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On approximation properties by exponential type of Bernstein-Stancu Operators

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Abstract

In the paper, we introduced a generalization of Bernstein-Stancu-Kantorovich operators that reproduces exponential functions. For appropriate function spaces, both the uniform and L^p convergence have been established. We proved that the new operators satisfy the Korovkin tests with the exponential functions and calculated the operators' analytical expressions evaluated on various powers of $e^{\mu x}$ which is necessary to get the uniform convergence conclusion using the well-known Korovkin Theorem. Consequently, the convergence theorem for the new operators, which transfer the weighted space $L^p_\mu([0,1])$ to itself, has been established. Additionally, using the usual modulus of continuity of the estimated function in the continuous case, we provide quantitative estimates for the approximation error.

Keywords: *Bernstein-Kantorovich operators, Exponential polynomials, Modulus of continuity.*

Üstel tip Bernstein-Stancu Operatörlerinin yaklaşım özellikleri üzerine

Öz

Bu çalışmada üstel fonksiyonları yeniden üreten Bernstein-Stancu-Kantorovich operatörlerinin bir genellemesi sunulmuştur. Uygun fonksiyon uzayları için hem düzgün hem de L^p yakınsaması kurulmuştur. Yeni operatörlerin üstel fonksiyonu sağladığını kanıtladık ve iyi bilinen Korovkin Teoremini kullanarak düzgün yakınsaklık sonucunu elde etmek için gerekli olan $e^{\mu x}$in çeşitli kuvvetlerine göre değerlendirilen operatörlerin analitik ifadelerini hesapladık. Sonuç olarak $L^p_\mu([0,1])$ ağırlıklı uzayını kendisine aktaran yeni operatörler için yakınsama teoremi kurulmuştur. Ek olarak, sürekli durumda tahmin

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edilen fonksiyonun olağan süreklilik modülünü kullanarak, yaklaşık hatası için niceliksel tahminler verilmiştir.

Anahtar kelimeler: *Bernstein-Kantorovich operatörleri, Üstel polinomlar, Süreklilik modülü.*

1. Preliminaries

The well-known polynomials developed by Bernstein, described as

$$B_v(f;x)=\sum_{l=0}^{v} p_{v,l}(x) f\left(\frac{l}{v}\right),\ x\in[0,1],\ v\in\mathbb{N}.$$

Here, $p_{v,l}(x)=\binom{v}{l}x^l(1-x)^{v-l}$, and f be a continuous function on the interval $[0,1]$ were presented to demonstrate the fundamental theorem of Weierstrass (see [1]). The following are the Kantorovich operators that are constructed from the traditional Bernstein operators

$$K_v(f;x)=\sum_{l=0}^{v} p_{v,l}(x)(v+1)\int_{\frac{l}{v+1}}^{\frac{l+1}{v+1}} f(t)dt\,, x\in[0,1],\ v\in\mathbb{N}.$$

In [2], D. D. Stancu introduced the following polynomials for each real α, β such that $0\le\alpha\le\beta$

$$B_{v,\alpha,\beta}(f;x)=\sum_{l=0}^{v} p_{v,l}(x) f\left(\frac{l+\alpha}{v+\beta}\right).$$

In [6], $K_v^{\alpha,\beta}(f;x): L_1([0,1])\to C([0,1])$defined for any $f\in L_1([0,1])$ Kantorovich-Stancu type operators were described as follows

$$K_v^{\alpha,\beta}(f;x)=(v+\beta+1)\sum_{l=0}^{v}\binom{v}{l}x^l(1-x)^{v-l}\int_{\frac{l+\alpha}{v+\beta+1}}^{\frac{l+\alpha+1}{v+\beta+1}} f(s)ds.$$

In [3], an exponential variation of Bernstein polynomials was presented for continuous functions on the interval [0,1], demonstrating uniform convergence. Recent research has focused on using exponential-type polynomials in the approximation theory. In [4], the exponential forms of Bernstein operators are presented as:

$$G_v(f;x)=\sum_{l=0}^{v} e^{-\mu l/v}e^{\mu x}p_{v,l}\big(z_v(x)\big)f(l/v),\ x\in[0,1],\mu>0,\ v\in\mathbb{N},$$

here $p_{v,l}\big(z_v(x)\big)=\binom{v}{l}\big(z_v(x)\big)^l\big(1-z_v(x)\big)^{v-l}$, and $z_v(x)=\frac{e^{\mu x/v}-1}{e^{\mu/v}-1}$. Here, $z_v(x)$ is defined as increasing, continuous and convex functions in $[0,1]$ with $z_v(0)=0$ and $z_v(1)=1$. The connection between their operators and the traditional Bernstein operators was defined as

$$G_v(f;x)=exp_\mu(x)B_v\left(\frac{f}{exp_\mu};z_v(x)\right).$$

The exponential function with a real parameter $\mu>0$ is denoted as $exp_\mu(x)=e^{\mu x}$.

