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Examining Prime Numbers/Components of Diophantine $D(\mp 3)$ Sets

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ABSTRACT

Mathematicians have long been interested in Diophantine sets. They have good ways to analyze the calculations and results. The aim of this paper is to explore the enigmatic world of diophantine $D(\mp 3)$ set shapes, revealing a new emphasis on its complex specifications and deep correlations. The Diophantine $D(\mp 3)$ sets, defined as integer values in this work, represent significant domain ripe for examinations. Our study analyzes these sets in detail, ignoring their cardinals, and aims to reveal hidden patterns and unique characteristics. By scrutinizing their structure, our intention is to reveal the high mathematics content of these collections. In our discussion we highlight basic principles of basic algebraic number theory, invoking the law of quadratic reciprocity, Diophantine equations, and the enduring grace of major mathematicians like Gauss, Dirichlet and Fermat. These tools and logic serve as viewers of our discussion, ultimately Diophantine provides a deeper appreciation of the concepts in the $D(\mp 3)$ sets and their importance in the broader mathematical terrain.

Diophantine D(∓3) Kümelerinin Asal Sayılarını/Bileşenlerini İnceleme

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ÖZ

Matematikçiler uzun zamandır Diophantine kümeleriyle ilgilenmektedir. Hesaplamaları ve sonuçları analiz etmek için iyi yollara sahiptirler. Bu makalenin amacı, Diophantine $D(\mp 3)$ küme şekillerinin gizemli dünyasını keşfetmek ve onun karmaşık özelliklerine ve derin ilişkilerine yeni bir vurgu yapmaktır. Bu çalışmada tam sayı değerleri cinsinden tanımlanan Diophantine $D(\mp 3)$ kümeleri, incelemeler için önemli bir alanı temsil eder. Çalışmamız, bu kümeleri ayrıntılı bir şekilde analiz ederek kardinal sayılarını göz ardı etmekte ve gizli kalıpları ile benzersiz özellikleri ortaya çıkarmayı amaçlamaktadır. Bu tip kümelerin yapılarını inceleyerek, bu tarz çalışmaların yüksek matematik içeriğini ortaya çıkarmak hedeflenir. Tartışmamızda, temel cebirsel sayı teorisinin temel prensiplerini vurguluyor, ikinci dereceden karşılıklılık yasasını, Diophantine denklemlerini ve Gauss, Dirichlet ve Fermat gibi önemli matematikçilerin kalıcı çalımalarını öne çıkarıyoruz. Bu araçlar ve mantık, çalışmaya hizmet ederek nihayetinde Diophantine $D(\mp 3)$ kümelerindeki kavramların ve daha geniş matematiksel alandaki önemlerinin daha derin bir şekilde anlaşılmasını sağlamaktadır.

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1. Introduction

In the sphere of number propositions (refer to the bibliography), Diophantine sets have long charmed mathematicians and presented rich ground for fine inquiry.

Prime numbers, their characteristics, and their connections to composite numbers have captivated mathematicians for millennia. However, it wasn't until the 1700s that Leonardh Euler made the first significant breakthrough in understanding prime numbers. The Quadratic Reciprocity Theorem was initially demonstrated by Carl Friedrich Gauss in the early 1800s and subsequently reaffirmed numerous times (at least eight times by Gauss himself).

We approach our concise examination of number theory with an elegant proof by the brilliant young mathematician Gotthold Eisenstein. This approximation serves as an appropriate endpoint for our study of number theory, as it alludes to the subject's wonderful, challenging, and nuanced aspects, and we hope it inspires you to delve deeper into number theory.

The ancient mathematician Diophantus of Alexandria was the first to prove the problem of discovering four figures that, when their pairwise products are increased by one, affect in perfect places. He successfully linked a set of four positive rational figures enjoying this property {1/16,33/16,17/4,105/16}. After all, Fermat was credited with discovering the original set of four positive integers that fulfilled this condition {1, 3, 8, 120}. Euler latterly linked an horizonless family of analogous sets using a formula involving integers a, b and r, where $ab + 1 = r^2$.

Several extensions of the original problem studied by Diophantus and Fermat have been explored. One notable extension involves replacing the number one (-1-) in the description of Diophantine m- tuples with an arbitrary integers. Also, multitudinous experimenters have excavated into the actuality of Diophantine quadruples with the property D(n), achieving partial results. These discoveries led to the expression of delineations for Diophantine m- tuples, which involve sets of positive integers or non-zero rationals that satisfy specific fine conditions.

The Diophantine-D(∓3) sets filled with integer values in this work represent an interesting area awaiting discovery. Our discussion plunges into these frameworks, for their special reasons, with a view to revealing the detached patterns and the peculiar characteristics they hold. By scrutinizing their structure, we are within ourselves to reveal the reality of the superior mathematics of these combinations. For these reasons, we consider the literature on this topic as follows:

The book of Apostol (Apostol, 1976) serves as an entry point to the fascinating world of logical number proposition. It covers abecedarian generalities, similar as high number proposition, Dirichlet series, and zeta functions, making it essential reading for those seeking the complex relationship between number propositions and analysis. It is a valuable resource for scholars and mathematicians who interested in the deep relationship between number propositions and analysis Baumgart's work (Baumgart, 2015) with Duke's work (Duke et al., 2005) introduces important methods and concepts that form the basis for further study in the number theory's topic named as Quadratic Reciprocity Law.

