One Dimensional Cellular Automa Under Null Boundary Condition

Sıfır Sınır Şartı Altında Bir Boyutlu Hücresel Dönüşümler

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Öz

Bu çalışmada sıfır sınır şartı altında, bir boyutlu hücresel dönüşümleri inceliyoruz. Her hücrenin durumunu, yerel kural yardımıyla temsil eden bir matris oluşturulur. Bu matris, her hücrenin durum geçişlerini tanımlayan katsayılar içerir. Temsili matris sistemin durumunu bir vektör olarak temsil eder ve dönüşüm kuralları matris çarpımlarıyla uygulanır. Bu yöntem, sistemin zamanla nasıl evirildiğini incelemeyi sağlar. Sonlu cisimler üzerindeki hesaplamalar yapılırken matris cebirlerinden faydalanıyoruz. İlk önce bir boyutlu hücresel dönüşümlerin tanımını veriyoruz. Daha sonra yerel kuralımızı tanımlıyoruz. Tanımladığımız yerel kural ve matris cebirlerini kullanarak temsili matrisimizi elde ediyoruz. Temsili matrisi elde ederken doğal tabanlardan faydalanıyoruz. Bundan önceki çalışmalarda genellikle yarıçap 1 alınarak temsili matrisler elde edildi. Diğer çalışmalardan farklı olarak yarıçapı 2 alıyoruz. Yarıçap 2 alınarak elde edeceğimiz tüm sonuçlar diğer çalışmalara göre farklılık gösterecektir. Bu şart altında elde edilen temsili matrisimiz çok daha orijinal olacaktır. Ayrıca elde edilen temsili matrisimizin, alt matrislerini de elde ediyoruz. Son olarak, matrisimizin en genel halini elde ediyoruz.

Anahtar Kelimeler: Hücresel Dönüşümler, Terslenebilirlik, Temsili Matris.

Abstract

In this study, we examine one-dimensional cellular automata under the null boundary condition. A matrix is created to represent the state of each cell using a local rule. This matrix contains coefficients that define the state transitions of each cell. The representative matrix depicts the system's state as a vector and transformation rules are applied through matrix multiplications. This method allows us to analyze how the system evolves over time. When performing calculations over finite fields, we utilize matrix algebra. First, we define one-dimensional cellular automata. Then, we define our local rule. Using the local rule and matrix algebra, we obtain our representative matrix. While obtaining the representative matrix, we utilize natural bases. In previous studies, representative matrices were generally obtained by taking the radius as 1. Unlike other studies, we take the radius as 2. By taking the radius as 2, all the results we obtain will differ from those in other studies. Under this condition, our representative matrix will be much more original. Additionally, we obtain the submatrices of our representative matrix. Finally, we derive the most general form of our matrix.

Keywords: Cellular Automata, Reversibility, Representative Matrice.

INTRODUCTION

Cellular Automata (CAs) have been a subject of significant research since the 1940s when John von Neumann initiated the foundational studies in this field. According to von Neumann, a CA is a discrete model consisting of a grid of cells, each in one of a finite number of states, that evolves over discrete time steps according to a set of rules based on the states of neighboring cells. Von Neumann's pioneering work demonstrated the potential of CAs to model self-reproducing systems, laying the groundwork for extensive research into their application for modeling complex system behavior (von Neumann, 1966). Stephen Wolfram and colleagues, in the 1980s, made substantial contributions to the understanding of one-dimensional CAs. Wolfram used polynomial algebra to analyze these systems, characterizing them within the framework of statistical mechanics using elementary mathematical models. This approach provided a robust method for studying the dynamic behavior of Cas (Wolfram, 1983).

Further advancements were made by Das et al., who employed matrix algebra to characterize one-dimensional CAs. Their innovative techniques included the development of methods to examine linear CAs and a novel exploration of CA structures using polynomial algebra. In their work, they paid particular attention to hybrid CAs and provided an algorithm to determine the invertibility of the CA representation matrices, thereby addressing a critical aspect of CA behavior (Das and Chaudhuri, 1993).