In [5], the authors introduced an exponential polynomial with Kantorovich type as follows

$$K_v(f;x) := \sum_{l=0}^{v} e^{\mu x}\, p_{v,l}\big(z_{v+1}(x)\big)(v+1)\int_{l/(v+1)}^{(l+1)/(v+1)} f(t)e^{-\mu t}\, dt,\ x \in [0,1],$$

here $p_{v,l}(x)$ and $z_v(x)$ be described as shown above. Approximation results using these families of operators have been extensively investigated and also the nonlinear positive operators have been introduced in place of linear positive operators (see [9]-[16]).

2. Convergence results

Firstly, we introduce the exponential type of Bernstein-Stancu Kantrovich polynomials as

$$\mathcal{K}_v(f;x) := K_v^{\alpha,\gamma}(f;x) = (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{n,l}\big(z_{v+1}(x)\big)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} f(s)e^{-\mu s}\, ds,$$

where $v \in \mathbb{N}$, and $s \in [0,1]$, and $z_v(x) = \frac{e^{\mu x/(v+\gamma)}-1}{e^{\mu/(v+\gamma)}-1}, 0 \le \alpha \le \gamma$.

By taking into account that $\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big) = 1$,for every $x \in [0,1]$, we obtain

$$\mathcal{K}_v exp_\mu(x) = (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} ds$$
$$=e^{\mu x}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big) = e^{\mu x}$$

Therefore, our operators fix the exponential function $exp_\mu(x) := e^{\mu x}$.

Lemma 2.1. There exist the following equalities:

$$\mathcal{K}_v e_0(x) = \frac{v+\gamma+1}{\mu}e^{\mu x}.\left(e^{-\mu\frac{\alpha+v}{v+\gamma+1}} - e^{-\mu\frac{\alpha+v+1}{v+\gamma+1}}\right)\left(e^{\mu/(v+\gamma+1)} + 1 - e^{-\mu x/(v+\gamma+1)}\right)^v (1)$$

$$\mathcal{K}_v exp_\mu(x) = exp_\mu(x) \qquad (2)$$

$$\mathcal{K}_v exp_\mu^2(x) = \frac{v+\gamma+1}{\mu}\left(e^{\mu(\alpha+1)/(v+\gamma+1)} - e^{\mu\alpha/(v+\gamma+1)}\right)e^{\mu x+\mu x v/(v+\gamma+1)} \qquad (3)$$

$$\mathcal{K}_v exp_\mu^3(x) = \frac{v+\gamma+1}{2\mu}e^{\mu x}\left(e^{2\mu(\alpha+1)/(v+\gamma+1)} - e^{\mu\alpha/(v+\gamma+1)}\right)$$
$$\cdot\left(e^{2\mu(x+1)/(v+\gamma+1)} + e^{\mu x/(v+\gamma+1)} - e^{\mu/(v+\gamma+1)}\right)^v \qquad (4)$$

Proof. As we have already noted, (2) is immediate. Now, we establish (1) and (3).

$$\mathcal{K}_v e_0(x) = (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)\int_{\frac{(l+\alpha)}{v+\gamma+1}}^{\frac{(l+\alpha+1)}{v+\gamma+1}} e^{-\mu s}\, ds$$
$$= \frac{v+\gamma+1}{\mu}e^{\mu x}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)\left(e^{-\mu((l+\alpha)/(v+\gamma+1))} - e^{-\mu((l+\alpha+1)/(v+\gamma+1))}\right)$$
$$= \frac{v+\gamma+1}{\mu}e^{\mu x}e^{-\mu\frac{\alpha}{v+\gamma+1}}\left(1 - e^{-\mu/(v+\gamma+1)}\right)\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)e^{-\mu l/(v+\gamma+1)}.$$

