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RESEARCH ARTICLE

# Applications of Cantor Set to Fractal Geometry

İpek Ebru Karaçay * and Salim Yüce

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## ABSTRACT

**Fractal geometry is a subfield of mathematics that allows us to explain many of the complexities in nature. Considering this remarkable feature of fractal geometry, this study examines the Cantor set, which is one of the most basic examples of fractal geometry. First of all, the Cantor set is one of the basic examples and important structure of it. First, the generalization of Cantor set in on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are taken into consideration. Then, the given structures are examined over curve and surface theory. This approach enables to given a relationship between fractal geometry and differential geometry. Finally, some examples are established.**

***Keywords:*** *Fractal geometry, Cantor set, iterated function system.*
***AMS Subject Classification (2020):*** *Primary: 28A80 ; Secondary: 53A04; 53A05.*

## 1. Introduction

The first mathematical concept of fractal was first used by K. Weierstrass in 1861 with a curve that is continuous but not differentiable at any point. The first definition of fractal geometry emerged in 1975 with B.Mandelbrot's discovery of the Mandelbrot set, formed by assigning complex numbers to quadratic polynomials. The terms fractal means is meaning fragmented or divided. Fractal geometry has been studied in many different disciplines. It has been a branch of mathematics that has common studies with many other disciplines, especially medicine, art, architecture, and economics. In the emergence of fractal geometry, Mandelbrot's question, 'What is the length of the coast of England?' is important. Mandelbrot thought that he could not find the answer to this question with classical Euclidean geometry and started new studies [9].

Although fractal geometry was defined by Mandelbrot in 1975, fractal-like structures were defined long before Mandelbrot defined fractal geometry, without being called fractal at that time. The Cantor set is one of these structures. The Cantor set was defined by G.Cantor in 1883 [4].

With the definition of fractal geometry by Mandelbrot, self-similarity, which is an important feature for fractals, was defined. Later, the concepts of repeated function systems, which are important structures in obtaining fractals, and fractal dimension, which are distinctive features of fractals, were defined.

Mandelbrot defines fractals as objects with the properties of self-similarity, iterative formation and fractional dimension. Self-similarity is expressed as follows: The set of points $a = (a_1, a_2, \dots)$ in a space $A$ is transformed into a set $rA$ consisting of points $ra = (ra_1, ra_2, \dots)$ when a similarity transformation is applied with a shrinkage rate $r$ such that $0 < r < 1$. A set $A$ is exactly self-similar if it is a union of $X$ distinct sets $ra$. The iterated function system is a general method for generating fractals. A point or a shape is taken and replaced with multiple shapes of different scales that are identical to the original shape, resulting in a fractal.

Moreover, the Cantor set was first introduced by G. Cantor in 1883. It is defined for closed interval $[0, 1]$, which is a subset of the set of real numbers, being reduced by $\frac{1}{3}$ reduction rate infinitely [4]. When the length calculation of this structure was made, the result was zero, and it was concluded that the concept of length does not indicate a characteristic feature for this fractal structure. Then, the concepts of the self-similarity and

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** Corresponding author*

fractal dimension, which are two features specific to fractal structures, were defined. Fractals are created using iterated function systems by making use of the self-similarity feature [2]. The generalized Cantor set is defined for $[0, 1]$, which is a subset of the set of real numbers. For this structure, length and dimension calculations are examined, and also fractal structure is created with iterated function systems. The generalized Cantor Set is defined for $[0, 1]$, calculated dimension, and is created with iterated function systems [6, 7, 8].

The aim of this article is to analyze the generalized Cantor set in a wider range, and analysed in terms of differential geometry. With this purpose, in section 2 basic concepts are examined. In the section 3, Cantor structures are generalized and examined in terms of analytic geometry and differential geometry. Firstly, to facilitate the application of Cantor sets in different fields, Cantor sets are analysed in the closed interval [a,b], which is a wider interval. Based on Mandelbrot's fractal definition, self-similarity, iterated function systems are obtained. Then, the concept of fractal dimension was defined by showing that length does not indicate a characteristic feature when length is calculated. After these were demonstrated on the example, they were analysed in the theory of curves. Then $\mathbb{R}^2$ was analysed and similarly iterated function systems were created. When the area is calculated, it is concluded that the area does not indicate a characteristic feature and the concept of fractal dimension is defined. Similar operations were performed on $\mathbb{R}^3$ and Cantor structure was created with the results obtained in $\mathbb{R}^3$. An example is given using Olin Rodrigues on the created structure. In the section 4, the conclusions are given.

## 2. Preliminaries

In this section, the basic concepts that will be used throughout the article are given.

**Definition 2.1.** Mandelbrot introduced the term fractal as objects with the following characteristics [9]:

- Self-Similarity,
- Iterative formation,
- Fractional dimension.

These characteristics mean that the components of a fractal object are identical to the whole of this object and, most importantly, are generated through iterative processes. When the dimensions of fractal objects are calculated, it is seen that their dimensions are fractional numbers [9] (see in Fig. 1 ).