Cox's work (Cox, 2013) explores the interesting relationship between Fermat's last theorem, square field propositions, and complex integration. Here is a detailed explanation of the propositions behind quadratic forms and finds their functions in number propositions. The book is an invaluable resource for those interested in understanding the complex relationships between algebraic numbers, elliptic angles, and quadratic propositions. A work by Gauss (Gauss, 1966) using mathematical generalizations gives a detailed discussion of, including music, quadratures, and integer propositions. His published book provides a detailed discussion of general mathematical expressions, including especially number theoretic tools. Gauss's pioneering work laid the groundwork for many discoveries in a clever way. Gauss's logic and subtle rigor continue to influence and inspire mathematicians, leading to a wealth of knowledge in pure mathematics.

Focusing primarily on the Pell equation, Gopalan, Özer and their colleagues (Gopalan et al., 2018) provide results and a comprehensive summary of this particular Diophantine equation for the Pell equation has fascinated mathematicians for centuries, and this book explores its interesting and complex parts in depth, presenting results and their properties. The books of Hardy (Hardy et al., 2008) and Grosswald (Grosswald, 1984) enable readers to go into a certain depth, and makes it a must for mathematicians interested in this field read and deals extensively with quantitative representations.

Foundational books written seperately by Rosen, Nathanson and Zukkerman in number proposition give a gentle preface to crucial generalities, including divisibility, high figures, and Diophantine equations. Bridging classical principles with ultramodern advancements, these books offer a comprehensive approach to the study of number proposition. They link literal perceptivity with contemporary developments, feeding to a broad followership of mathematicians seeking a well- rounded understanding of the subject.

Separate foundational books on numbers and number theory by Nathanson (Nathanson, 2010) , Ireland (Ireland et al., 2018), Niven (Niven et al., 2008) and Kuroki (Kuroki et al., 2009) offer gentle examples of important general concepts including division, higher scores, and Diophantine equations. Real emotions are associated with contemporary developments, giving them a large following of mathematicians seeking a fuller understanding of the subject

The author Özer (Özer, 2018-2022 and 2023) examined a selection of Diophantine P_{400} triplets, quadruples and P_2 to triplets exercising styles concerning Diophantine equations. The issues deduced in these studies set up out significance in demonstrating the operation of ways and unveiling new perceptivity in Diophantine proposition within the literature. Covering computation from a broad perspective, the book from Serre (Serre, 1996) offers a comprehensive course, including number proposition, algebra and more. It aims to give a well- rounded understanding of computation, making it suitable for scholars and mathematicians at colorful levels. For the work of the Rødseth (Rødseth, 1994) Brown and Shiue’s paper was considered and found out on a remark related to the Frobenius problem.

Focused on algebraic figure, the book of Shafarevich (Shafarevich, 2013) concentrates on the study of kinds in projective space. It provides a foundational understanding of algebraic figure and its connections to colorful fine generalities."Elliptic Angles Number Theory and Cryptography" explores the intricate connections between elliptic angles, number proposition, and their operations in cryptography.

For additional information on classical proofs of the Quadratic Reciprocity Law, we refer the reader to Baumgart's (Baumgart, 2015) work. Theorem which is stated as "If a and b are relatively primes, the number of natural numbers which changed in the form $au + bv$ for nonnegative integers u and v is equal to $[(a-1)(b-1)]/2$" was showed by Sylvester (Sylvester, 1882) in 1882. In 1884, he (Sylvester, 1884) presented it as a significant problem, and Tripathi (Tripathi, 2000) later worked on a concise proof utilizing generating functions by reviewing some key pace in the Gauss-Eisenstein proof of the quadratic reciprocity law for the use of Legendre symbols.

Schering (Schering, 1882) extended Gauss' Lemma to the Jacobi symbol. After all, a deictic proof of the Gauss-Schering Lemma (in the work of Kuroki et all. and Schering's papers) appears to be plenty of theory based. Zolotarev (Zolotarev, 1872) noted that Legendre and Jacobi symbols are related to the signings of innately united permutations. This aproximation has led to other proofs (Duke and Hopkins, 2005) demonstrating that Gauss' Lemma can be generalized to the Jacobi symbol. These ways provide direct proofs of the quadratic reciprocity law for Jacobi symbols but require the introduction of some auxiliary concepts from abstract algebra. For positive coprime integers a and b, and any positive number n, let $N(a,b;n)$ represent the number of positive integer solutions of $au+bv=n$.

Additionally, it is well-known that $N(a,b;n+ab)=N(a,b;n)+1$ (see Tripathi's work, Lemma 1). For $l < ab$, the equation $au+bv= l$ has not more than one solution (see the paper of Tripathi, Lemma 2 and Lemma 4).

This paper begins to explore a journey into the esoteric world of diophantine D(∓3) sets' theories, shedding new light on their complex nature and deep correlations of Diophantine-D(∓3) sets using integer standards the species trend remains an interesting round ripe for investigation. Our analysis examines these frameworks in more detail (regardless of their specificity) in an attempt to separate the extracted patterns from the distinctive characteristics they hold. By analyzing their compositions, we aim to reveal the high mathematics of these devices.

2. Materials and Method

Detailed eloquent explanations are essential to ensure clarity and coherence of the paper. The theory and description below detail the aforementioned section and are illustrated by the literature. Each of these terms will be used interchangeably in proofs of the proposition of our text.

Definition 2.1. A set $\mathcal{H} = \{\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_r\}$ of r positive integers is called a Diophantine r-Tuples with property $D(s)$ if $(\hbar_i . \hbar_j + s)$ is a perfect square for all $1 \leq i \neq j \leq r$.

The first set of four positive integers with the above property was found as $\{1, 3, 8,120\}$ by Fermat.