The reversibility of one-dimensional finite linear CAs has been a focal point of research due to its implications in various scientific fields. A CA is considered reversible if its evolution can be uniquely reversed, meaning that each state has a unique predecessor. The reversibility of a linear CA can be analyzed by constructing a rule(representative) matrix that depends on the local rule of the CA and the length of the initial vector. If this rule matrix is invertible, then the CA is reversible.

Akın et al. have contributed significantly to this area by investigating the conditions under which linear CAs are reversible. They constructed rule matrices for linear CAs defined over finite fields and finite rings, considering different boundary conditions such as null boundary, periodic boundary, and reflective boundary. Their work has provided deeper insights into the reversibility of CAs under various constraints (Akın et al., 2011,2012,2014,2017).

In this study, we aim to further this research by constructing the rule matrix for a one-dimensional finite linear CA under the null boundary condition. This construction will depend on a 2-radius local rule, which will be defined in the subsequent sections. By focusing on this specific boundary condition, we aim to contribute to the understanding of the conditions that determine the reversibility of linear CAs.

MATERIAL AND METHOD

Let $\mathbb{Z}_p = \{0,1, \dots, p-1\}$. Assume that $x = (x_n)_{n=-\infty}^{n=\infty}$ an infinite sequence with two sides. It represents $\mathbb{Z}_p^Z$. Let f be the local rule and r be the radius. Thus, the local rule is defined as follows:

$$f: \mathbb{Z}_p^{r+1} \to \mathbb{Z}_p, \qquad f(x_{-r}, \dots, x_r) = \left(\sum_{i=-r}^{i=r} k_i\, x_i\right) \ (mod\ p) \qquad (1).$$

where $k_i \in \mathbb{Z}_p$. This local rule generates the function $F: \mathbb{Z}_p^Z \to \mathbb{Z}_p^Z$.Thus, it is known as 1D additive CA. The definition of this transformation is as follows:

$$Fx = (y_n)_{n=-\infty}^{n=+\infty},\ y_n = f(x_{-r}, \dots, x_r) = \left(\sum_{i=-r}^{i=r} k_i\, x_{n+i}\right) \ (mod\ p) \qquad (2).$$

A dimensional CA structure defined over the field $\mathbb{Z}_2$ can be regarded as a grid of cells or blocks that take the value of each cell as 0 or 1. If r=2 is taken, then the next transition state of the cell can be obtained depending on itself and its other four neighbors.. Cells can only evaluation in separate time stages according to certain local neighborhood rules.

In terms of mathematics, the current states of the $(i)th$, $(i-1)th$ and $(i+1)th$ cells can be used to describe the next state change of the $(i)th$ cell:

$$\wp_i(t+1) = f(\wp_i(t), \wp_{i+1}(t), \wp_{i-1}(t))$$

where $\wp$ is known as the rule of a CA (Khan et. al.,1993).
Now, we give some important definitions.

Definition 1. In a cellular automata, the time step always consists of a sequence of geometric shapes of the same kind. Each of these sequences is called the configuration at the time.

Definition 2. In a cellular automata, the function that converts a configuration of the time step t to another configuration in the time step $t+m$ is called the global transition function and is shown by . Here is the number of cells in the starting sequence and $f_i, i = 1,2, \dots, n$ are global transition functions.

Definition 3. In the finite 1D CA configuration, the CA is said to as single or regular if the same rule is applied to every cell.

Definition 4. CA is referred to be a hybrid CA, if various rules are applied to distinct cells within the CA.

The value of the next state of a cell may not depend on all neighbors. Limiting a local relationship requires a precise interpretation of the rules on the edge of the cells. It is possible to observe the neighborhood problem of the cells on the edge. There are no neighbors at the ends of the edge cells.

