^k$ shows that $r \equiv 0$ for $(\mathcal{M}, g)$ and so $Riem = E$ and the curvature range is *dual invariant*. From Table 1, potential holonomy types admitting (real or complex) recurrent vector fields are $1(a)$, $1(b)$, $2(a)$, $2(b)$, $2(c)$, $2(d)$, $2(e)$, $2(f)$, $2(h)$, $2(j)$, $2(k)$, $3(a)$, $3(b)$, $3(d)$ $(\eta \neq 0)$ and $4(c)$. However, $(\mathcal{M}, g)$ cannot be conformally flat for $1(a)$ and $1(b)$ as for these types $r \neq 0$ and the condition (4) forces $Riem$ to be zero. Moreover, $G = l \wedge n - L \wedge N \in \overset{+}{S}$, $H = l \wedge N \in \overset{+}{S}$ generate the holonomy type $2(a)$ yielding the recurrence of l and N. But it was proven in [14] that $(\mathcal{M}, g)$ is automatically Ricci-flat, i.e., $Ricc \equiv 0$. Thus, the condition (4) gives flatness and so $2(a)$ is not possible for a conformally flat space according to our assumption. For holonomy types $2(b)$ and $2(e)$, the cross term in $Riem$ vanishes by considering the identity $R_{i[jkh]} = 0$ and so, $Riem$ is of the form $R_{ijkh} = \psi F_{ij} F_{kh} + \varrho \overset{*}{F}_{ij} \overset{*}{F}_{kh}$ for some smooth functions ψ, ϱ and a dual pair $(F, \overset{*}{F})$. For these types, the condition (4) is satisfied if $\psi = -\varrho$. For the type $2(c)$ where ϕ is generated by $l \wedge L + n \wedge N \in \overset{-}{S}$, $l \wedge n - L \wedge N \in \overset{+}{S}$, conformal flatness is possible if the coefficient of cross term in $Riem$ is not zero and other coefficients are zero. In this case, $R_{ij} = \vartheta(l_i L_j + L_i l_j - n_i N_j - N_i n_j)$ for a nowhere zero function ϑ and $r = 0$ so that $Ricc$ has a special Segre type denoted by $\{(zz)(\bar{z}\bar{z})\}$ (complex eigenvalues with degeneracies) occurring only for neutral signature. On the other hand, considering that l and L are recurrent for the type $2(d)$ and applying Equations (4) and (8), it is obtained that $(\mathcal{M}, g)$ could be conformally flat if the coefficient of cross term, say ρ, in $Riem$ is not zero and other coefficients are zero where $Ricc$ is of the form $R_{ij} = -2\rho(l_i L_j + L_i l_j)$ (Segre type $\{(22)\}$ with eigenvalue zero). Analogously, for holonomy type $2(f)$, conformal flatness is possible so that $Ricc$ is of type $\{(22)\}$ having zero eigenvalue. Considering the generators from Table 1, it can be seen that holonomy type $2(h)$ admits recurrent vector field(s) but it does not give rise to a conformally flat space as the conditions $\eta \neq \pm\zeta$ and Equation (4) force $Riem$ to be zero. Similar comments can be made for types $2(j)$ and $2(k)$. For types $3(a)$, $3(b)$ and $4(c)$ admitting recurrent vector fields (which are l, N for $3(a)$, l for $3(b)$ and $4(c)$), the condition (4) is satisfied when $r = 0$ as for these cases, by taking into account the generators presented in Table 1 it yields that if $F \in \Lambda_p \mathcal{M}$ is in the range of the curvature map, so is its dual F^{*}. For $3(d)$ $(\eta \neq 0)$, the condition (4) imposes $Rg(\tilde{f})$ to be generated by the bivectors $l \wedge N$ and $l \wedge L$ which gives rise to the contradiction by using the Ambrose-Singer theorem. Furthermore, there can be conformally flat spaces which admit no parallel or recurrent vector fields. The above argument shows that if the space is conformally flat with $r=0$, the range of $Riem$ is dual invariant. Therefore, considering the Ambrose-Singer theorem one also gets potential types $4(a)$, $4(b)$, 5 satisfying Equation (4) as the generators of these types are members of either $\overset{-}{S}$ or $\overset{+}{S}$. In conclusion, the following result is proven:

Theorem 1. *Let $\mathcal{M}$ be a connected and simply connected $4-$dimensional space admitting a metric g of neutral signature. Suppose that $(\mathcal{M}, g)$ is not Ricci-flat. If $(\mathcal{M}, g)$ is a conformally flat space, then the holonomy group Φ of $\mathcal{M}$ is one of the types* $2(b)$, $2(c)$, $2(d)$, $2(e)$, $2(f)$, $2(g)$, $3(a)$, $3(b)$, $3(c)$, $4(a)$, $4(b)$, $4(c)$, 5 *or* 6.