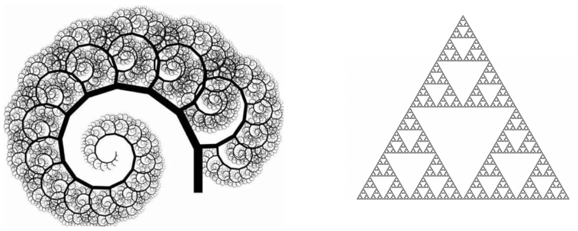

**Figure 1.** Examples of Fractal Structure, [5].

Let us give Mandelbrot's definition iterative function system. In Mandelbrot's definition, iterative formation holds an important place. Fractal structures are formed from the property of iterative formation through iterative function systems.

**Definition 2.2.** Iterated function systems are a type of function system used to create fractals. For any repeating part of the fractal, it is defined as follows:

$$f_{i,j} : \mathbb{R} \to \mathbb{R} \tag{2.1}$$

$$x \to f_{i,j}(x) = rx + (1 - r)\alpha, \tag{2.2}$$

where $i$ represents the step number and $j$ represents the part number in each step, denoting the iterated function system $f_{i,j}$. $r > 0$ denotes the contraction ratio, and $\alpha \in \mathbb{R}$ indicates the center of each part [2, 5].

**Definition 2.3.** Let $N \in \mathbb{Z}^+$ and $r > 0$. When creating the fractal $F$, the self-similarity dimension $D$ of the fractal is calculated as follows:

$$D = \lim_{k \to 0} \frac{\log N}{\log(\frac{1}{r})}, \tag{2.3}$$

where $r$ is the shrink rate and $N$ is the repetition number. The similarity dimension described above is applied to fractals obtained by reducing all parts by the same ratio. If the fractal has more than one reduction ratio, this formula needs to be generalized as [9]:

$$d = \frac{\log N}{\log(\frac{1}{r})}. \tag{2.4}$$

**Definition 2.4.** Let $C_0 = [0, 1]$ be closed interval in $\mathbb{R}$ (where $C_0$ is the first piece of the structure). If the piece in each step is divided into three equal parts and the centre piece is discarded, let obtain the set $C_1$ by remaining

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Similarly, the set $C_2$ is given by:

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

By a similar method, the obtained sets are in the decreasing order such that:

$$C_0 \supset C_1 \supset C_2 \supset \dots$$

The Cantor set is a subset of $\mathbb{R}$, (see details in [2, 5]).

**Definition 2.5.** Let us take the surface,

$$\begin{aligned} \phi : [a, b] \times [c, d] &\to E^3 \\ (u, v) &\to \phi(u, v). \end{aligned}$$

Then the area of this surface is calculated by:

$$A = \int_a^b \int_c^d (\sqrt{EG - F^2}) du\, dv, \tag{2.5}$$

where $E$, $F$, and $G$ are coefficients of first fundamental form [1, 12].

**Definition 2.6.** The Rodrigues rotation formula gives an efficient method for computing the rotation matrix $R \in SO(3)$ corresponding to a rotation by an angle $\theta$ about a fixed axis specified by the unit vector $N = (n_1, n_2, n_3) \in \mathbb{R}^3$. Then $R(n, \theta)$ is given by

$$R(n, \theta) = I_3 + \sin\theta N + (1 - \cos\theta) N^2, \tag{2.6}$$

where $I$ is the $3 \times 3$ identity matrix and $N$ denotes the antisymmetric matrix with entries [3, 10, 11];

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

## 3. A New Approach to the Generalized Cantor Set

In this section, a more general form of the Cantor set, which is generalized in the $[0, 1]$ range, is generalized to the $[a, b]$ range. In addition, not only for $\mathbb{R}$, but also for $\mathbb{R}^2$ and $\mathbb{R}^3$, these original approaches are studied. Also curves and surfaces according to this idea has been examined.

### 3.1. A New Approach to the Generalized Cantor Set in $\mathbb{R}$

Considering Definition 2.4, let $\mathfrak{C}_0 = [a, b]$ be a closed interval in $\mathbb{R}$ and $\mathfrak{C}_0$ breaks down into $(2n + 1)$ pieces, where $n \in \mathbb{Z}^+$. Let obtain the set $\mathfrak{C}_1$ by remaining

$$\mathfrak{C}_1 = \left[a, \frac{2na + b}{2n + 1}\right] \cup \cdots \cup \left[\frac{2nb + a}{2n + 1}, b\right] .$$

Similarly the set $\mathfrak{C}_2$ is given by:

$$\mathfrak{C}_2 = \left[a, \frac{4na(n + 1) + b}{(2n + 1)^2}\right] \cup \cdots \cup \left[\frac{4nb(n + 1) + a}{(2n + 1)^2}, b\right] .$$

By similar method, the obtained generalized Cantor sets and discarded sets as follows:

$$\mathbb{V} = \cup_{k=1}^{\infty} \mathbb{V}_m = \mathbb{V}_1 \cup \mathbb{V}_2 \cup \ldots$$

and

$$\mathfrak{C} = \cap_{k=1}^{\infty} \mathfrak{C}_m = \mathfrak{C}_1 \cap \mathfrak{C}_2 \cap \ldots ,$$

where $\mathfrak{C}$ is a generalized Cantor set (see in Fig. 2) and $\mathbb{V}$ is a discarded sets.