We can write the following by some computations

$$\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)e^{-\mu l/(v+\gamma+1)} = e^{-\mu v/(v+\gamma+1)}\big(e^{\mu/(v+\gamma+1)} + 1 - e^{-\mu x/(v+\gamma+1)}\big)^{v},$$

and then we have the equality (1). Now,

$$\mathcal{K}_v exp_{\mu}^2(x) = (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} e^{\mu s}\,ds$$

$$= \frac{v+\gamma+1}{\mu}e^{\mu x}\big(e^{\mu(\alpha+1)/(v+\gamma+1)} - e^{\mu\alpha/(v+\gamma+1)}\big)\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)e^{-\mu l/(v+\gamma+1)}.$$

Here, $\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)e^{-\mu l/(v+\gamma+1)} = e^{\mu x v/(v+\gamma+1)}$, we obtain the equality (3). Using analogous reasons, it is easy to demonstrate the equality (4).

Theorem 2.2 If $f \in C([0,1])$, then $\mathcal{K}_v f$ converges to f uniformly on [0,1].

Proof. According to the well-known Korovkin Theorem (see in [7],[8]), since $\mathcal{K}_v$ are positive linear operators, it must be verified to confirm uniform convergence on a Korovkin subset of $C([0,1])$ to obtain uniform convergence for every $f \in C([0,1])$. Simply confirming the uniform convergence for the Korovkin subset $\{1, exp_{\mu}, exp_{\mu}^2\}$ is sufficient (see in [8]). Using Lemma 2.1, and the recognizing that $\mathcal{K}_v e_1(x) = e_1(x)$ for any $x \in [0,1]$, we can quickly see that $\mathcal{K}_v e_i$ converges uniformly to e_i, $i = 0, 2$, and the conclusion is thus verified.

The convergence of our operators in L^p will be examined. In accordance with its definition, it is inherent to derive a consequence of convergence in $L_{\mu}^{p}([0,1])$, which is a weighted L^p -space defined as the set of the measurable functions $f: [0,1] \to \mathbb{R}$ such that

$$\| f \|_{p,\mu} := \left\{\int_0^1 | e^{-\mu x} f(x) |^p \, d\,x\right\}^{1/p} < +\infty.$$

Theorem 2.3 Let $f \in L_{\mu}^{p}([0,1])$, then we obtain

$$\| \mathcal{K}_v f \|_{p,\mu} \le \frac{v+\gamma+1}{v+1}\frac{e^{\mu}-1}{\mu}\| f \|_{p,\mu}, \tag{5}$$

for every $v \in \mathbb{N}$. Furthermore, $\| \mathcal{K}_v f - f \|_{p,\mu} \to 0$ as $v \to +\infty$.

Proof. Consider the function $f \in L_{\mu}^{p}([0,1])$, and then apply Jensen's inequality

$$\| \mathcal{K}_v f \|_{p,\mu}^p = \int_0^1 \left|\sum_{l=0}^{n} p_{v,l}\big(z_{v+1}(x)\big)(v+\gamma+1)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} e^{-\mu s} f(s)ds\right|^p dx$$

$$\le \int_0^1 \sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)\left|(v+\gamma+1)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} e^{-\mu s} f(s)ds\right|^p dx.$$

If we apply again Jensen's inequality in $\left[\frac{l+\alpha}{v+\gamma+1}, \frac{l+\alpha+1}{v+\gamma+1}\right]$, then

$$\| \mathcal{K}_v f \|_{p,\mu}^p \le \sum_{l=0}^{v}\int_0^1 p_{v,l}\big(z_{v+1}(x)\big)dx(v+\gamma+1)\int_{(l+\alpha)/(v+\gamma+1)}^{(l+\alpha+1)/(v+\gamma+1)} e^{-\mu p s} | f(s) |^p \, ds.$$

We now estimate the terms $I_{v,l} = (v+\gamma+1)\int_0^1 p_{v,l}\big(z_{v+1}(x)\big)dx$. If we put