Note 2.1. More generally, this definition can be given for rational numbers as follows:

A set n consisting of nonzero rational numbers $\{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n\}$ is referred to as a rational Diophantine n-tuple if for all $1 \leq i < j \leq n$ the product $\mathcal{S}_i . \mathcal{S}_j + 1$ is a perfect square.

Also, some conjectures and useful theorems have been used for these type of sets in the literature as you seen in the below:

Conjectures. (i) A Diophantine quintuple does not exist.

(ii)If a nonzero integer r is not a perfect square, then there are only a finite number of $D(r)$ −quadruples.

Useful Theorems. (i) If r is an integer of the form $r = 4s + 2$, then there is no Diophantine quadruple with the property $D(r)$.

(ii) If an integer r does not have the form $4s + 2$ and $r \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one Diophantine quadruple with the property $D(r)$.

The following question has been examined related with these sets

Does a Diophantine triple exist where $n \neq 1$ (i.e. $D(1)$ −triple) ?

Dujella and his collaborators have discovered examples of Diophantine triples for several different values of n where $n \neq 1$. For instance, the set $\{4, 12, 420\}$ is a $D(1)$ −triple as well as a $D(436), D(3796)$ and $D(40756) -$ triple whether such triples exist in infinite numbers remains an open question.

Dujella and other researchers have also explored the existence of sets of positive integers, for a given integer $k \geq 3$ where the product of any two elements plus 1 results in a $k - th$ power.

These sets are called "$k - th$ power Diophantine tuples". Examples of such triples for $k = 3$ and $k = 4$ are given by the sets $\{2, 171, 25326\}$ and $\{1352, 8539880, 9768370\}$ respectively.

Definition 2.2. Let *a* and *m* be integers, with conditions *m* > 1 and *(a, m)* = 1. It is said that *a* is a quadratic residue modulo *m* if the congruence $x^2 \equiv a$ (mod *m*) has a solution; it is said that *a* is a quadratic nonresidue modulo *m* if it is not a quadratic residue.

For example, the quadratic residues modulo 7 are determined as 1, 2 and 4, and the quadratic nonresidues are found as 3, 5 and 6; 0 is neither residue nor nonresidue.

Quadratic reciprocity law, even apparently one of Gauss' favorite subjects, it was a corresponding with Dirichlet, the famous mathematician. Gauss and Dirichlet considered two different quadrant reciprocity, however, could not be verified. However, Dirichlet was able to show that the quadratic reciprocity in the concept of binary reciprocity. It works for Gaussian integers as well as rational integers. As mentioned earlier, Dirichlet discovered fine shapes leading to a simple proof of some basic assumptions in Number theory. For stating theorem, however, it is given as follows named by Dirichlet's theorem.

Theorem 2.1. If two positive integers q and r are coprimes, then there are infinitely many primes of the form $r + kq$ with ıntegers k.

In 1837, this theorem was found out by Dirichlet and before that, there were several mathematicians whose work dealt closely with the achievements related to this significant and useful theorem. It can be easily proved by contradiction that there exist infinitely many primes and by constructing a converging alternating series, it may also proved that there are infinitely many primes in the form $4k + 1$.

Dirichlet's theorem has numerous implications and applications in various other number-theoretic quantitative theories, methods, investigations, and problems. It serves as a crucial tool for analysis in many aspects of quantitative research.

Definition 2.3. Let p be an odd prime and suppose that a is an integer such that $(p,a) = 1$ (relatively primes). The *Legendre symbol* (a/p) is defined as follows:

$$(a/p) = 1 \text{ if } x^2 \equiv a \pmod{p} \text{ has a solution } x \in \mathbb{Z}$$

or

$$(a/p) = -1 \text{ if there is no such solution}$$

The Legendre symbol can be computed by using Euler's Criterion. Besides, the Legendre symbol has many important properties such as $(a.b/p) = (a/p).(b/p)$ where a,b are integers.

Theorem 2.2. (Euler's criterion). Let p be an odd prime and suppose that a is an integer such that $(p,a) = 1$. Then, following congruent is satisfied.

$$(a/p) \equiv a^{(p-1)/2} \pmod{p}.$$

Corollary 2.1. Assume that p is an odd prime number. The product of two quadratic residues or of two quadratic non-residues (modulo p) is a quadratic residue; the product of a quadratic residue and a quadratic non-residue is a quadratic non-residue.

The Law of Quadratic Reciprocity is a key theorem in number theory, used to determine if an integer is a quadratic residue under a modulus p, where p is an odd prime number. The Law of Quadratic Reciprocity was first proposed by Euler in 1744, but he was unable to prove the main theorem. Legendre made partial progress in 1785 but his proof had gaps. The first complete proof came in 1796, when 18-year-old Gauss provided it, calling it the "Golden Theorem." Gauss later developed eight different proofs.

This theorem is especially important in cryptography, such as in the Goldwasser-Micali system, where the chosen key must not be a quadratic residue in the modulus of large prime numbers p and q. It is also applied in prime number tests like the Euler test.

Over the past 300 years, various methods have been used to prove this theorem. Gauss extended it to higher-order reciprocity laws. Mathematicians continue to explore new proofs of this significant theorem, not only for its technical complexity but also for its aesthetic elegance. It is also noted as "For those who regard number theory as the 'Queen of Mathematics,' this law is one of the crown's jewels" in the literature.