There are some well-known approaches to solving this situation. Why, when working with cellular automata, should we restrict the extreme cells of the lattice? We impose different boundary conditions on the extreme cells to get better results. In this work, we employ null boundary conditions, on the contrary other types. Here, let's give you the definition of the null boundary condition, which is one of them.

Definition 5. A CA is defined null boundary condition if both the left and right of CA are connected to 0.

Example 1. A table is given related to a null boundary condition for a finite one-dimensional cellular automata named f, e, r, h, a, and t as follows. Since we take the radius as 2, we add 2 zeros to the far right and far left of the sequence.

0	0	f	e	r	h	a	t	0	0

FINDINGS AND DISCUSSION

$C^t = [x_1^t, x_2^t, , \dots, x_n^t]$ is a configuration of a one-dimensional cellular automata in t time. Let C^0 be initial configuration. There are various neighborhoods for one-dimensional cellular automata. The study used the closest neighborhood model with the greatest range of application. In this study, the representative matrix of one-dimensional cellular automata was taken as radius 2. A well-defined local rule under addition and multiplication can be given as follows for $m \geq 3$. We define the local rule as follows:

$$x_i^{t+1} = \begin{cases} k_3x_1^t + k_4x_2^t + k_5x_3^t \ ; & mod\, m, \ for \ \ i = 1 \\ k_2x_1^t + k_3x_2^t + k_4x_3^t \ + k_5x_4^t; & mod\, m, \ for \ i = 2 \\ k_1x_{i-2}^t + k_2x_{i-1}^t + k_3x_i^t \ + k_4x_{i+1}^t + k_5x_{i+2}^t; & mod\, m, \ for \ 3 \leq i \leq n-2 \\ k_1x_{n-3}^t + k_2x_{n-2}^t + k_3x_{n-1}^t \ + k_4x_n^t; & mod\, m, \ for \ \ i = n-1 \\ k_1x_{n-2}^t + k_2x_{n-1}^t + k_3x_n^t \ ; & mod\, m, \ for \ i = n \end{cases} \quad (3)$$

where $k_1, k_2, k_3, k_4, k_5 \in Z_m - \{0\}$. The variable x_i^t represents the cell's condition at time t. The number of cells is finite.

Let's define the 1D-CA representation matrix T using null boundary condition. (Shortly; NBC).

$$00[x_1^t, x_2^t, .., x_n^t]00 \underset{\rightarrow}{T} [x_1^{t+1}, x_2^{t+1}, .., x_n^{t+1}] \qquad (4)$$

To find the most general state of the representative matrix with the help of local rule, we obtain some specific cases for values n.

Example 2. Let, $n = 4$. Then we get the rule matrix $\boldsymbol{T}$ of order **4**. We investigate a configuration of size 1×4 with null boundary conditions.

$$00[x_1^t \ x_2^t \ x_3^t \ x_4^t]00$$

This configuration is represented by the information matrix shown below:

$$[X]_{1\times4}^t = [x_1^t \quad x_2^t \quad x_3^t \quad x_4^t]$$

If we apply the local rule to whole cells of the vector $[X]_{1\times4}^t$, we get a new information matrix $[X]_{1\times4}^{t+1}$ containing the following entries.

$$k_3x_1 + k_4x_2 + k_5x_3$$

$$k_2x_1 + k_3x_2 + k_4x_3 + k_5x_4$$

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4$$

$$k_1x_2 + k_2x_3 + k_3x_4$$

To obtain the rule matrix T_4 using basis vectors, you typically follow these steps:

1. **Identify Basis Vectors**: Determine the set of basis vectors for the vector space in which the transformation or rule is applied. Basis vectors are linearly independent vectors that span the vector space.

2. **Apply Transformation to Basis Vectors**: Apply the given transformation (rule) to each of the basis vectors. This involves taking each basis vector and seeing where it maps to under the transformation.

3. **Express Transformed Vectors in Terms of Basis**: Write the resulting transformed vectors as linear combinations of the original basis vectors. This step helps to identify the coefficients needed to construct the rule matrix.