$s = z_{v+1}(x) = \frac{e^{\mu x/(v+\gamma+1)}-1}{e^{\mu/(v+\gamma+1)}-1}$, so that $x = \frac{v+\gamma+1}{\mu} ln\left(s\left(e^{\mu/(v+\gamma+1)} - 1\right) + 1\right)$ and $dx = \frac{v+\gamma+1}{\mu} \frac{e^{\mu/(v+\gamma+1)}-1}{\left(e^{\mu/(v+\gamma+1)}-1\right)s+1} ds$, the integral becomes

$$I_{v,l} = (v + \gamma + 1)\int_0^1 p_{v,l}(s)\frac{v + \gamma + 1}{\mu}\frac{e^{\mu/(v+\gamma+1)} - 1}{(e^{\mu/(v+\gamma+1)} - 1)s + 1} ds$$
$$\leq \frac{e^{\mu/(v+\gamma+1)} - 1}{\mu/(v + \gamma + 1)}(v + \gamma + 1)\int_0^1 p_{v,l}(s)ds\,.$$

Since

$$(v + \gamma + 1)\int_0^1 p_{v,l}(s)ds = (v + \gamma + 1)\int_0^1 \binom{v}{l} s^l(1 - s)^{v-l}\, ds = 1 + \frac{\gamma}{v + 1},$$

we get

$I_{v,l} \leq \left(e^{\mu/(v+\gamma+1)} - 1\right)\frac{(v+\gamma+1)^2}{\mu(v+1)}$, and

$$\| \mathcal{K}_v f \|_{p,\mu}^p \leq \frac{e^{\mu/(v+\gamma+1)} - 1}{(v + 1)\mu/(v + \gamma + 1)^2}\sum_{l=0}^{v}\int_{(l+\alpha)/(v+\gamma+1)}^{(l+\alpha+1)/(v+\gamma+1)} e^{-\mu ps} \mid f(s) \mid^p ds$$
$$\leq \frac{e^{\mu/(v+\gamma+1)} - 1}{\mu/(v + \gamma + 1)}\frac{v + \gamma + 1}{v + 1}\int_0^1 e^{-\mu ps} \mid f(s) \mid^p ds \leq 2\frac{e^{\mu/(v+\gamma+1)} - 1}{\mu/(v + \gamma + 1)} \| f \|_{p,\mu}^p.$$

Considering that $\frac{\mu}{v+\gamma+1} \leq \mu$, for every $v, \gamma \geq 0$, and the function $h(t) = \frac{e^t-1}{t}$ is increasing, we get $\| \mathcal{K}_v f \|_{p,\mu}^p \leq \frac{v+\gamma+1}{v+1}\frac{e^{\mu}-1}{\mu} \| f \|_{p,\mu}^p$, that is the proof of the inequality given in ([5].)

Define $\epsilon > 0$ as fixed. If $f \in L_\mu^p([0,1])$ then by the density of $C([0,1])$ in $L_\mu^p([0,1])$, there exists $h \in C([0,1])$ such that $\| f - h \|_p < \frac{\epsilon}{2(K_\mu+1)}$, where $K_\mu = \frac{v+\gamma+1}{v+1}\frac{e^{\mu}-1}{\mu}$. Additionally,

$$\| f - h \|_{p,\mu} = \left\{\int_0^1 (e^{-\mu x}\left(f(x) - h(x)\right)^p dx\right\}^{\frac{1}{p}} \leq \left\{\int_0^1 \left(f(x) - h(x)\right)^p dx\right\}^{\frac{1}{p}}$$
$$= \| f - h \|_p < \frac{\epsilon}{2(K_\mu+1)}. \tag{6}$$

Then

$\| \mathcal{K}_v f - f \|_{p,\mu} \leq \| \mathcal{K}_v f - \mathcal{K}_v h \|_{p,\mu} + \| \mathcal{K}_v h - h \|_{p,\mu} + \| h - f \|_{p,\mu}$.

Now,

$$\| \mathcal{K}_v h - h \|_{p,\mu} \leq \left\{\int_0^1 \mid \mathcal{K}_v h(x) - h(x) \mid^p dx\right\}^{\frac{1}{p}} \leq \| \mathcal{K}_v h - h \|_\infty$$

and so, there is $\tilde{v} \in \mathbb{N}$ such that, for each $v \geq \tilde{v}$, $\| \mathcal{K}_v h - h \|_{p,\mu} \leq \frac{\epsilon}{2}$ from Theorem 2.2. Furthermore, by (5), $\| \mathcal{K}_v f - \mathcal{K}_v h \|_{p,\mu} \leq K_\mu \| h - f \|_{p,\mu}$ and thus, by inequality (6), $\| \mathcal{K}_v f - f \|_{p,\mu} \leq \left(K_\mu + 1\right) \| f - h \|_{p,\mu} + \frac{\epsilon}{2} = \epsilon$, for each $v \geq \tilde{v}$. It is immediately apparent that all of the conclusions of Theorem 2.3 can be reformulated using the standard L^p-norm $\|.\|_p$ instead of its weighted version $\|.\|_{p,\mu}, 1 \leq p < +\infty$, by applying the following simple inequalities: $e^{-\mu} \| f \|_p \leq \| f \|_{p,\mu} \leq \| f \|_p$.