It will be useful to interpret what is proved in Theorem 1 and compare the results achieved for the neutral signature with the case when g has Lorentz signature $(+,+,+,-)$. Certain remarks and interpretations can be given as follows:

Remark 1. For the Lorentz case, a bivector and its dual bivector must be independent, but this is not true for neutral signature (for example, one can consider the members of $\overset{+}{S}$ and $\overset{-}{S}$). This implies that for Lorentz signature, if the space is conformally flat with $r=0$, $\dim\phi$ must be even. Nevertheless, it is *false* for neutral signature as proved in Theorem 1. For instance, $3-$dimensional types $3(a)$ and $3(b)$ yield recurrent vector fields and ϕ is dual invariant but it is not even-dimensional for these types.

Remark 2. It can be observed from Theorem 1 that even if no parallel or recurrent vector fields arise in the holonomy, $(\mathcal{M}, g)$ could be conformally flat, e.g., the case when Φ is one of the types $4(a)$, $4(b)$ or 5.

Remark 3. For a conformally flat space of dimension $n \geq 4$, it is known that the Cotton tensor whose components are given by

$$C_{jkh} = \nabla_h R_{jk} - \nabla_k R_{jh} + \frac{1}{2(n-1)}\left(\nabla_k r\, g_{jh} - \nabla_h r\, g_{jk}\right) \quad \textbf{(9)}$$

vanishes. The steps and calculations carried out in the proof of Theorem 1 indicated that r is zero unless Φ is one of the holonomy types $3(c)$ or 6. It then follows from Equation (9) that in cases where the potential holonomy types $2(b)$, $2(c)$, $2(d)$, $2(e)$, $2(f)$, $2(g)$, $3(a)$, $3(b)$, $4(a)$, $4(b)$, $4(c)$ and 5 satisfy the conformally flat condition (4), $Ricc$ must be a Codazzi tensor, i.e., the condition $\nabla_h R_{jk} = \nabla_k R_{jh}$ holds.

Finally, suppose that $(\mathcal{M}, g)$ is a proper Einstein space, in other words, $Ricc = \xi\, g$ where $0 \neq \xi = \frac{r}{4}$, and $E=0$ in Equation (3). Then, Segre type of $Ricc$ is $\{(1111)\}$, r is constant and $\nabla Ricc = 0$. In this case, it was proven in [14] that Φ could be one of the types $2(b)$, $2(c)$, $2(e)$, $2(f)$, $3(a)$, $3(b)$, $4(a)$, $4(b)$, $4(c)$, 5 or 6. Moreover, if the space is also conformally flat, then it is clear from Equation (4) that $\nabla Riem = 0$ and it has constant sectional curvature. Bringing together the aforementioned result and Theorem 1, the following corollary can be stated:

Corollary 1. *Let $\mathcal{M}$ be a connected and simply connected $4-$dimensional proper Einstein space equipped with a neutral metric g. If $(\mathcal{M}, g)$ is also a conformally flat space, then it has constant sectional curvature and possible holonomy types could be* $2(b)$, $2(c)$, $2(e)$, $2(f)$, $3(a)$, $3(b)$, $4(a)$, $4(b)$, $4(c)$, 5 *or* 6.

IV. EXAMPLES

This section is devoted to give some examples of conformally flat $4-$dimensional spaces of neutral signature.

Example 1. Consider the following metric on $\mathcal{M} = \mathbb{R}^4$ with coordinates (u, v, x, y):

$$ds^2 = a(u)(x^2 + \epsilon y^2)du^2 + 2dudv + dx^2 + \epsilon dy^2 \quad \textbf{(10)}$$

where a is a nowhere zero function. If $\epsilon = 1$, the metric (10) has Lorentz signature which is known as the *plane wave metric* in the general relativity theory (see [11] pages 248–249) whilst it has neutral signature if $\epsilon = -1$ (see also [15]). It can be calculated that such a space is conformally flat and that the rank of $Riem$ is 2. In addition, the vector field $\frac{\partial}{\partial v}$ is parallel and the Ricci tensor is given as $Ricc = -2a(u)dudu$. Therefore, it has Segre type of $\{(211)\}$ (with eigenvalue zero). In that case, the holonomy group of $(\mathcal{M}, g)$ is $2(g)$ (for $\epsilon = -1$).

Example 2. Consider now the product manifold $\mathcal{M} = \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ with the following metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g'_{ab}dx^{a'}dx^{b'} \quad \textbf{(11)}$$

where $\mu, \nu = 1,2$ and $a, b = 3,4$ and also, $g_{\mu\nu}$ and g'_{ab} denote the components of the metrics g and g' in the coordinates x^μ and $x^{a'}$, respectively (in other words, the $g_{\mu\nu}$ are independent of x^3, x^4 and the g'_{ab} are independent of x^1, x^2). Suppose that the $2-$dimensional manifolds $(\mathbb{R}^2, g)$ and $(\mathbb{R}^2, g')$ have constant curvatures and that the metrics g and g' both have Lorentz signatures $(+,-)$. On the other hand, if we denote the metric given in Equation (11) by $\tilde{g}$, then $\tilde{g} = g \times g'$ and it has neutral signature. Now, one can set up a null basis $\{l, n, L, N\}$ in some neighbourhood of $p \in \mathcal{M}$ so that l, n are tangent to the manifold admitting the metric g, and L, N are tangent to the manifold admitting the metric g'. In this case, $Riem$ can be expressed as follows:

$$R_{ijkh} = \gamma(l_i n_j - n_i l_j)(l_k n_h - n_k l_h) + \lambda(L_i N_j - N_i L_j)(L_k N_h - N_k L_h) \quad \textbf{(12)}$$

where $\gamma = -\lambda \neq 0$ and the simple bivectors $H \equiv l \wedge n$, $K \equiv L \wedge N$ are dual pairs. It follows from Equation (12) that $Ricc$ has Segre type $\{(11)(11)\}$ and $r = 0$ and that $W \equiv 0$. In this case, the bivectors H and K generate the range of the curvature map. Therefore, the holonomy of $(\mathbb{R}^4, \tilde{g})$ is $2(b)$ from Table 1. Besides, the simple bivectors H and K are parallel and their blades are also orthogonal.

Example 3. Next, consider the following metric expressed in a coordinate system $(t, \tau, \theta, \varphi)$:

$$-dt^2 + (1 - \Lambda\tau^2)^{-1}d\tau^2 + \tau^2(d\theta^2 - \sin^2\theta d\varphi^2) \quad \textbf{(13)}$$

where Λ is a non-zero positive constant. Firstly, it is useful to note that the metric (13) is the neutral signature equivalent of the *Einstein static universe* metric in the theory of general relativity, that is, the case when $(\mathcal{M}, g)$ is a space-time (see [11], page 249). For the metric (13), it is true that $W \equiv 0$, more precisely, the space is conformally flat having a non-zero, parallel vector field $\frac{\partial}{\partial t}$ which is timelike and also, $r = 6\Lambda$. It then follows that Φ is of holonomy type $3(c)$.

V. CONCLUSION

In this study, $4-$dimensional conformally flat spaces with a metric of neutral signature were described by the holonomy structure. Besides the remarks given in Section III, it is useful to briefly mention about the cases Lorentz and positive definite signatures as well. Regarding conformally flat space-times, it is found in [3] that if $(\mathcal{M}, g)$ is conformally flat (but not flat), then it can be one of the holonomy types R_7, R_8, R_{10}, R_{13}, R_{14} or R_{15} where the labelings (up to isomorphism) are tabulated in [16]. Several examples were also presented in [3]. It is useful to note that the standard Friedmann-Robertson-Walker space-time is of holonomy type R_{15}, the Einstein static universe metric yields the holonomy type R_{13} which are both conformally flat (for details, see, [3]).

Finally, for positive definite signature, if $(\mathcal{M}, g)$ is conformally flat (but not flat and not Ricci-flat), then the holonomy group Φ of $\mathcal{M}$ is one of the types S_2, S_3, $\overset{+}{S}_4$ or S_6 where the labelings (up to isomorphism) are tabulated in [17].

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