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SPLIT $(s,t)-$LUCAS QUATERNIONS

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Abstract. In this paper, we introduce a new class of split $(s,t)-$ Lucas quaternions that generalizes the split Lucas quaternions. Additionally, we derive Binet-like formulas, generating functions, binomial sums and Honsberger-like, d'Ocagne-like, Catalan's-like and Cassini's-like identities.

1. Introduction

Quaternions are a number system that extends real numbers to one real and three imaginary dimensions. They were first defined by the Irish mathematician Sir William Rowan Hamilton in 1843 and have been applied to mathematics in $3-$dimensional space. Quaternions do not possess the commutative property ($ab = ba$). Nowadays, many researchers are relating quaternions to Fibonacci and other special number sequences. Halıcı investigated the Fibonacci and Lucas quaternions, and gave the generating functions, Binet formulas and some sum formulas for these quaternions [6]. İpek studied on the quaternions of the $(p,q)-$Fibonacci sequence, which are generalizations of the Fibonacci sequence [4]. Likewise, Çimen and İpek also investigated Pell quaternions and Pell-Lucas quaternions [20]. Taşçı defined Padovan and Pell-Padovan quaternions, and gave Binet-like formulas, generating functions, sums formulas and the matrix representation of the Padovan and Pell-Padovan quaternions [21]. Dişkaya and Menken worked on the quaternions of the $(s,t)-$Padovan and $(s,t)-$Perrin sequences, which are generalizations of the Padovan and Perrin sequences [13]. Refer to [5, 9–11, 14] for more details on their research. Split quaternions are a variation of quaternions where the standard basis elements satisfy slightly different multiplication rules. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements and nontrival idempotent. The split quaternions were defined by James Cockle in 1849. A split quaternion is defined by

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

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where q_0, q_1, q_2 and q_3 are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standart basis in $\mathbb{R}^4$. Then we can write

$$q = S_q + V_q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where $S_q = q_0e_0$ and $V_q = q_1e_1 + q_2e_2 + q_3e_3$. S_q is called the scalar part of the split quaternion q and V_q is called the vector part of the split quaternion q. The split quaternion multipication is defined using the rules;

$$e_0^2 = -1, \quad e_1^2 = e_2^2 = e_3^2 = 1$$

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = -e_1 \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2.$$

This algebra is associative and non-comutative . Let $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ and $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ be any two split quaternions. Then the addition and subtraction of the split quaternions is

$$q \mp p = (q_0 \mp p_0)e_0 + (q_1 \mp p_1)e_1 + (q_2 \mp p_2)e_2 + (q_3 \mp p_3)e_3$$

and multiplication of the split quaternions is

$$\begin{aligned} qp &= (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3)(p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \\ &= (q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3)e_0 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)e_1 \\ &+ (q_0p_2 + q_2p_0 - q_1p_3 + q_3p_1)e_2 + (q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1)e_3 \\ &= S_qS_p + < V_q, V_p > + S_qV_p + S_pV_q + V_q\text{x}V_p \end{aligned}$$

where

$$< V_q, V_p >= q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3$$

and

$$V_q\text{x}V_p = \begin{vmatrix} -e_1 & e_2 & e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

and for $k \in \mathbb{R}$ the multiplication by scalar is

$$kq = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3$$

and conjugate and norm of split quaternion q are

$$\overline{q} = q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3$$

and

$$||q|| = \sqrt{|q\overline{q}|} = \sqrt{q_0^2 + q_1^2 - q_2^2 - q_3^2}$$

The basic operations on the two split quaternions given above can also be seen in [1, 8, 12, 15–19].