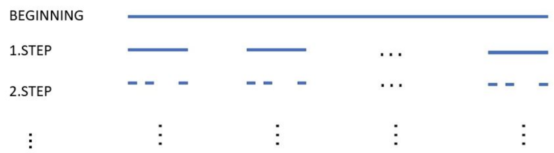


**Figure 2.** Generalized Cantor Set for $[a, b]$.

In order to calculate the length of the pieces, the lengths of the discarded pieces are calculated first. The lengths of the pieces thrown for $m$-th step as:

$$m(\mathbb{V}_m) = \frac{(b - a)n(n + 1)^{(m-1)}}{(2n + 1)^m}$$

and generally since every $V_i$ is disjoint, so we have:

$$\begin{aligned} m(\mathbb{V}) &= m(\mathbb{V}_1) + m(\mathbb{V}_2) + m(\mathbb{V}_3) + \ldots \\ &= \frac{(b - a)n}{2n + 1} + \frac{(b - a)n(n + 1)}{(2n + 1)^2} + \frac{(b - a)n(n + 1)^2}{(2n + 1)^3} + \ldots , \end{aligned}$$

where $\mathbb{V}$ is the discarded sets and $\mathfrak{C}$ is the remaining sets.
Therefore, $m(\mathbb{V})$ is found as $(b - a)$. We know that $m([a, b])$ is equal to $(b - a)$. Hence, $m(\mathfrak{C})$ is found $0$.

**Corollary 3.1.** *If the operations are transferred to the theory of curves, the result is the same and the curve length is found to be zero.*

### 3.2. Iterated function system for $\mathbb{R}$

The iterated function system is a method used to obtain fractals (see in Definition 2.1). The parts in each step of construction of the Cantor set, which are generalized in the interval $[a, b]$, are obtained by iterated function systems.

Considering Definition 2.2, the following theorem can be given:

**Theorem 3.1.** *The iterated functions system for $\mathbb{R}$ is obtained as follow:*

$$F_{i,j} : \mathbb{R} \to \mathbb{R}$$
$$x \to F_{i,j}(x) = r_i x + (1 - r_i)\alpha_i,$$

*where $i$ represents the step number and $j$ represents the part number in each step, denoting the iterated function system $F_{i,j}$. The parameter $r > 0$ denotes the contraction ratio, and $\alpha \in \mathbb{R}$ indicates the center of each part, where $\alpha_i$ is a center and $r_i$ is a rate. For every $r_i = r = \frac{1}{2n+1}$, $n = 1, 2, 3, \ldots$ , where $i$ is the number of pieces in each step:*

***Step 1***

***For part 1*** *Let $\alpha_1 = a$ and $r = \frac{1}{2n+1} \cdot$ Then*

$$F_{1,1}(x) = \frac{1}{2n+1}x + \left(1 - \frac{1}{2n+1}\right) a = \frac{x + 2na}{2n+1} \cdot$$

$$\vdots$$

***For part (n+1)*** *Let $\alpha_{(n+1)} = b$ and $r = \frac{1}{2n+1} \cdot$ Then*

$$F_{1,(n+1)}(x) = \frac{1}{2n+1}x + \left(1 - \frac{1}{2n+1}\right) b = \frac{x + 2nb}{2n+1} \cdot$$

***Step 2***

***For part 1*** *Let $\alpha_1 = a$ and $r = \frac{1}{(2n+1)} \cdot$ Then*

$$F_{2,1}(x) = F(F_{1,1}(x)) = \frac{1}{(2n+1)} \frac{x + 2na}{2n+1} + \left(\frac{2n}{(2n+1)}\right) a$$
$$= \frac{x + 2na}{(2n+1)^2} + \frac{2na}{(2n+1)} \cdot$$

$$\vdots$$

***For part** $(n+1)^2$* *Let $\alpha_{(n+1)^2} = b$ and $r = \frac{1}{(2n+1)} \cdot$ Then*

$$F_{2,(n+1)^2}(x) = F(F_{1,(n+1)}(x)) = \frac{1}{(2n+1)} \frac{x + 2nb}{2n+1} + \left(\frac{2n}{(2n+1)}\right) b$$
$$= \frac{x + 2nb}{(2n+1)^2} + \frac{2nb}{(2n+1)} \cdot$$

*It can be found by doing similar operations in other steps.*

### 3.3. Dimension for $\mathbb{R}$

**Theorem 3.2.** *Let $m \in \mathbb{Z}^+$ be the number of steps, the number of pieces in each step $(n+1)^m$ and the shrinkage rate $\frac{1}{(2n+1)^m}$, dimension calculation of the generalized Cantor set for the interval $[a, b]$ is defined as:*

$$d_{(2n+1)} = \lim_{m \to 0} \frac{\log (n+1)^m}{\log (2n+1)^m} = \frac{\log(n+1)}{\log(2n+1)} \cdot$$

**Corollary 3.2.** *Since the dimension calculation does not depend on the interval but on the number of pieces and the reduction (growth) rate of the number of pieces, the dimension of the generalized Cantor set defined in the interval $[0, 1]$ and the dimension calculated in the interval $[a, b]$ give the same result.*

**Example 3.1.** Let consider the curve:

$$\alpha : [0, 2\pi] \to S'$$
$$t \to \alpha(t) = (\cos t, \sin t).$$

Calculate the length and dimension of the fractal structure obtained by the circle curve with a reduction ratio of $\frac{1}{5}$ (see in Fig. 3).

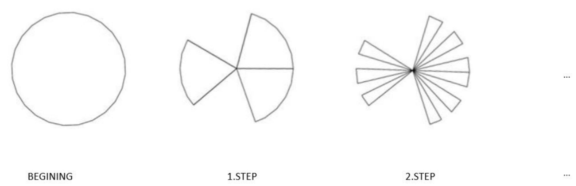


**Figure 3.** Example of a circle formed by transferring the interval $[a, b]$ on the curve.

Calculating the initial length of the circle curve, the result is $2\pi$. Later, when the length of all the pieces that are thrown in the structure shrinks in the ratio of $\frac{1}{5}$ as shown in the figure in each step are calculated, the result is again $2\pi$. Thus, the length of the remaining fractal structure is found to be zero.

The reduction ratio $(\frac{1}{5})^n$, and the number of shapes formed in each step $3^n$, so the dimension is obtained as follows:

$$d = \frac{\log 3}{\log 5} = \frac{\log 3}{\log 5} = 0,682.$$

### *3.4. A New Approach to the Generalized Cantor Set in $\mathbb{R}^2$*

Considering Definition 2.4, let $\mathfrak{C}_0{}^2 = [a, b] \times [a, b]$ be closed interval in $\mathbb{R}^2$. Each interval breaks down into $(2n + 1)$ pieces for $\mathfrak{C}_0$, where $n \in \mathbb{Z}^+$. Let obtain the set $\mathfrak{C}_1{}^2$ by remaining

$$\mathfrak{C}_1{}^2 = \left[a, \frac{2na + b}{2n + 1}\right] \times \left[c, \frac{2nc + d}{2n + 1}\right] \cup \cdots \cup \left[\frac{2nb + a}{2n + 1}, b\right] \times \left[\frac{2nd + c}{2n + 1}, d\right].$$

Similarly, the set $\mathfrak{C}_2{}^2$ is given by:

$$\begin{aligned} \mathfrak{C}_2{}^2 &= \left[a, \frac{4na(n + 1) + b}{(2n + 1)^2}\right] \times \left[c, \frac{4nc(n + 1) + d}{(2n + 1)^2}\right] \cup \ldots \\ &\cup \left[\frac{4nb(n + 1) + a}{(2n + 1)^2}, b\right] \times \left[\frac{4nd(n + 1) + c}{(2n + 1)^2}, d\right]. \end{aligned}$$

By similar method, the obtained generalized Cantor sets and discarded sets as follows:

$$\mathbb{V}^2 = \cup_{k=1}^{\infty} \mathbb{V}_m{}^2 = \mathbb{V}_1{}^2 \cup \mathbb{V}_2{}^2 \cup \ldots$$

and

$$\mathfrak{C}^2 = \cap_{k=1}^{\infty} \mathfrak{C}_m{}^2 = \mathfrak{C}_1{}^2 \cap \mathfrak{C}_2{}^2 \cap \ldots,$$

where $\mathfrak{C}^2$ is a generalized Cantor set for $\mathbb{R}^2$ and $\mathbb{V}^2$ is a discarded sets (see in Fig. 4).
In order to calculate the area of the region, the areas of the discarded regions are calculated first.
The area of the region thrown for $m$-th step

$$A(\mathbb{V}_m^2) = \frac{(b - a)(d - c)(3n^2 + 2n)(n + 1)^{(2m-2)}}{(2n + 1)^{2m}}$$

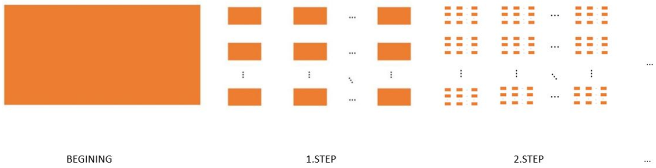


**Figure 4.** Generalized Cantor Set for $[a, b] \times [c, d]$

and generally since every ${V_m}^2$ is disjoint, so we have:

$$\begin{aligned} A(\mathbb{V}^2) &= A(\mathbb{V}_1^2) + A(\mathbb{V}_2^2) + A(\mathbb{V}_3^2) + \dots \\ A(\mathbb{V}^2) &= \frac{(b-a)(d-c)(3n^2+2n)}{(2n+1)^2} \\ &\quad + \frac{(b-a)(d-c)(3n^2+2n)(n+1)^2}{(2n+1)^4} \\ &\quad + \frac{(b-a)(d-c)(3n^2+2n)(n+1)^4}{(2n+1)^6} + \dots, \end{aligned}$$

where $\mathbb{V}^2$ is the discarded areas and $\mathfrak{C}^2$ is the remaining areas. Thus, $A(\mathbb{V}^2)$ is found $(b-a) \times (d-c)$. We know that $A([a, b] \times [a, b])$ is equal to $(b-a) \times (d-c)$. Hence, $A(\mathfrak{C}^2)$ is found $0$.

*Remark* 3.1. When this examination is done for $[a, b] \times [a, b]$ rather than $[a, b] \times [c, d]$ result is found similar.

### *3.5. Surface Review*

Using Eq. (2.1), the property $\sqrt{EG - F^2} = ||\phi_u \times \phi_v|| = 1$ is specifically used. Then area of this surface is,

$$A = \int_a^b \int_c^d du\, dv = (d-c)(b-a).$$

This surface is divided into $(2n+1)$ parts. If the pieces are discarded similarly to the previous ones, the areas of the discarded pieces are obtained as follows:

$$A_1 = \int_{\frac{2na+2b-a}{2n+1}}^{\frac{2na+b}{2n+1}} \int_{\frac{2nc+2d-c}{2n+1}}^{\frac{2nc+d}{2n+1}} du\, dv + \dots + \int_{\frac{2nb+a}{2n+1}}^{\frac{2nb-b+2a}{2n+1}} \int_{\frac{2nd+c}{2n+1}}^{\frac{2nd-d+2c}{2n+1}} du\, dv$$

or

$$A_1 = (3n^2+2n)\left[\frac{(b-a)(d-c)}{(2n+1)^2}\right].$$

Similarly,

$$A_2 = (3n^2+2n)(n+1)^2\left[\frac{(b-a)(d-c)}{(2n+1)^4}\right].$$

If this is continued infinitely, the areas of the discarded surfaces are obtained as follows:

$$A = (b-a)(c-d).$$

Since the result obtained is equal to the initial area calculation, the area of the remaining fractal structure is found to be zero.

**Corollary 3.3.** *As can be seen from the results obtained, when the processes are transferred to the theory of surfaces, the same results are obtained and the area of the fractal structure obtained is found to be zero.*

### *3.6. Iterated function system for* $\mathbb{R}^2$

The generalization of iterated function systems used to obtain fractals in $\mathbb{R}^2$ is obtained as follows:

**Theorem 3.3.** *Let us take the iterated function system*

$$F_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2,$$

*where $i$ represents the step number and $j$ represents the part number in each step, denoting the iterated function system $F_{i,j}$. Then the first step is obtained as follows:*

$$\begin{aligned}
F_{1,1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \\
F_{1,2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2na+2b-a}{2n+1} \\ 0 \end{bmatrix}, \\
&\vdots \\
F_{1,r}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2na+2b-a}{2n+1} \end{bmatrix}, \\
&\vdots \\
F_{1,(n+1)^2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{2nb+a}{2n+1} \\ \frac{2nb+a}{2n+1} \end{bmatrix},
\end{aligned}$$

*where a is the starting point, b is the ending point and $n \in \mathbb{Z}^+$.*
*Similarly, the other steps are obtained by using the ones obtained in the first step in the second step.*

*3.7. Dimension for $\mathbb{R}^2$*

**Theorem 3.4.** *Let $m \in \mathbb{Z}^+$ be the number of steps, $(n+1)^{2m}$ be the number of pieces in each step, and $\dfrac{1}{(2n+1)^m}$ be the reduction rate. The dimension calculation of the generalised Cantor set for the interval $[a, b] \times [c, d]$ is follows:*

$$d_{(2n+1)} = \lim_{m \to 0} \frac{\log (n+1)^{2m}}{\log (2n+1)^m} = \frac{2\log(n+1)}{\log(2n+1)}.$$

**Example 3.2.** For the fractal structure formed by the region $[0, 5] \times [0, 5]$, let us create iterated function systems by calculating the area and dimension (see Fig. 5). In order to calculate the area for this region, let us calculate

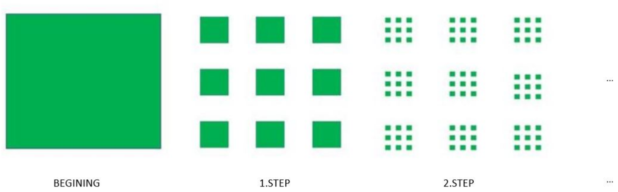


**Figure 5.** Example of $[0, 5] \times [0, 5]$

firstly the areas of the discarded regions as follows:

$$\begin{aligned}
A(V^2) &= A(V_1^2) + A(V_2^2) + A(V_3^2) + \dots \\
&= \frac{5^2.16}{5^2} + \frac{5^2.16.3^2}{5^4} + \frac{5^2.163^4}{5^6} + \dots \\
&= \frac{5^2.16}{5^2}\left[1 + \frac{3^2}{5^2} + \frac{3^4}{5^4} + \dots\right].
\end{aligned}$$

Then it gives

$$A(V^2) = \frac{5^2.16}{5^2}.\frac{5^2}{16} = 5^2.$$

Since the result obtained is equal to the initial area calculation, the area of the remaining fractal structure is found to be zero. The dimension is found as follows:

$$d_5 = \lim_{m \to 0} \frac{\log 3^{2m}}{\log 5^m} = \frac{2 \log 3}{\log 5} = 1,365.$$

Iterated function systems for the first step, with 9 parts in each step

$$F_{1,1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$
$$F_{1,2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$
$$F_{1,3}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$
$$F_{1,4}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$
$$F_{1,5}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$
$$F_{1,6}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$
$$F_{1,7}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$
$$F_{1,8}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$
$$F_{1,9}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

are found. Iterated function systems for parts in other steps are found similarly.

**Example 3.3.** Let us consider the surface

$$\theta : [0, h] \times [0, 2\pi] \to E^3$$
$$(u, v) \to \theta(u, v) = (\cos u, \sin u, v).$$

Calculate the area and dimension of the fractal structure with $\frac{1}{3}$ shrinking ratio for the cylinder (see in Fig. 6). Calculating the initial area of the cylinder, the result is $2\pi h$. When the areas of all the pieces that are thrown in

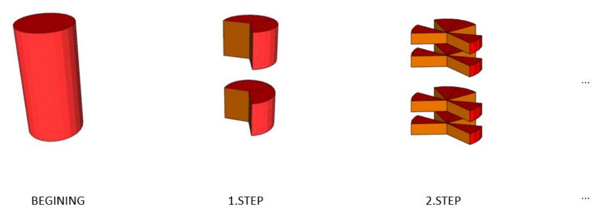


**Figure 6.** Example of cylinder

the structure that shrinks in the ratio of $\frac{1}{3}$ as shown in the figure in each step are calculated, the result is again $2\pi h$. Therefore, the area of the remaining fractal structure is found to be zero.

For the dimension calculation, the dimension is obtained as follows, with the reduction ratio $(\frac{1}{3})^n$ and the number of shapes formed in each step $4^n$

$$d = \frac{\log 4}{\log 3} = \frac{2 \log 2}{\log 3} = 1,261.$$

### 3.8. A New Approach to the Generalized Cantor Set in $\mathbb{R}^3$

Considering Definition 2.4, let ${C_0}^3 = [a,b] \times [c,d] \times [e,f]$ in $\mathbb{R}^3$, and each interval breaks down into $(2n+1)$ pieces for ${C_0}^3$, where $n \in \mathbb{Z}^+$. Let obtain the set ${C_1}^3$ by remaining

$$\begin{aligned} I_{1,1} &= \left[a, \frac{2na+b}{2n+1}\right] \times \left[c, \frac{2nc+d}{2n+1}\right] \times \left[e, \frac{2ne+f}{2n+1}\right], \\ &\vdots \\ I_{1,(n+1)^3} &= \left[\frac{2nb+a}{2n+1}, b\right] \times \left[\frac{2nd+c}{2n+1}, d\right] \times \left[\frac{2nf+e}{2n+1}, f\right]. \end{aligned}$$

Then,

$$\mathfrak{C}_1^3 = I_{1,1} \cup I_{1,2} \cup \cdots \cup I_{1,(n+1)^3}$$

is obtained. Similarly, if done in other steps, generally the parts that are discarded and the remaining parts are obtained as follows:

$$\mathbb{V}^3 = \cup_{k=1}^{\infty} \mathbb{V}_m^3 = \mathbb{V}_1^3 \cup \mathbb{V}_2^3 \cup \ldots$$

and

$$\mathfrak{C}^3 = \cap_{k=1}^{\infty} \mathfrak{C}_m^3 = \mathfrak{C}_1^3 \cap \mathfrak{C}_2^3 \cap \ldots,$$

where $\mathfrak{C}^3$ is a generalized Cantor set for $\mathbb{R}^3$ and $\mathbb{V}^3$ is a discarded sets (see in Fig. 7). In order to calculate the volume of the region, first calculate the volume of the discarded regions. The volume of the regions thrown for $m$-th step is,

$$V(\mathbb{V}_m^3) = \frac{(b-a)(d-c)(f-e)(7n^3+9n^2+3n)(n+1)^{3(m-1)}}{(2n+1)^{3m}}$$

and generally since every $\mathbb{V}_m^3$ disjoint. Thus, by calculating the volume of this region,

$$V(\mathbb{V}^3) = (b-a)(d-c)(f-e)$$

is obtained. Since this result is equal to the volume of the initial region, the volume of the fractal structure is found to be zero.

*Remark* 3.2. When this examination is done for $[a,b] \times [a,b] \times [a,b]$ rather than $[a,b] \times [c,d] \times [e,f]$ result is found similar (see in Fig. 8).

### 3.9. Iterated function system for $\mathbb{R}^3$

The generalization of iterated function systems used to obtain fractals in $\mathbb{R}^3$ is obtained as follows:

**Theorem 3.5.** *Let us take the iterated function system*

$$F_{i,j} : \mathbb{R}^3 \to \mathbb{R}^3,$$

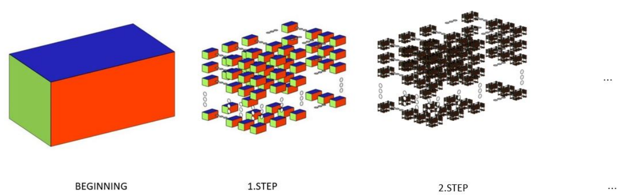


**Figure 7.** Generalized Cantor Set for $[a,b] \times [c,d] \times [e,f]$

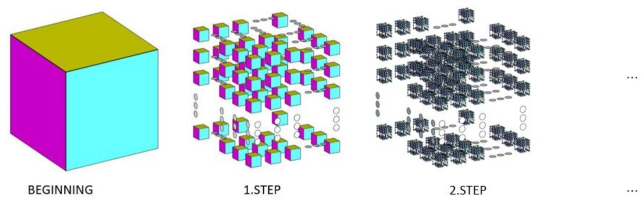


**Figure 8.** Generalized Cantor Set for $[a, b] \times [a, b] \times [a, b]$

*where $i$ represents the step number and $j$ represents the part number in each step, denoting the iterated function system $F_{i,j}$. Then the first step is obtained as follows:*

$$F_{1,1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$F_{1,2}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \frac{2na+2b-a}{2n+1} \\ 0 \\ 0 \end{bmatrix},$$

$$\vdots$$

$$F_{1,r}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2na+2b-a}{2n+1} \\ 0 \end{bmatrix},$$

$$\vdots$$

$$F_{1,m}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{2na+2b-a}{2n+1} \end{bmatrix},$$

$$\vdots$$

$$F_{1,(n+1)^3}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{b-a}{2n+1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \frac{2nb+a}{2n+1} \\ \frac{2nb+a}{2n+1} \\ \frac{2nb+a}{2n+1} \end{bmatrix},$$

*where a is the starting point, b is the ending point and $n \in \mathbb{Z}^+$.*
*Similarly, the other steps are obtained by using the ones obtained in the first step in the second step.*

### *3.10. Rotation in iterated function systems*

In this section, the rotation of the obtained fractal structure at a certain angle is analysed. We will use the Olin-Rodrigues formula (see details in Definition 2.6).

**Example 3.4.** Let us write the iterated function system for a Cantor cube with an angle of rotation $\theta = \frac{\pi}{3}$ around the $x$ axis (see in Fig. 9). Then (using Eq.(2.6)) the rotation matrix is,

$$R\left(n, \frac{\pi}{3}\right) = I_3 + \left(\sin\frac{\pi}{3}\right)N + \left(1 - \cos\frac{\pi}{3}\right)N^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

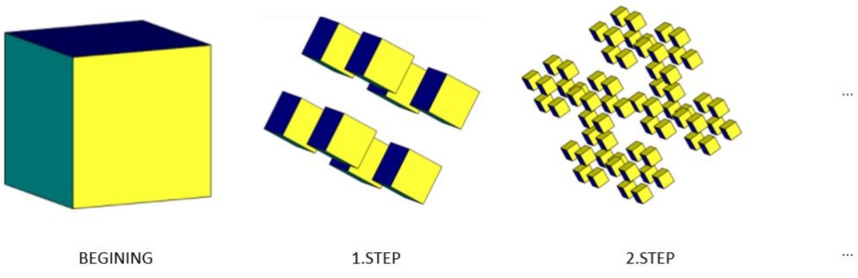


**Figure 9.** Example of rotation

and for first step iterated function system

$$F_{1,1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$F_{1,2}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \end{bmatrix},$$

$$F_{1,3}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{3} \\ 0 \end{bmatrix},$$

$$F_{1,4}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{2}{3} \end{bmatrix},$$

$$F_{1,5}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix},$$

$$F_{1,6}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2}\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + \begin{bmatrix}\frac{2}{3}\\0\\\frac{2}{3}\end{bmatrix},$$

$$F_{1,7}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2}\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + \begin{bmatrix}0\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix},$$

$$F_{1,8}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2}\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} + \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$$

are obtained.

### *3.11. Dimension for $\mathbb{R}^3$*

**Theorem 3.6.** *Let $m \in \mathbb{Z}^+$ be the number of steps, $(n+1)^{3m}$ be the number of pieces in each step and $\frac{1}{(2n+1)^m}$ be the reduction rate. The dimension calculation of the generalised Cantor set for the interval $[a,b] \times [c,d] \times [e,f]$ is done as follows:*

$$d_{(2n+1)} = \lim_{m\to 0} \frac{\log{(n+1)^{3m}}}{\log{((2n+1)^m)}} = \frac{3\log(n+1)}{\log(2n+1)}.$$

**Example 3.5.** For the fractal structure formed by the region $[0,5] \times [0,5] \times [0,5]$, let us create iterated function systems by calculating the volume and dimension (see in Fig. 10). In order to calculate the volume for this

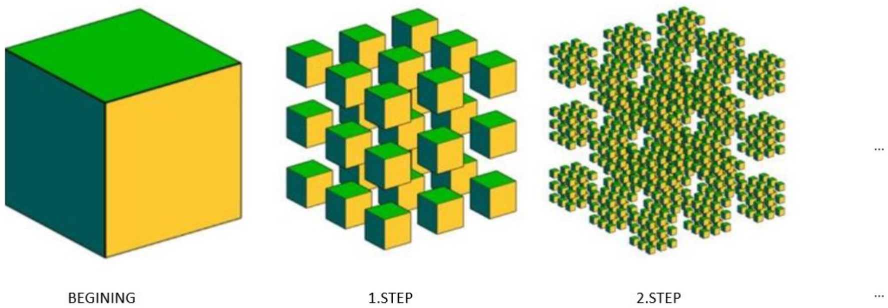


**Figure 10.** Example of $[0,5] \times [0,5] \times [0,5]$

region, first calculate the volumes of the discarded regions. Then,

$$\begin{aligned} V(\mathbb{V}^3) &= V(\mathbb{V}_1^3) + V(\mathbb{V}_2^3) + V(\mathbb{V}_3^3) + \ldots \\ &= \frac{5^3.98}{5^3} + \frac{5^3.98.3^3}{5^6} + \frac{5^3.98.3^6}{5^9} + \ldots \\ &= \frac{5^3.98}{5^3}\left[1 + \frac{3^3}{5^3} + \frac{3^6}{5^6} + \ldots\right] \end{aligned}$$

and finally

$$V(\mathbb{V}^3) = \frac{5^3.98}{5^3}.\frac{5^3}{98} = 5^3$$

is obtained. Since the result obtained is equal to the initial volume calculation, the volume of the remaining fractal structure is found to be zero. The dimension is found as follows:

$$d_{[0,5]\times[0,5]\times[0,5]} = \lim_{m\to 0} \frac{\log 3^{3m}}{\log (5^m)} = \frac{3\log 3}{\log 5} = 2,047.$$

The iterated function systems for the first step, with 27 parts in each step,

$$F_{1,1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$F_{1,2}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix},$$

$$F_{1,3}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix},$$

$$F_{1,4}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

$$\vdots$$

$$F_{1,27}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

are obtained. Iterated function systems for parts in other steps are found similarly.

*Remark* 3.3. As in the Cantor set, it is seen that the length, area and volume are not indicate a characteristic feature in the generalized Cantor sets, and the characteristic feature for these structures are the concept of dimension.

## 4. Conclusions

This paper provides a definition of fractal structures and generalizes one of the most fundamental fractal structures, the Cantor set in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. Initially, in the generalization examined in the closed interval $[a, b]$, iterative function systems are created using fractal properties such as self-similarity, iterative formation, and dimension concepts. Later, the concept of dimension is defined by observing that the concept of length approaches zero and characteristic properties are not determined. Similar processes are applied to $\mathbb{R}^2$, where it is seen that curves and surfaces in differential geometry can also be examined. When extending to $\mathbb{R}^3$, similar fractal property definitions are given, and the structure is rotated using Olin Rodrigues formula from analytical geometry. The obtained results show that the properties of the generalized structure conform to the fractal definition. This study allows Cantor sets to be examined in a broader context.

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### Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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## Affiliations


İPEK EBRU KARAÇAY
**ADDRESS:** İstanbul Gelisim University, Gelisim Vocational School, Department of Computer Programming, 34310, İstanbul-Türkiye.
**E-MAIL:** iekaracay@gelisim.edu.tr
**ORCID ID: 0000-0002-5289-6457**

SALİM YÜCE
**ADDRESS:** Yildiz Technical University, Faculty of Arts and Sciences, Department of Mathematics, 34220, İstanbul-Türkiye.
**E-MAIL:** sayuce@yildiz.edu.tr
**ORCID ID: 0000-0002-8296-6495**