Definition 2.4. The Law of Quadratic Reciprocity is defined for distinct odd prime numbers p and q. The law of quadratic reciprocity is usually complemented with a formula for the Legendre symbol given by

$$\left(\frac{q}{p}\right).\left(\frac{p}{q}\right) = (-1)^{\frac{1}{2}.(p-1).\frac{1}{2}.(q-1)}$$

Let's define the ring of integers $\mathbb{Z}_n$ as the set $\{0,1,\ldots,n-1\}$ under modulo n. Zolotarev's Lemma states that if a, b are relatively prime positive integers then $\left[\frac{a}{b}\right]$ is equal to the Jacobi symbol $\left(\frac{a}{b}\right)$ and this symbol gives quadratic reciprocity (used for fast computation of Legendre symbols). In general, $\left[\frac{a}{b}\right]$ represents the sign of multiplication by a in the set $\mathbb{Z}/b\mathbb{Z}_n$ but it is not the sign of multiplication in $(\mathbb{Z}/b\mathbb{Z})^*$; however, when b is a prime these cases coincide. The lemma is stated as follows:

Lemma 2.1. (Zolotarev). For any prime number p and any $m \in \mathbb{Z}_P^*$ the Legendre Symbol $\left(\frac{m}{p}\right)$ is equivalent to the sign of the permutation $\tau_m : x \longmapsto mx$ of $\mathbb{Z}_P^*$.

More broadly, quadratic reciprocity is key to explicitly expressing the Dedekind zeta functions of quadratic number fields, and attempting to generalize this leads to class field theory and other advanced topics.

Theorem 2.3. (Gauss' lemma). Let p be an odd prime and $(a / p) = 1$. Assume that $\mathscr{s}$ is the number of least positive residues of the integers $a;\ 2a;\ 3a;\ \ldots\ ;\ [(p-1).a]/2$ modulo p that are greater than $p/2$. Then, following equality is satisfied.

$$\left(\frac{a}{p}\right) = (-1)^{\mathscr{s}}.$$

Diophantine equations are the study of solutions of polynomial equations in integers or general number rings. Originating in ancient texts, this branch of number theory is one of the oldest in mathematics. The fascination of the subject lies in the difficulty of solving these problems, often involving sophisticated mathematical tools.

Definition 2.5. A Diophantine equation is defined by $f(y_1, y_2, \ldots y_m) = 0$ where $y_1, y_2, \ldots y_m$ are variables and f presents a set of polynomial equations with integer coefficients and solutions.

Diophantine equations are the simplest of degree 1, The two-variable case is well known: the equation $ax + by = c$ has a solution in integers x, y if and only if $gcd(a, b)$ divides c. As we know any particular solution is easier to obtain with the extended Euclidean algorithm.

Specifically, the Pell Equation, named after the mathematician John Pell, is a type of Diophantine equation of the form $u^2 - Dv^2 = 1$ as classically. Then, D is a nonsquare positive integer, and the task is to find integer results for u and v. Pell equations are a special case of a broader class of equations known as the generalized Pell equations. Some of the resulting methods for Pell Equation can be mentioned as Continued Fraction Expansions, Rush Relations, Algebraic and Number Theoretic Styles, the Method Samasa, Brahmagupta's Method etc. (They are trivially known to mathematicians from multitudinous books on mathematics in particular in mathematical proposition).

3. Main Results and Discussion

In this section, it is proven in detail which forms the primes in the Diophantine D(3) or D(-3) sets should take, with the help of the definitions and theorems available in the literature and expressed in the preliminaries section.

Note 3.1. If we consider the prime numbers of the Diophantine D(+3) sets from the literature and references, it is seen that p = 2, 3, 11, 13, 23, 37, 47, 59, 61, 71, 73, 83, 97, 107, 109,… so on are in the Diophantine sets showcasing the D(+3) property even this compendium comprises high figures conforming to the given criteria.

So, these primes can be given in the special form exactly as follow:

Theorem 3. 1. Let p be an odd prime number (greater than 3) that belongs to the Diophantine sets characterized by property D(+3). The prime numbers $(p \neq 2, 3)$ in the set D(+3) are of the form p≡∓1 (mod 12) and conversely, the prime numbers of the form $p \equiv \mp 1 \ (mod\ 12)$ also exist in the set D(+3).

Proof. Let $p \equiv \mp 1 \ (mod\ 12)$ be a prime number and ₪ be a positive integer in the Diophantine sets with property D(+3). Using the definition of the Diophantine sets with property D(+3) given in the preliminaries section, following equation is obtained:

$$\text{₪} . p + 3 = \mathfrak{X}^2$$

and

$$\mathfrak{X}^2 \equiv 3 (mod\ p).$$

Using Legendre symbol, quadratic reciprocity and $p \equiv \mp 1 \ (mod\ 12)$ for following,

$$\left(\frac{3}{p}\right) . \left(\frac{p}{3}\right) = (-1)^{\frac{1}{2}.(p-1).\frac{1}{2}.(3-1)}$$

then, we have $\left(\frac{3}{p}\right) = +1$. It is demonstrated that primes in the form of $p \equiv \mp 1 \ (mod\ 12)$ belongs to Diophantine sets with property D(+3).

On the other hand, for any odd prime number p, if it is in the Diophantine sets with the property D(+3) then these primes are of the form $p \equiv \mp 1 \ (mod\ 12)$. This can be easily seen using Gauss Lemma, Quadratic reciprocity and the Legendre symbol.