In [2], the Lucas sequence $\{L_n\}_{n\geq 0}$ is

$$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \tag{1.1}$$

for all $n \geq 2$. Here, L_n is the $n-$th Lucas number. The first few terms of the Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364.$$

A generalization of the Lucas sequence $\{L_n\}_{n\geq 0}$, which are called the $(s,t)-$Lucas sequence $\{L_{s,t,n}\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and $s,t \geq 1$ such that $s^2+4t>0$;

$$L_{s,t,0}=2, \quad L_{s,t,1}=s \quad \text{and} \quad L_{s,t,n+2}=sL_{s,t,n+1}+tL_{s,t,n} \tag{1.2}$$

$(s,t)-$Lucas sequence refer to reader to [3]. The first few terms of the $(s,t)-$Lucas numbers are

$$2, s, s^2+2t, s^3+3st, s^4+4s^2t+2t^2.$$

To simplify notation, take $L_{s,t,n}=\mathcal{L}_n$. In [3], for every $x\in\mathbb{N}$, one can write the Binet-like formula for the $(s,t)-$Lucas sequence as the form

$$\mathcal{L}_n=\alpha^n+\beta^n \tag{1.3}$$

where $\alpha=\frac{s+\sqrt{s^2+4t}}{2}$ and $\beta=\frac{s-\sqrt{s^2+4t}}{2}$ are the roots of the characteristic equation

$$x^2-sx-t=0 \tag{1.4}$$

associated with the recurrence relation (1.2). Moreover, it can be observed that

$$\alpha^n=\alpha\mathcal{L}_n+t\mathcal{L}_{n-1},$$

$$\beta^n=\beta\mathcal{L}_n+t\mathcal{L}_{n-1},$$

$$\alpha+\beta=s,$$

$$\alpha-\beta=\sqrt{s^2+4t},$$

$$\alpha\beta=-t.$$

The following properties hold [7]:

(1) $\mathcal{L}_m\mathcal{L}_{n+1}+t\mathcal{L}_{m-1}\mathcal{L}_n=\left(s^2+4t\right)\mathcal{F}_{m+n}, \quad m\geq n$
(2) $\mathcal{L}_{n-r}\mathcal{L}_{n+r}-\mathcal{L}_n^2=(-t)^{n-r}\left(s^2+4t\right), \quad n\geq r$
(3) $\mathcal{L}_{n-1}\mathcal{L}_{n+1}-\mathcal{L}_n^2=(-t)^{n-1}\left(s^2+4t\right), \quad n\geq 1$

2. Split $(p,q)-$Lucas Quaternions

In this section, we define new split quaternions that are split $(p,q)-$Lucas quaternions. Then, we give their Binet-like formula, generating functions, certain binomal sums and Honsberg, d'Ocagne, Catalan's and Cassini's identites.

Definition 2.1. The split $(p,q)-$Lucas quaternion $\{\mathcal{QL}_{s,t,n}\}_{n\geq 0}$ is defined by

$$\mathcal{QL}_{s,t,n}=\mathcal{L}_ne_0+\mathcal{L}_{n+1}e_1+\mathcal{L}_{n+2}e_2+\mathcal{L}_{n+3}e_3 \tag{2.1}$$

where $\mathcal{L}_n$ is the $n-$th $(s,t)-$Lucas number.

To simplify notation, take $\mathcal{QL}_{s,t,n}=\mathcal{QL}_n$.

Theorem 2.2. *The Binet-like formula for the $n-$th split $(s,t)-$Lucas quaternion is*

$$\mathcal{QL}_n=\hat{\alpha}\alpha^n+\hat{\beta}\beta^n, \qquad n\geq 0 \tag{2.2}$$

where $\hat{\alpha}=e_0+\alpha e_1+\alpha^2e_2+\alpha^3e_3$ *and* $\hat{\beta}=e_0+\beta e_1+\beta^2e_2+\beta^3e_3$.

Proof. From the definition of $n-$th split $(s,t)-$Lucas quaternion $\mathcal{L}_n$, we obtain

$$\begin{aligned}\mathcal{QL}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 \\ &= (\alpha^n + \beta^n)\, e_0 + \left(\alpha^{n+1} + \beta^{n+1}\right) e_1 + \left(\alpha^{n+2} + \beta^{n+2}\right) e_2 + \left(\alpha^{n+3} + \beta^{n+3}\right) e_3 \\ &= \alpha^n (e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3) + \beta^n (e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3) \\ &= \hat{\alpha}\alpha^n + \hat{\beta}\beta^n.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.3. *The generating function for the $n-$th split $(s,t)-$Lucas quaternions is*

$$\mathcal{G}_{\mathcal{L}}(x) = \frac{2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3 + (-se_0 + 2te_1 + ste_2 + (s^2t + 2t^2)e_3)x}{1 - sx - tx^2}$$

Proof. Let

$$\mathcal{G}_{\mathcal{L}}(x) = \sum_{n=0}^{\infty} \mathcal{QL}_n x^n = \mathcal{QL}_0 + \mathcal{QL}_1 x + \mathcal{QL}_2 x^2 + \mathcal{QL}_3 x^3 + \ldots + \mathcal{QL}_n x^n + \ldots$$

be generating function of the split $(p,q)-$Lucas quaternions. This function is multipied every side with $-sx$ such as

$$-sx\mathcal{G}_{\mathcal{L}}(x) = -s\mathcal{QL}_0 x - s\mathcal{QL}_1 x^2 - s\mathcal{QL}_2 x^3 - s\mathcal{QL}_3 x^4 - \ldots - s\mathcal{QL}_n x^{n+1} - \ldots$$

and that is multipied every side with $-tx^2$ such as

$$-tx^2\mathcal{G}_{\mathcal{L}}(x) = -t\mathcal{QL}_0 x^2 - t\mathcal{QL}_1 x^3 - t\mathcal{QL}_2 x^4 - t\mathcal{QL}_3 x^5 - \ldots - t\mathcal{QL}_n x^{n+2} - \ldots$$

Then, we write

$$\begin{aligned}(1 - sx - tx^2)\mathcal{G}_{\mathcal{L}}(x) &= \mathcal{QL}_0 + (\mathcal{QL}_1 - s\mathcal{QL}_0)x + (\mathcal{QL}_2 - s\mathcal{QL}_1 - t\mathcal{QL}_0)x^2 + \ldots \\ &\quad + (\mathcal{QL}_n - s\mathcal{QL}_{n-1} - t\mathcal{QL}_{n-2})x^n\end{aligned}$$

Now using

$$\mathcal{QL}_0 = 2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3,$$

$$\mathcal{QL}_1 = se_0 + (s^2+2t)e_1 + (s^3+3st)e_2 + (s^4+4s^2t+2t^2)e_3,$$

$$\mathcal{QL}_2 = (s^2+2t)e_0 + (s^3+3st)e_1 + (s^4+4s^2t+2t^2)e_2 + (s^5+5s^3t+5st^2)e_3$$

and

$$\mathcal{QL}_n - s\mathcal{QL}_{n-1} - t\mathcal{QL}_{n-2} = 0$$

we obtain

$$\mathcal{G}_{\mathcal{L}}(x) = \frac{2e_0 + se_1 + (s^2+2t)e_2 + (s^3+3st)e_3 + (-se_0 + 2te_1 + ste_2 + (s^2t + 2t^2)e_3)x}{1 - sx - tx^2}.$$

Thus, the proof is completed. □

Remark 2.4. Let m be a positive integer. Then,

$$(a+b)^m = \sum_{n=0}^{m} \binom{m}{n} a^n b^{m-n} \tag{2.3}$$

where a and b are any real numbers.

Theorem 2.5. *Let m be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\mathcal{QL}_{n}=\mathcal{QL}_{2m}.$$

Proof. Applying the Binet-like formula (2.2) and combining this with (1.4) and (2.3) we obtain the identity

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\mathcal{QF}_{n}&=\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\left(\hat{\alpha}\alpha^{n}+\hat{\beta}\beta^{n}\right)\\&=\sum_{n=0}^{m}\binom{m}{n}\left(\hat{\alpha}(\alpha s)^{n}t^{m-n}+\hat{\beta}(\beta s)^{n}t^{m-n}\right)\\&=\hat{\alpha}(s\alpha+t)^{m}+\hat{\beta}(s\beta+t)^{m}\\&=\hat{\alpha}\alpha^{2m}+\hat{\beta}\beta^{2m}\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.6. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QL}_{n-k}=\mathcal{QL}_{n+m}$$

Proof. Applying the Binet-like formula (2.2) and combining this with (1.4) and (2.3) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QL}_{n-k}&=\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\left(\hat{\alpha}\alpha^{n-k}+\hat{\beta}\beta^{n-k}\right)\\&=\sum_{k=0}^{m}\binom{m}{k}\left(\hat{\alpha}(s\alpha)^{m-k}t^{k}\alpha^{n-m}+\hat{\beta}(s\beta)^{m-k}t^{k}\beta^{n-m}\right)\\&=\hat{\alpha}(s\alpha+t)^{m}\alpha^{n-m}+\hat{\beta}(s\beta+t)^{m}\beta^{n-m}\\&=\hat{\alpha}\alpha^{n+m}+\hat{\beta}\beta^{n+m}.\end{aligned}$$

Thus, the proof is completed. □

Henceforth, we will get

$$\mathcal{A}_{n}\text{ instead of }(\hat{\alpha})^{2}\alpha^{n}+(\hat{\beta})^{2}\beta^{n},$$
$$\mathcal{B}_{n}\text{ instead of }\hat{\beta}\hat{\alpha}\alpha^{n}-\hat{\alpha}\hat{\beta}\beta^{n},$$

in the following theorems.

Theorem 2.7. ***(Hosberg-like İdentity)*** *Let $\mathcal{QL}_{n}$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied*

$$\mathcal{QL}_{n+1}\mathcal{QL}_{m}+t\mathcal{QL}_{n}\mathcal{QL}_{m-1}=\mathcal{A}_{n+m}\sqrt{s^{2}+4t}.$$

Proof.

$$
\begin{aligned}
&\mathcal{QL}_{n+1}\mathcal{QL}_m + t\mathcal{QL}_n\mathcal{QL}_{m-1}\\
&= \left(\hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}\right)\left(\hat{\alpha}\alpha^{m} + \hat{\beta}\beta^{m}\right) + t\left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right)\left(\hat{\alpha}\alpha^{m-1} + \hat{\beta}\beta^{m-1}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} + \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} + (\hat{\beta})^2\beta^{n+m+1}\\
&+ t\left((\hat{\alpha})^2\alpha^{n+m-1} + \hat{\alpha}\hat{\beta}\alpha^{n}\beta^{m-1} + \hat{\beta}\hat{\alpha}\beta^{n}\alpha^{m-1} + (\hat{\beta})^2\beta^{n+m-1}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} + \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} + (\hat{\beta})^2\beta^{n+m+1}\\
&- (\hat{\alpha})^2\alpha^{n+m}\beta - \hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m} - \hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m} - (\hat{\beta})^2\alpha\beta^{n+m}\\
&= (\hat{\alpha})^2\alpha^{n+m}(\alpha-\beta) + (\hat{\beta})^2\beta^{n+m}(\alpha-\beta)\\
&= \left((\hat{\alpha})^2\alpha^{n+m} + (\hat{\beta})^2\beta^{n+m}\right)(\sqrt{s^2+4t})
\end{aligned}
$$

□

Theorem 2.8. ***(d'Ocagne-like Identity)*** *Let $\mathcal{QL}_n$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied,*

$$\mathcal{QL}_m\mathcal{QL}_{n+1} - \mathcal{QL}_{m+1}\mathcal{QL}_n = (-t)^m\mathcal{B}_{n-m}\ \sqrt{s^2+4t}$$

Proof.

$$
\begin{aligned}
&\mathcal{QL}_m\mathcal{QL}_{n+1} - \mathcal{QL}_{m+1}\mathcal{QL}_n\\
&= \left(\hat{\alpha}\alpha^{m} + \hat{\beta}\beta^{m}\right)\left(\hat{\alpha}\alpha^{n+1} + \hat{\beta}\beta^{n+1}\right) - \left(\hat{\alpha}\alpha^{m+1} + \hat{\beta}\beta^{m+1}\right)\left(\hat{\alpha}\alpha^{n} + \hat{\beta}\beta^{n}\right)\\
&= (\hat{\alpha})^2\alpha^{n+m+1} + \hat{\alpha}\hat{\beta}\alpha^{m}\beta^{n+1} + \hat{\beta}\hat{\alpha}\beta^{m}\alpha^{n+1} + (\hat{\beta})^2\beta^{n+m+1}\\
&- (\hat{\alpha})^2\alpha^{n+m+1} - \hat{\alpha}\hat{\beta}\alpha^{m+1}\beta^{n} - \hat{\beta}\hat{\alpha}\beta^{m+1}\alpha^{n} - (\hat{\beta})^2\beta^{n+m+1}\\
&= -\hat{\alpha}\hat{\beta}\alpha^{m}\beta^{n}(\alpha-\beta) + \hat{\beta}\hat{\alpha}\beta^{m}\alpha^{n}(\alpha-\beta)\\
&= (-t)^m\left(\hat{\beta}\hat{\alpha}\alpha^{n-m} - \hat{\alpha}\hat{\beta}\beta^{n-m}\right)\sqrt{s^2+4t}
\end{aligned}
$$

□

Theorem 2.9. ***(Catalan's Identity)*** *Let $\mathcal{QL}_n$ be the split $(s,t)-$Lucas quaternion. The following relations are satisfied,*

$$\mathcal{QL}_{n-r}\mathcal{QL}_{n+r} - \mathcal{QL}_n^2 = (-t)^n\mathcal{B}_0F_r\sqrt{s^2+4t} \tag{2.4}$$

where the Binet formula of the r. Fibonacci number F_r is $\dfrac{\alpha^r-\beta^r}{\alpha-\beta}$.

Proof.

$$
\begin{aligned}
&\mathcal{QL}_{n-r}\mathcal{QL}_{n+r}-\mathcal{QL}_n^2\\
&=\left(\hat{\alpha}\alpha^{n-r}+\hat{\beta}\beta^{n-r}\right)\left(\hat{\alpha}\alpha^{n+r}+\hat{\beta}\beta^{n+r}\right)-\left(\hat{\alpha}\alpha^{n}+\hat{\beta}\beta^{n}\right)^2\\
&=(\hat{\alpha})^2\alpha^{2n}+\hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r}+\hat{\beta}\hat{\alpha}\beta^{n-r}\alpha^{n+r}+(\hat{\beta})^2\beta^{2n}\\
&-(\hat{\alpha})^2\alpha^{2n}-\hat{\alpha}\hat{\beta}\alpha^{n}\beta^{n}-\hat{\beta}\hat{\alpha}\beta^{n}\alpha^{n}-(\hat{\beta})^2\beta^{2n}\\
&=\hat{\alpha}\hat{\beta}(\alpha\beta)^n(\beta^r-\alpha^r)+\hat{\beta}\hat{\alpha}(\alpha\beta)^n(\alpha^r-\beta^r)\\
&=(-t)^n\left(\hat{\beta}\hat{\alpha}-\hat{\alpha}\hat{\beta}\right)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)(\alpha-\beta)\\
&=(-t)^n\mathcal{B}_0F_r\sqrt{s^2+4t}
\end{aligned}
$$

□

Theorem 2.10. ***(Cassini's Identity)*** *Let $\mathcal{QF}_n$ be the split $(p,q)-$Lucas quaternion. The following relations are satisfied*

$$\mathcal{QL}_{n-1}\mathcal{QL}_{n+1}-\mathcal{QL}_n^2=(-t)^n\mathcal{B}_0\sqrt{s^2+4t}.$$

Proof. We take 1 instead of r in (2.4) to prove the this theorem. □

3. Conclusion

In this paper, we introduced a new class of split $(s,t)-$Lucas quaternions, extending the concept of split Lucas quaternions. We derived Binet-like formulas, generating functions, binomial sums, and various identities analogous to those of Honsberger, d'Ocagne, Catalan, and Cassini.

By embedding the $(s,t)-$Lucas sequence within the quaternion algebra, we demonstrated its broader applicability and utility. The derived formulas and identities provide powerful tools for explicit calculations and deeper insights into the sequence's behavior.

Our findings open up new possibilities for future research, particularly in exploring geometric interpretations and potential applications in theoretical physics and computer science. We believe our contributions will stimulate further discovery in this intriguing area of mathematical inquiry.

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The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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