4. **Construct the Rule (Representative) Matrix**: Form the rule matrix by using the coefficients obtained from the previous step. Each column of the matrix represents the image of a basis vector under the transformation, expressed in terms of the basis vector

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}k_3\\k_2\\k_1\\0\end{pmatrix}, \qquad T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}k_4\\k_3\\k_2\\k_1\end{pmatrix},$$

$$T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}k_5\\k_4\\k_3\\k_2\end{pmatrix}, \qquad T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\k_5\\k_4\\k_3\end{pmatrix}.$$

Here, each basis vector and its representation under transformation can be seen. The rule matrix T_{NBC} of order 4 is then formed by using these transformed basis vectors as the columns of T_{NBC}:

$$T_{NBC} = \begin{pmatrix} k_3 & k_4 & k_5 & 0 \\ k_2 & k_3 & k_4 & k_5 \\ k_1 & k_2 & k_3 & k_4 \\ 0 & k_1 & k_2 & k_3 \end{pmatrix}_{4\times 4}$$

Example3. Let, $n = 7$**.** Then we get the rule matrix $\boldsymbol{T}$ of order $\boldsymbol{7}$. We investigate a configuration of size 1×7 with null boundary conditions.

$$00[x_1^t\ x_2^t\ x_3^t\ x_4^t\ x_5^t\ x_6^t\ x_7^t]00$$

This configuration is represented by the information matrix shown below:

$$[X]_{1\times 7}^t = [x_1^t \quad x_2^t \quad x_3^t \quad x_4^t \quad x_5^t \quad x_6^t \quad x_7^t]$$

If we apply the local rule to whole cells of the vector $[X]_{1\times 7}^t$, we get a new information matrix $[X]_{1\times 7}^t$ containing the following entries.

$$k_3x_1 + k_4x_2 + k_5x_3$$

$$k_2x_1 + k_3x_2 + k_4x_3 + k_5x_4$$

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5$$

$$k_1x_2 + k_2x_3 + k_3x_4 + k_4x_5 + k_5x_6$$

$$k_1x_3 + k_2x_4 + k_3x_5 + k_4x_6 + k_5x_7$$

$$k_1x_4 + k_2x_5 + k_3x_6 + k_4x_7$$

$$k_1x_5 + k_2x_6 + k_3x_7$$

$$T\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}k_3\\k_2\\k_1\\0\\0\\0\\0\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}k_4\\k_3\\k_2\\k_1\\0\\0\\0\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}k_5\\k_4\\k_3\\k_2\\k_1\\0\\0\end{pmatrix},\ T\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\k_5\\k_4\\k_3\\k_2\\k_1\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\k_5\\k_4\\k_3\\k_2\\k_1\end{pmatrix},\ T\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\k_5\\k_4\\k_3\\k_2\end{pmatrix},T\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\k_5\\k_4\\k_3\end{pmatrix}.$$

Here, each column corresponds to the transformation of a respective basis vector. The rule matrix T_{NBC} of order $\boldsymbol{7}$ is then formed by using these transformed basis vectors as the columns of T_{NBC}:

$$T_{NBC}=\begin{pmatrix}k_3 & k_4 & k_5 & 0 & 0 & 0 & 0\\ k_2 & k_3 & k_4 & k_5 & 0 & 0 & 0\\ k_1 & k_2 & k_3 & k_4 & k_5 & 0 & 0\\ 0 & k_1 & k_2 & k_3 & k_4 & k_5 & 0\\ 0 & 0 & k_1 & k_2 & k_3 & k_4 & k_5\\ 0 & 0 & 0 & k_1 & k_2 & k_3 & k_4\\ 0 & 0 & 0 & 0 & k_1 & k_2 & k_3\end{pmatrix}_{7\times 7}$$

For, $n \geq 3$ ($n \in Z^+$) the general form of the rule (representation) matrix T_{NBC} of order $n \times n$ is as follows: