Journal of Universal Mathematics
Vol.8 No.1 pp.40-51 (2025)
ISSN-2618-5660
DOI: 10.33773/jum.1551436

FRACTIONAL ORDER LORENZ CHAOS MODEL AND NUMERICAL APPLICATION

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Abstract. The most complex steady-state behaviour known in dynamical systems is that which is characterised as "chaos". The three-dimensional Lorenz system, which is linear and non-periodic, is a chaotic system that is used to study the properties of a two-dimensional liquid layer that is homogeneously heated from below and cooled from above. In this study, the fractional order Lorenz Chaos model is considered and mathematically analysed. This model consists of three compartments: x orbit, y orbit and z orbit. The fractional derivative is used in the sense of Caputo. The numerical results for the fractional Lorenz Chaos model are obtained with the help of the Euler method, and graphs are drawn.

1. Introduction

Chaos is science that helps to explain non-linear phenomena, defined, in its shortest definition, as the order of disorder. It is a complex process, but one with its own internal order. It is important to note that chaos is not randomness. Chaos is a unique "order" that shows complex behavior. The most complex steady-state behavior known in dynamical systems is "chaos". The study of chaos is part of the theory of nonlinear dynamical systems [1].

Chaos and chaotic signals are characterized by irregularity in the time dimension, sensitive dependence on initial conditions, an unlimited number of different periodic oscillations, a wide noise-like power spectrum, a fractal dimension of the limit set, and signals whose amplitude and frequency cannot be determined but vary in a limited area [2].

The scientific term "chaos" speaks of an interconnectedness that exists within and underlies seemingly random events. Chaos science focuses on hidden patterns of form, subtle differences, the "sensitivity" of things and the "rules" of how the unpredictable gives rise to the new. Chaos is a science that seeks to understand the movements that create the complex patterns of form, from lightning storms, foaming rivers, hurricanes, jagged mountain peaks, jagged coastlines and river deltas to the nerves and blood vessels in our bodies. Chaos is a pattern of behavior that reaches a regular state or repeats itself endlessly. In phase space, the state of all the information of a dynamic

Date: **Received:** 2024-09-17; **Accepted:** 2025-01-31.

Key words and phrases. Fractional Lorenz Chaos Model, Mathematical Modeling, Euler Method, Caputo Derivative.

Mathematics Subject Classification: 34A08, 92B05.

system at a given moment in time is reduced to a single point. This point is the dynamical system itself at that exact moment. In contrast, in the next state following this moment, the system will change, albeit very slightly, and the point will be displaced. The strange attractor occurs in phase space, one of the most important discoveries of modern science. Edward Lorenz was a pioneer of chaos science. In 1963, M.I.T. scientist E. N. Lorenz, while simulating fluid heat-radiation in the atmosphere to predict the weather, observed a new type of irregular oscillations and proposed a model. The mathematics used by Lorenz in his model of the atmosphere was widely investigated in the 1970s, and over time it became known that a fundamental property of a chaotic system is that the smallest difference in two different sets of initial conditions can lead to large differences in the state of the system [3].

The existence of chaos in various branches of engineering and other sciences such as nuclear physics, solid state physics, laser optics, chemistry, biology, medicine, ecology, astronomy, sociology, economics, international relations, history, hydraulics, atmospherics, electricity, electronics, machinery, etc., intensive studies on the subject and the developments in the field have led to the emergence of many application areas related to chaos and chaotic systems. The application areas related to chaos and chaotic systems include; chaotic parallel distributed processing, deterministic nonlinear prediction, identification and modeling of nonlinear systems, nonlinear filtering, biomedical and medical applications, dynamic information compression and coding, chaotic reliable communication, precise pattern recognition, use of chaotic dynamics for music and art, artificial generation of chaotic oscillations, realization of chaotic systems electronically, optically and optoelectronically, detection and control of chaotic vibrations and oscillations, control of lasers, turbulence control, control of crane and ship oscillations, weather forecasting [4].

For numerical modeling and simulation of a physical system with block diagrams, a mathematical model including one or more differential equations and initial conditions on the variables is required. The system can be of linear or nonlinear type. Block diagrams can be modeled and simulated with electronic circuit programs using analog operational elements. Again, the same simulation results can be obtained by setting up the real electronic circuit of the electronic circuit that is numerically modeled and simulated. The system resulting from the implementation of block diagrams as electronic circuits can also be called an "analog computer". The mathematical model of the analog computer created to model a specific physical system is identical to the mathematical model of the system [5, 6].

This paper consists of four parts. In the first part, information about Chaos science and its application area is given. In the second part, the formation of the fractional Lorenz Chaos model, mathematical analysis of the existence, uniqueness and non-negativity of the system and the Generalized Euler method are presented. In the third section, the fractional Lorenz model is applied with the Generalised Euler method and numerical results are obtained and graphs are drawn. In the fourth section, conclusions are given.

2. Fractional Derivation and Fractional Lorenz Chaos Model

The most commonly used definitions of the fractional derivative are Riemann-Liouville, Caputo, Atangana-Baleanu and the Conformable derivative. In this study, because the classical initial conditions are easily applicable and provide ease of calculation, the Caputo derivative operator was preferred and modeling was created. The definition of the Caputo fractional derivative is given below.

Definition 2.1. ([4]) Let $f(t)$ be a function. It can be continuously differentiable n times. The value of the function $f(t)$ for the value of α that satisfies the condition $n-1<\alpha<n$. The Caputo

fractional derivative of $\alpha-$th order $f(t)$ is defined by $D_t^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{(n-\alpha-1)} f^n(x)dx$. These comparisons show that the Caputo fractional-order model presented is more representative of the system than its integer-ordered form. Mathematical modelling based on enhanced models naturally leads to differential equations of fractional order and to the necessity of the formulation of initial conditions to such equations. The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations, contain the limit values of integer-order derivatives of unknown functions at the terminal $t = \alpha$.

Definition 2.2. [4] The Riemann-Liouville (RL) fractional-order integral of a function $A(t) \in C_n$ $(n \geq -1)$ is given by

$$\mathrm{J}^{\gamma} A(t)=\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\gamma-1)} A(s)ds,\ J^0 A(t) = A(t). \tag{2.1}$$

Definition 2.3. [4] The series expansion of two-parametrized form of Mittag-Leffler function for $a, b > 0$ is given by

$$\mathrm{E}_{a,b}(t) = \sum_{t=0}^{\infty} \frac{t^i}{\Gamma(ai+b)}. \tag{2.2}$$

2.1. **The Fractional Lorenz Chaos Model.** The chaotic Lorenz system is the most famous chaotic system for two-dimensional fluid behavior. The chaotic Lorenz system is described by the following system of equations:

$$\begin{aligned} \frac{d^{\alpha}X}{dt^{\alpha}} &= \delta(X-Y) \\ \frac{d^{\alpha}Y}{dt^{\alpha}} &= X(\gamma - Z) - Y \\ \frac{d^{\alpha}Z}{dt^{\alpha}} &= XY - \epsilon Z. \end{aligned} \tag{2.3}$$

Here $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Caputo fractional derivative of α-th order with respect. The initial values are defined as,

$$\mathrm{X}(0)=\mathrm{X}_0,\ Y(0) = Y_0,\ Z(0) = Z_0$$

$0 < \alpha \leq 1$ time t.

Since fractional order models have a memory feature in time-dependent events, they produce more realistic and accurate results than integer order models. For this reason, the established model was created as fractional order. By taking α=1 in system (2.3), the differential equation of fractional order is reduced to a differential equation of full order.

Here δ, γ and ϵ are system parameters, X, Y and Z are dynamic variables. As can be seen from the equations, this chaotic system is a 3rd order system where nonlinearity is ensured by linear product terms. The system is characterized by the generation of non-periodic oscillations whose spectrum is spread over a wide frequency region. Since these oscillations resemble noise and depend on initial conditions in an unpredictable way, it has been realized that they can be used in covert communications [5-23]. Chaotic systems are characterized by "extreme sensitivity to initial conditions". If two chaotic systems of similar structure start to operate with a small difference in initial values, they will soon drift apart.

2.2. **Existence, Uniqueness and Non-Negativity of the System.** We investigate the exintence and uniqueness of the solutions of the fractional-order system (2.3) in the region $B \times [t_0, T]$ where

$$B = \{(X, Y, Z) \in R^3_+ : max\{\mid X \mid, \mid Y \mid, \mid Z \mid\} \leq \Psi, min\{\mid X \mid, \mid Y \mid, \mid Z \mid \geq \Psi_0\} \tag{2.4}$$

and $T < +\infty$.

Theorem 2.4. *For each $H_0 = (X_0, Y_0, Z_0) \in B$ there exists a unique solution $H(t) \in B$ of the fractional-order system (2.3) with intial condition H_0, which is defined for all $t \geq 0$.*

Proof: We denote $H = (X, Y, Z)$ and $\bar{H} = (\bar{X}, \bar{Y}, \bar{Z})$. Consider a mapping $M(H) = (M_1(H), M_2(H), M_3(H))$

$$\begin{aligned} M_1(H) &= \delta(X - Y) \\ M_2(H) &= X(\gamma - Z) - Y \\ M_3(H) &= XY - \epsilon Z. \end{aligned} \tag{2.5}$$

For any $H, \bar{H} \in B$ it follows from equation (2.5) that

$$\| M(H) - M(\bar{H}) \| = \mid M_1(H) - M_1(\bar{H}) \mid + \mid M_2(H) - M_2(\bar{H}) \mid + \mid M_3(H) - M_3(\bar{H}) \mid \tag{2.6}$$

$$\begin{aligned} \mid M_1(H) - M_1(\bar{H}) \mid &= \mid \delta(X - Y) - \delta(\bar{X} - \bar{Y}) \mid \\ &= \mid \delta(X - \bar{X}) - \delta(Y - \bar{Y}) \mid \\ &\leq \delta \mid X - \bar{X} \mid + \delta \mid Y - \bar{Y} \mid \end{aligned}$$

$$\begin{aligned} \mid M_2(H) - M_2(\bar{H}) \mid &= \mid X(\gamma - Z) - Y - \bar{X}(\gamma - \bar{Z}) + \bar{Y} \mid \\ &= \mid \gamma(X - \bar{X}) - (XZ - \bar{X}\bar{Z}) - (Y - \bar{Y}) \mid \\ &\leq \gamma \mid X - \bar{X} \mid + \Psi \mid X - \bar{X} \mid + \Psi \mid Z - \bar{Z} \mid + \mid Y - \bar{Y} \mid \end{aligned}$$

$$\begin{aligned} \mid M_3(H) - M_3(\bar{H}) \mid &= \mid XY - \epsilon Z - \bar{X}\bar{Y} + \epsilon \bar{Z} \mid \\ &= \mid (XY - \bar{X}\bar{Y}) - \epsilon(Z - \bar{Z}) \mid \\ &\leq \Psi \mid X - \bar{X} \mid + \Psi \mid Y - \bar{Y} \mid + \epsilon \mid Z - \bar{Z} \mid \end{aligned}$$

Then equation (2.6) becomes,

$$\| M(H) - M(\bar{H}) \| \leq \delta \mid X - \bar{X} \mid + \delta \mid Y - \bar{Y} \mid + \gamma \mid X - \bar{X} \mid + \Psi \mid X - \bar{X} \mid$$

$$+\Psi \mid Z-\bar{Z} \mid + \mid Y-\bar{Y} \mid +\Psi \mid X-\bar{X} \mid +\Psi \mid Y-\bar{Y} \mid +\epsilon \mid Z-\bar{Z} \mid$$

$$\leq (\delta+\gamma+2\Psi) \mid X-\bar{X} \mid +(1+\delta+\Psi) \mid Y-\bar{Y} \mid +(\Psi+\epsilon) \mid Z-\bar{Z} \mid$$

$$\parallel M(H)-M(\bar{H}) \parallel \leq L \parallel H-\bar{H} \parallel$$

where $L = max(\delta+\gamma+2\Psi, 1+\delta+\Psi, \Psi+\epsilon)$.

Therefore $M(H)$ obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (2.3).

Theorem 2.5. $\forall\ t \geq 0,\ X(0) = X_0 \geq 0,\ Y(0) = Y_0 \geq 0,\ Z(0) = Z_0 \geq 0,$ *the solutions of the system in (2.3) with initial conditions* $(X(t), Y(t), Z(t)) \in R^3_+$ *are not negative.*

Proof: (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(\alpha) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\epsilon)(x-a)^\alpha \tag{2.7}$$

with $0 \leq \epsilon \leq x$, $\forall x \in (a,b]$.
The existence and uniqueness of the solution (2.3) in $(0,\infty)$ can be obtained via Generalized Mean Value Theorem. We need to show that the domain R^3_+ is positively invariant. Since

$$\begin{aligned} D^\alpha X &= \delta(X-Y) \geq 0 \\ D^\alpha Y &= X(\gamma - Z) - Y \geq 0 \\ D^\alpha Z &= XY - \epsilon Z \geq 0 \end{aligned}$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R^3_+.

2.3. **Generalized Euler Method.** In this paper, we used the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many of the mathematical models consist of nonlinear systems, and finding solutions to these systems can be quite difficult. In most cases, analytical solutions cannot be found and a numerical approach should be considered for this. One of these approaches is the Generalized Euler method [15].
$D^\alpha y(t) = f(t, y(t)), y(0) = y_0, 0 < \alpha \leq 1, 0 < t < \alpha$ for the initial value problem, $h = \frac{a}{n}$ impending $[t_j, t_{j+1}]$ is divided into n sub with $j = 0, 1, .., n-1$. Suppose that $y(t), D^\alpha y(t)$ and $D^{2\alpha} y(t)$ are continuous in range $[0, a]$ and using the generalized Taylor's formula, the following equation is obtained [15].

$$y(t_1) = y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)).$$

This process will be repeated to create an array. Let $t_j = t_{j+1} + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_j, y(t_j)$$

$$\begin{aligned} D^\alpha X(t) &= \delta(X(0) - Y(0)) \\ D^\alpha Y(t) &= X(0)(\gamma - Z(0)) - Y(0) \\ D^\alpha Z(t) &= X(0)Y(0) - \epsilon Z(0). \end{aligned} \tag{2.8}$$

$j = 0, 1, .., n - 1$ the generalized formula in the form is obtained. For each $k = 0, 1, ..., n - 1$ with step size h. For $t \in [0, h)$, $\frac{t}{h} \in [0, 1)$ we have

$$\begin{aligned} D^\alpha X(t) &= \delta(X(0) - Y(0)) \\ D^\alpha Y(t) &= X(0)(\gamma - Z(0)) - Y(0) \\ D^\alpha Z(t) &= X(0)Y(0) - \epsilon Z(0). \end{aligned} \tag{2.9}$$

The solution of (2.9) reduces to

$$\begin{aligned} X(1) &= X(0) + \frac{h^\alpha}{\Gamma(\alpha+1)}(\delta(X(0) - Y(0))) \\ Y(1) &= Y(0) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(0)(\gamma - Z(0)) - Y(0)) \\ Z(1) &= Z(0) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(0)Y(0) - \epsilon Z(0)). \end{aligned} \tag{2.10}$$

For $t \in [h, 2h)$, $\frac{t}{h} \in [1, 2)$, we get

$$\begin{aligned} X(2) &= X(1) + \frac{h^\alpha}{\Gamma(\alpha+1)}(\delta(X(1) - Y(1))) \\ Y(2) &= Y(1) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(1)(\gamma - Z(1)) - Y(1)) \\ Z(2) &= Z(1) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(1)Y(1) - \epsilon Z(1)). \end{aligned} \tag{2.11}$$

Repeating the process n times, we obtain

$$\begin{aligned} X(n+1) &= X(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}(\delta(X(n) - Y(n))) \\ Y(n+1) &= Y(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(n)(\gamma - Z(n)) - Y(n)) \\ Z(n+1) &= Z(n) + \frac{h^\alpha}{\Gamma(\alpha+1)}(X(n)Y(n) - \epsilon Z(n)). \end{aligned} \tag{2.12}$$

3. Numerical Simulation of Fractional Lorenz Chaos Model

In the chaotic Lorenz system, the weather at a given instant is represented by a point in the three-dimensional phase space and the course of the weather over time is represented by a trajectory passing through these points. This trajectory represents the history of the dynamical system.

Since chaotic systems are nonlinear, their trajectories are very complex but not random. As time progresses, trajectories begin to fill the phase space and never close over; they repeat. This kind of behavior is a sign of chaos. The set of possible weather states obtained by running the system is called the Lorenz attractor. The Lorenz attractor does not occupy any volume in three-dimensional space.

Let $X = 0,001, Y = 0,0, Z = 0,0, \gamma = 28, \delta = 10, \epsilon = \frac{8}{3}$ and let's take size of step $h = 0.1$. Hence we get the following results and tables. Using the Euler method, we obtain the following tables.

TABLE 1. The values of X, Y and Z at the moment t for $\alpha = 1$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	0,0015	0,0028	0,00
2	0,0028	0,00252	0,00
3	0,0025	0,0101	0,0000007
4	0,0101	0,0161	0,000003
5	0,0161	0,0428	0,0000182
6	0,0428	0,0837	0,0000828
7	0,0837	0,1950	0,000419
8	0,1950	0,4100	0,0019
9	0,4100	0,9160	0,00944
10	0,9160	1,9730	0,0445
11	1,9730	4,3370	0,2130
12	4,3370	9,3870	1,0120
13	9,3870	20,1550	4,8150
14	20,1500	39,9000	22,4500

TABLE 2. The values of X, Y and Z at the moment t for $\alpha = 0.9$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	0,0020	0,00366	0,00
2	0,00489	0,00205	-0,0000001
3	0,00117	0,0197	0,00000121
4	0,0254	0,0214	0,00000382
5	0,0202	0,1110	0,0000739
6	0,1402	0,1710	0,000344
7	0,1809	0,6620	0,00336
8	0,8110	1,2390	0,01780
9	1,3710	4,0500	0,1430
10	4,8770	8,5190	0,8200
11	9,6440	24,7500	5,9740
12	29,4280	49,3200	35,1500
13	55,4700	15,3100	212,910
14	2,9100	-132,4000	250,00

TABLE 3. The values of X, Y and Z at the moment t for $\alpha = 0.8$.

t	$X(t)$	$Y(t)$	$Z(t)$
0	0,001	0,00	0,00
1	-0,0007	0,00476	0,00
2	0,0086	0,00061	0,000000568
3	-0,00499	0,0414	0,000000582
4	0,0740	0,0106	-0,0000349
5	-0,0330	0,3610	0,000114
6	0,6390	0,1380	-0,00212
7	-0,2120	3,1620	0,0139
8	5,5300	1,6110	-0,106
9	-1,1380	27,7800	1,458
10	48,0800	17,9100	-4,585
11	-3,2480	281,500	144,110
12	481,300	297,790	-76,740
13	169,030	882,770	243,470
14	148,800	-692,170	267,184

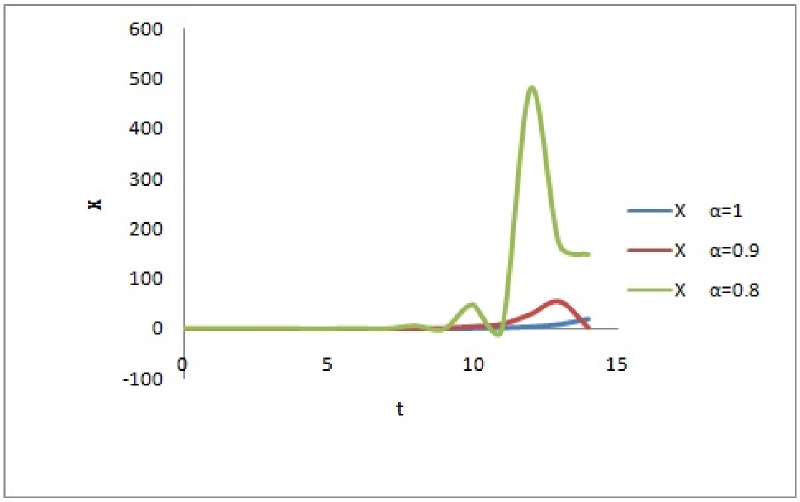

FIGURE 1. Graph of X phase plane change with time.

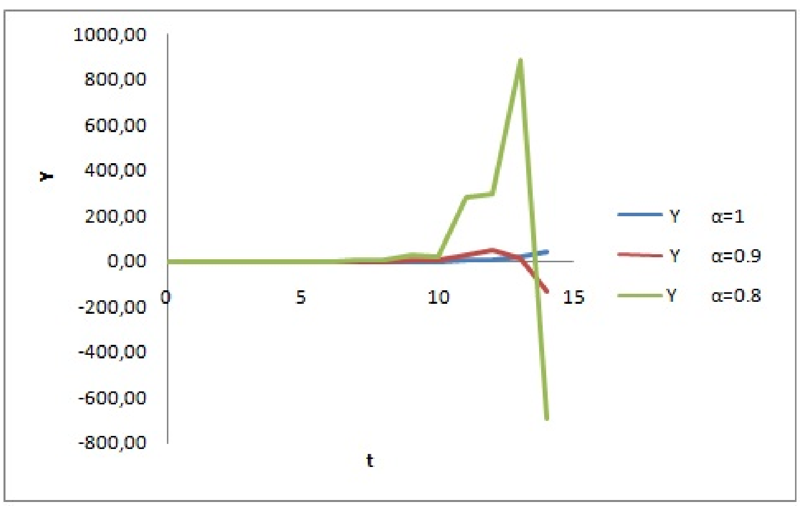

FIGURE 2. Graph of Y phase plane change with time.

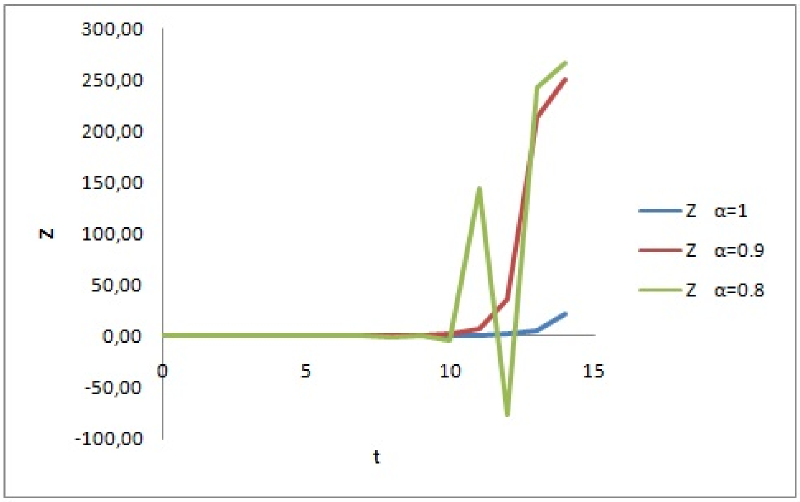

FIGURE 3. Graph of Z phase plane change with time.

Table 3, Table 4 and Table 5 show the changes of x, y and z for different cases of α .

4. Conclusions and Comments

Chaos-based reliable communication systems have become an alternative to the standard spread spectrum communication systems in the literature because they can spread the spectrum of information signals over a wide area, have a noise-like structure and can be realized with simple, inexpensive chaotic circuitry. In this study, the existence, uniqueness and non-negativity of the fractional order Lorenz Chaos model system were mathematically analysed. In the obtained graphs, it is observed that while the x phase plane is constant for α=1 and α=0.9, it starts to decrease after reaching a maximum value at a certain point for α=0.8. While the Y phase plane is constant for α=1 and α=0.9, it is observed that for α=0.8 it starts to decrease rapidly after taking the maximum value at a certain point. In the Z phase plane, it is observed that it progresses steadily for α=1, increases rapidly after a certain point for α=0.9, and increases rapidly after taking the minimum value at a certain point for α=0.8.

5. ACKNOWLEDGMENTS

The author would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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