Alternative Proof for $p \equiv 11 \ (mod\ 12)$: When we examine the Diophantine equation $x^2 - ny^2 = p$, where n is a nonzero integer and p is a prime number, and if $gcd(p, n) = 1$ is satisfied, then reducing this equation modulo p leads to the conclusion $\left(\frac{n}{p}\right) = 1$ as derived from the law of Quadratic reciprocity. Using this law, it has been proven that the following arithmetic sequences contain infinitely many prime numbers:

Consider the function $f(x) = 3x^2 - 1$. If p divides $f(n)$, then $\left(\frac{3}{p}\right) = 1$. However, in order to show that there are infinitely many primes that divide f and are congruent to $3\ mod 4$, we can modify the proof of the corresponding results. Assume that there is a finite set of primes $p_1, .. p_n$ that satisfy this property and examine $f(2p_1 ... p_n) \equiv 3\ mod 4$. This implies that there are infinitely many primes p for which $\left(\frac{3}{p}\right) = 1$ and $p \equiv 3\ mod 4$. According to the Law of Quadratic Reciprocity $\left(\frac{p}{3}\right) = -1$, which means $p \equiv 2\ mod 3$. Therefore, $p \equiv 11\ mod 12$.

Corollary 3.1. The following table contains some numerical results for Diophantine sets with property $D(+3)$ as triples. The results in these tables can be given as a simple example to see the validity of the theorems for numerical values between 1 and 1000.

(1, 6, 13)	(1, 481, 526)	(2, 263, 311)	(3, 362, 431)	(11, 26, 71)	(13, 177, 286)	(23, 66, 167)	(33, 262, 481)	(46, 373, 681)	(69, 142, 409)	(83, 386, 827)	(131, 383, 962)
(1, 13, 22)	(1, 526, 573)	(2, 311, 363)	(3, 431, 506)	(11, 66, 131)	(13, 241, 366)	(23, 122, 251)	(33, 334, 577)	(46, 457, 793)	(69, 169, 454)	(94, 177, 529)	(138, 227, 719)
(1, 22, 33)	(1, 573, 622)	(2, 363, 419)	(3, 506, 587)	(11, 71, 138)	(13, 286, 421)	(23, 167, 314)	(33, 481, 766)	(47, 74, 239)	(69, 409, 814)	(94, 249, 649)	(141, 166, 613)
(1, 33, 46)	(1, 622, 673)	(2, 419, 479)	(3, 587, 674)	(11, 131, 218)	(13, 366, 517)	(23, 251, 426)	(33, 577, 886)	(47, 143, 354)	(69, 454, 877)	(97, 118, 429)	(142, 241, 753)
(1, 46, 61)	(1, 673, 726)	(2, 479, 543)	(3, 674, 767)	(11, 138, 227)	(13, 421, 582)	(23, 314, 507)	(37, 73, 214)	(47, 239, 498)	(71, 138, 407)	(97, 349, 814)	(143, 179, 642)
(1, 61, 78)	(1, 726, 781)	(2, 543, 611)	(3, 767, 866)	(11, 218, 327)	(13, 517, 694)	(23, 426, 647)	(37, 94, 249)	(47, 354, 659)	(71, 183, 482)	(97, 429, 934)	(143, 291, 842)
(1, 78, 97)	(1, 781, 838)	(2, 611, 683)	(3, 866, 971)	(11, 227, 338)	(13, 582, 769)	(23, 507, 746)	(37, 214, 429)	(47, 498, 851)	(71, 407, 818)	(107, 146, 503)	(143, 354, 947)
(1, 97, 118)	(1, 838, 897)	(2, 683, 759)	(6, 13, 37)	(11, 327, 458)	(13, 694, 897)	(23, 647, 914)	(37, 249, 478)	(59, 83, 282)	(71, 482, 923)	(107, 359, 858)	(146, 191, 671)
(1, 118, 141)	(1, 897, 958)	(2, 759, 839)	(6, 37, 73)	(11, 338, 471)	(13, 769, 982)	(26, 47, 143)	(37, 429, 718)	(59, 194, 467)	(73, 121, 382)	(109, 229, 654)	(157, 334, 949)
(1, 141, 166)	(2, 3, 11)	(2, 839, 923)	(6, 73, 121)	(11, 458, 611)	(22, 33, 109)	(26, 71, 183)	(37, 478, 781)	(59, 282, 599)	(73, 214, 537)	(109, 262, 709)	(166, 193, 717)
(1, 166, 193)	(2, 11, 23)	(3, 11, 26)	(6, 121, 181)	(11, 471, 626)	(22, 69, 169)	(26, 143, 291)	(39, 59, 194)	(59, 467, 858)	(73, 382, 789)	(111, 143, 506)	(167, 314, 939)
(1, 193, 222)	(2, 23, 39)	(3, 26, 47)	(6, 181, 253)	(11, 611, 786)	(22, 109, 229)	(26, 183, 347)	(39, 122, 299)	(61, 78, 277)	(74, 107, 359)	(111, 386, 911)	(169, 313, 942)
(1, 222, 253)	(2, 39, 59)	(3, 47, 74)	(6, 253, 337)	(11, 626, 803)	(22, 169, 313)	(26, 291, 491)	(39, 194, 407)	(61, 213, 502)	(74, 239, 579)	(118, 141, 517)	(177, 286, 913)
(1, 253, 286)	(2, 59, 83)	(3, 74, 107)	(6, 337, 433)	(11, 786, 983)	(22, 229, 393)	(26, 347, 563)	(39, 299, 554)	(61, 277, 598)	(74, 359, 759)	(118, 429, 997)	(179, 219, 794)
(1, 286, 321)	(2, 83, 111)	(3, 107, 146)	(6, 433, 541)	(13, 22, 69)	(22, 313, 501)	(26, 491, 743)	(39, 407, 698)	(61, 502, 913)	(78, 97, 349)	(121, 181, 598)	(181, 253, 862)
(1, 321, 358)	(2, 111, 143)	(3, 146, 191)	(6, 541, 661)	(13, 37, 94)	(22, 393, 601)	(26, 563, 831)	(39, 554, 887)	(66, 131, 383)	(78, 277, 649)	(121, 382, 933)	(191, 242, 863)
(1, 358, 397)	(2, 143, 179)	(3, 191, 242)	(6, 661, 793)	(13, 69, 142)	(22, 501, 733)	(33, 46, 157)	(46, 61, 213)	(66, 167, 443)	(78, 349, 757)	(122, 251, 723)	(193, 222, 829)
(1, 397, 438)	(2, 179, 219)	(3, 242, 299)	(6, 793, 937)	(13, 94, 177)	(22, 601, 853)	(33, 109, 262)	(46, 157, 373)	(66, 383, 767)	(83, 111, 386)	(122, 299, 803)	(219, 263, 962)
(1, 438, 481)	(2, 219, 263)	(3, 299, 362)	(11, 23, 66)	(13, 142, 241)	(23, 39, 122)	(33, 157, 334)	(46, 213, 457)	(66, 443, 851)	(83, 282, 671)	(131, 218, 687)	(222, 253, 949)

Note 3.2. Similar results can be found for the Diophantine sets with the property D(-3) too. Primes such as p = 2, 3, 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, …so on belongs to the Diophantine sets with the property D(-3) and these can be classified like we mentioned above.

Theorem 3. 2**.** Let p be an odd prime number that belongs to the Diophantine sets characterized by property D(-3). The prime numbers in the set D(-3) are of the form $p \equiv 1\ (mod\ 3)$ or p = 3. On the other hand, the prime numbers of the form $p \equiv 1\ (mod\ 3)$ or p = 3 also are included in the Diophantine sets with the property D(-3).

Proof. Assume that $p \equiv 1 \ (mod\ 3)$ is a prime and $\mathfrak{S}$ is a positive integer in the Diophantine sets with property D(-3). Using the definition of the Diophantine sets with property D(-3), we get following

$$\mathfrak{S}.p - 3 = \xi^2$$

and

$$\xi^2 \equiv -3(mod\ p).$$

Using Legendre symbol, quadratic reciprocity and $p \equiv 1\ (mod\ 3)$ for the following equalities

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right).\left(\frac{3}{p}\right) \quad , \quad \left(\frac{3}{p}\right).\left(\frac{p}{3}\right) = (-1)^{\frac{1}{2}.(p-1).\frac{1}{2}.(3-1)} \quad \text{with} \quad \left(\frac{-1}{p}\right) = (-1)^{\frac{1}{2}.(p-1)}$$

it is obtained that $\left(\frac{-3}{p}\right) = +1$. It is demonstrated that primes in the form of the $p \equiv 1\ (mod\ 3)$ belongs to Diophantine sets with property D(-3).

Trivially, contrast of the statement is easy to prove.

The following conclusion can be given in the different way of what is stated in Theorem 3.2

Corollary 3. 2. Let p represent a prime number greater than 3. Then, following condition is satisfied. If p belongs to the Diophantine sets with the property D(-3), then p conforms to the $p = u^2 + 3v^2$, $u, v \in \mathbb{Z}$ (p is prime). Besides, the primes in the form of $p = u^2 + 3v^2$, $(u, v \in \mathbb{Z})$, then they are also found in the set D(-3).

Proof. Considering Dirichlet's theorem, Euler's approximations to demonstrate Fermat's result, this theorem can be readily ascertained.

The number minus three (-3) must be a quadratic residue with respect to p prime number in order to get the prime number p to be in the diophantine sets with property D(-3). Hence, Legendre symbol $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)$ value has to be equivalent to +1. From the properties of Legendre symbol and Quadratic Reciprocity Law with the definitions and lemmas mentioned in the Preliminaries section, we get that the prime value p is written of the type of $p = u^2 + 3v^2 \quad (u, v \in \mathbb{Z})$.

It is also known that

$$p = u^2 + 3v^2, \quad (u, v \in \mathbb{Z}) \Leftrightarrow \ p \equiv 1\ (mod\ 3) \ \text{ or } p = 3$$

and it completes the proof.

Corollary 3.3. The table below presents some numerical outcomes for Diophantine sets with the D(-3) property in the form of triples. These results serve as straightforward examples to demonstrate the validity of the theorems for numerical values ranging from 1 to 1000.

(1, 4, 7)	(1, 579, 628)	(2, 614, 686)	(4, 21, 43)	(4, 757, 871)	(7, 229, 316)	(12, 631, 817)	(14, 302, 446)	(21, 412, 619)	(28, 409, 651)	(38, 438, 734)	(49, 516, 883)	(67, 84, 301)	(84, 163, 481)	(109, 367, 876)	(148, 193, 679)
(1, 7, 12)	(1, 628, 679)	(2, 686, 762)	(4, 31, 57)	(4, 813, 931)	(7, 277, 372)	(12, 721, 919)	(14, 402, 566)	(21, 439, 652)	(28, 489, 751)	(38, 494, 806)	(52, 67, 237)	(67, 237, 556)	(84, 217, 571)	(111, 157, 532)	(148, 273, 823)
(1, 12, 19)	(1, 679, 732)	(2, 762, 842)	(4, 43, 73)	(4, 871, 993)	(7, 316, 417)	(13, 28, 79)	(14, 446, 618)	(21, 619, 868)	(28, 523, 793)	(39, 52, 181)	(52, 97, 291)	(67, 301, 652)	(84, 301, 703)	(111, 364, 877)	(151, 309, 892)
(1, 19, 28)	(1, 732, 787)	(2, 842, 926)	(4, 57, 91)	(6, 14, 38)	(7, 372, 481)	(13, 31, 84)	(14, 566, 758)	(21, 652, 907)	(28, 613, 903)	(39, 133, 316)	(52, 139, 361)	(73, 111, 364)	(84, 373, 811)	(114, 146, 518)	(151, 372, 997)
(1, 28, 39)	(1, 787, 844)	(3, 4, 13)	(4, 73, 111)	(6, 38, 74)	(7, 417, 532)	(13, 79, 156)	(14, 618, 818)	(26, 42, 134)	(28, 651, 949)	(39, 181, 388)	(52, 181, 427)	(73, 228, 559)	(84, 481, 967)	(114, 398, 938)	(156, 229, 763)
(1, 39, 52)	(1, 844, 903)	(3, 13, 28)	(4, 91, 133)	(6, 74, 122)	(7, 481, 604)	(13, 84, 163)	(14, 758, 978)	(26, 78, 194)	(31, 57, 172)	(39, 316, 577)	(52, 237, 511)	(73, 364, 763)	(86, 114, 398)	(122, 182, 602)	(156, 259, 817)
(1, 52, 67)	(1, 903, 964)	(3, 28, 49)	(4, 111, 157)	(6, 122, 182)	(7, 532, 661)	(13, 156, 259)	(19, 28, 93)	(26, 134, 278)	(31, 84, 217)	(39, 388, 673)	(52, 291, 589)	(74, 122, 386)	(86, 294, 698)	(122, 386, 942)	(157, 211, 732)
(1, 67, 84)	(2, 6, 14)	(3, 49, 76)	(4, 133, 183)	(6, 182, 254)	(7, 604, 741)	(13, 163, 268)	(19, 61, 148)	(26, 194, 362)	(31, 172, 349)	(39, 577, 916)	(52, 361, 687)	(74, 218, 546)	(86, 398, 854)	(124, 147, 541)	(158, 266, 834)
(1, 84, 103)	(2, 14, 26)	(3, 76, 109)	(4, 157, 211)	(6, 254, 338)	(7, 661, 804)	(13, 259, 388)	(19, 93, 196)	(26, 278, 474)	(31, 217, 412)	(42, 62, 206)	(52, 427, 777)	(74, 386, 798)	(91, 133, 444)	(124, 247, 721)	(163, 268, 849)
(1, 103, 124)	(2, 26, 42)	(3, 109, 148)	(4, 183, 241)	(6, 338, 434)	(7, 741, 892)	(13, 268, 399)	(19, 148, 273)	(26, 362, 582)	(31, 349, 588)	(42, 206, 434)	(52, 511, 889)	(76, 109, 367)	(91, 169, 508)	(124, 313, 831)	(169, 271, 868)
(1, 124, 147)	(2, 42, 62)	(3, 148, 193)	(4, 211, 273)	(6, 434, 542)	(7, 804, 961)	(13, 388, 543)	(19, 196, 337)	(26, 474, 722)	(31, 412, 669)	(42, 326, 602)	(52, 589, 991)	(76, 129, 403)	(91, 244, 633)	(127, 217, 676)	(172, 199, 741)
(1, 147, 172)	(2, 62, 86)	(3, 193, 244)	(4, 241, 307)	(6, 542, 662)	(12, 19, 61)	(13, 399, 556)	(19, 273, 436)	(26, 582, 854)	(31, 588, 889)	(42, 434, 746)	(57, 91, 292)	(76, 219, 553)	(91, 292, 709)	(127, 364, 921)	(182, 222, 806)
(1, 172, 199)	(2, 86, 114)	(3, 244, 301)	(4, 273, 343)	(6, 662, 794)	(12, 37, 91)	(13, 543, 724)	(19, 337, 516)	(28, 39, 133)	(31, 669, 988)	(42, 602, 962)	(57, 172, 427)	(76, 247, 597)	(91, 444, 937)	(129, 196, 643)	(182, 254, 866)
(1, 199, 228)	(2, 114, 146)	(3, 301, 364)	(4, 307, 381)	(6, 794, 938)	(12, 61, 127)	(13, 556, 739)	(19, 436, 637)	(28, 49, 151)	(37, 76, 219)	(43, 73, 228)	(57, 292, 607)	(76, 367, 777)	(93, 196, 559)	(129, 403, 988)	(183, 241, 844)
(1, 228, 259)	(2, 146, 182)	(3, 364, 433)	(4, 343, 421)	(7, 12, 37)	(12, 91, 169)	(13, 724, 931)	(19, 516, 733)	(28, 79, 201)	(37, 91, 244)	(43, 124, 313)	(57, 427, 796)	(76, 403, 829)	(93, 223, 604)	(133, 183, 628)	(186, 302, 962)
(1, 259, 292)	(2, 182, 222)	(3, 433, 508)	(4, 381, 463)	(7, 21, 52)	(12, 127, 217)	(13, 739, 948)	(19, 637, 876)	(28, 93, 223)	(37, 219, 436)	(43, 228, 469)	(61, 127, 364)	(78, 158, 458)	(97, 156, 499)	(133, 283, 804)	(193, 244, 871)
(1, 292, 327)	(2, 222, 266)	(3, 508, 589)	(4, 421, 507)	(7, 37, 76)	(12, 169, 271)	(14, 26, 78)	(19, 733, 988)	(28, 133, 283)	(37, 244, 471)	(43, 313, 588)	(61, 148, 399)	(78, 194, 518)	(97, 291, 724)	(133, 316, 859)	(196, 277, 939)
(1, 327, 364)	(2, 266, 314)	(3, 589, 676)	(4, 463, 553)	(7, 52, 97)	(12, 217, 331)	(14, 38, 98)	(21, 43, 124)	(28, 151, 309)	(37, 436, 727)	(43, 469, 796)	(61, 364, 723)	(78, 458, 914)	(98, 186, 554)	(134, 278, 798)	(199, 228, 853)
(1, 364, 403)	(2, 314, 366)	(3, 676, 769)	(4, 507, 601)	(7, 76, 129)	(12, 271, 397)	(14, 78, 158)	(21, 52, 139)	(28, 201, 379)	(37, 471, 772)	(43, 588, 949)	(61, 399, 772)	(78, 518, 998)	(98, 258, 674)	(134, 326, 878)	(211, 273, 964)
(1, 403, 444)	(2, 366, 422)	(3, 769, 868)	(4, 553, 651)	(7, 97, 156)	(12, 331, 469)	(14, 98, 186)	(21, 124, 247)	(28, 223, 409)	(38, 74, 218)	(49, 76, 247)	(62, 86, 294)	(79, 156, 457)	(103, 124, 453)	(139, 268, 793)	(222, 266, 974)
(1, 444, 487)	(2, 422, 482)	(3, 868, 973)	(4, 601, 703)	(7, 129, 196)	(12, 397, 547)	(14, 158, 266)	(21, 139, 268)	(28, 283, 489)	(38, 98, 258)	(49, 151, 372)	(62, 206, 494)	(79, 201, 532)	(103, 373, 868)	(139, 361, 948)	(228, 259, 973)

(1, 487, 532)	(2, 482, 546)	(4, 7, 21)	(4, 651, 757)	(7, 156, 229)	(12, 469, 631)	(14, 186, 302)	(21, 247, 412)	(28, 309, 523)	(38, 218, 438)	(49, 247, 516)	(62, 294, 626)	(79, 457, 916)	(103, 453, 988)	(146, 182, 654)	…
(1, 532, 579)	(2, 546, 614)	(4, 13, 31)	(4, 703, 813)	(7, 196, 277)	(12, 547, 721)	(14, 266, 402)	(21, 268, 439)	(28, 379, 613)	(38, 258, 494)	(49, 372, 691)	(62, 494, 906)	(84, 103, 373)	(109, 148, 511)	(147, 172, 637)	…

Combining Theorem 3.1 and Corollary 3.1, a significant result is obtained on primes for Diophantine sets with the properties both D(-3) and D(+3) as following:

Theorem 3.3. Assume that $p > 3$ is a prime number. Then, the following equivalence is satisfied.

$p \equiv 1\ (mod\ 24) \iff p$ is included to the Diophantine sets with the properties both D(-3) and D(+3).

Proof. Using Theorem 3.1 and Corollary 3.2 with their proofs associated with Dirichlet's Theorem, the following also can be found;

$$\left(\frac{3}{p}\right) = 1 \quad \text{if and only if} \quad p \equiv 1\ (mod\ 12) \text{ or } p \equiv 11\ (mod\ 12)$$

$$\text{if and only if} \quad p \equiv 1, 11, 13, 23\ (mod\ 24)$$

and

$$\left(\frac{-3}{p}\right) = 1 \quad \text{if and only if} \quad p \equiv 1\ (mod\ 3) \quad \text{if and only if} \quad p \equiv 1, 7, 13, 19\ (mod\ 24\ )$$

Using above mentioned results, it is obtained that primes $p \equiv 1\ (mod\ 24)$ are in the Diophantine sets with property D(± 3).

4. Conclusion

In substance, insights into the mysterious realm of the diophantine D(∓ 3) set have revealed a robust understanding of their complex particulars and deep correlations among them. The study highlighted the magical nature of these systems in terms of numerical, and demonstrated the possibility of a richly nuanced analysis.

This study explores the mysterious realm of diophantine D(±3) sets, revealing their complex particulars-deep interactions. These integerdominated sets have provided an interesting geographical settlement research fort his issue. However, our exploratory tests on these collections end up revealing their shapes, hidden shapes, and distinctive characteristics. By examining their structure more closely, we seek to determine the frequency of high-level figures embedded in these collections.

The focus in this study on the diophantine-D(∓ 3) sets described by integer values has provided an interesting platform for insight. Regardless of their cardinality, the purpose of our study is to reveal the examples drawn and the unique characteristics of these systems. A careful examination of their compositions accounted for the presence of the upper classes/ larger sets, and shed light on the absedary characteristics of these sets.

This excursion into the world of diophantine D(∓ 3) sets has not only expanded our understanding of those properties but also highlighted their importance in a broader quantitative context. The observations made throughout this experiment pave the way for far greater insight and research, leading to a greater appreciation of the intricate relationships and complexities required in these particles.

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Declaration

The paper "Examining Prime Numbers/Components of Diophantine D(∓3) Sets" suggests exploring efforts aimed at unmasking and understanding the complexity and structure requirements of Diophantine sets through the D(∓3) property the focus of attention.

This work implies identifying the figures in these categories, including possibly identifying patterns, combinations and unique characteristics of similar categories. This study does not need to carry the blessing of the Ethics Commission.

Conflict of interest

The author states that there is no conflict with the paper" Examining Prime Numbers/Components of Diophantine D(∓3) Sets."

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