Theorem 2.4 Let $f \in C([0,1])$, for every $v \in \mathbb{N}$, $n > 1$ the following inequality holds

$$\| \mathcal{K}_v f - f \|_\infty \leq \omega\left(exp_\mu^{-1} f, \frac{1}{\sqrt{v + \gamma + 1}}\right) e^{\mu}\left(1 + \frac{1}{\sqrt{v + \gamma + 1}\sqrt{\gamma + 1}} + \sqrt{\gamma + 1}\right)$$

$$+e^{\mu}\omega\left(exp_{\mu}^{-1}f, max_{x\in[0,1]}|z_{v+1}(x)-x|\right).$$

Proof. For every fixed $x \in [0,1]$, we have

$$\mathcal{K}_v f(x) - f(x) = \mathcal{K}_v f(x) - f(x)\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)$$

$$= \mathcal{K}_v f(x) - (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}} e^{-\mu x} f(x)ds$$

$$= (v+\gamma+1)e^{\mu x}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\left(e^{-\mu s}f(s) - e^{-\mu x}f(x)\right)ds.$$

By using properties of the modulus of continuity, we get

$$|\ \mathcal{K}_v f(x) - f(x)\ |$$

$$\leq (v+\gamma+1)e^{\mu}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}|e^{-\mu s}f(s) - e^{-\mu x}f(x)|ds$$

$$\leq (v+\gamma+1)e^{\mu}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\omega\left(exp_{\mu}^{-1}f, |s-x|\right)ds\,.$$

Additionaly, we can easily see that

$$\lim_{v\to\infty}\max_{x\in[0,1]}|\ z_{v+1}(x) - x\ | = 0,$$

and the following inequality exists

$$|s-x| \leq |s - z_{v+1}(x)| + |z_{v+1}(x) - x| \leq |s - z_{v+1}(x)| + \max_{x\in[0,1]}|\ z_{v+1}(x) - x\ |.$$

By using the above inequality, we obtain

$$|\ \mathcal{K}_v f(x) - f(x)\ |$$

$$\leq (v+\gamma+1)e^{\mu}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\omega\left(exp_{\mu}^{-1}f, |s - z_{v+1}(x)|\right)ds$$

$$+(v+\gamma+1)e^{\mu}\sum_{l=0}^{v} p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\omega(\ exp_{\mu}^{-1}f, \max_{x\in[0,1]}|\ z_{v+1}(x) - x\ |)ds$$

$$:= E_1 + E_2.$$

Firstly, by means of the Cauchy-Schwarz inequality, we need to estimate the following;

$$\sum_{l=0}^{v}\left|x - \frac{l+\alpha}{v+\gamma+1}\right| p_{v,l}(x) \leq \left(\sum_{l=0}^{v}\left|x - \frac{l+\alpha}{v+\gamma+1}\right|^{1/2} p_{v,l}(x)\right)^{1/2}\left(\sum_{l=0}^{v} p_{v,l}(x)\right)^{1/2}$$

$$= \left(\sum_{l=0}^{v}\left(x - \frac{l+\alpha}{v+\gamma+1}\right)^{2} p_{v,l}(x)\right)^{1/2}.$$

By applying some equalities associated with classical Bernstein-Stancu polynomials we have

$$\sum_{l=0}^{v}\left(x-\frac{l+\alpha}{v+\gamma+1}\right)^2 p_{v,l}(x)$$
$$=x^2+\sum_{l=0}^{v}\frac{(l+\alpha)^2}{(v+\gamma+1)^2}p_{v,l}(x)-2x\sum_{l=0}^{v}\frac{l+\alpha}{v+\gamma+1}p_{v,l}(x)$$
$$=x^2+\frac{v(v-1)}{(v+\gamma+1)^2}x^2+\frac{(1+2\alpha)v}{(v+\gamma+1)^2}x+\frac{\alpha^2}{(v+\gamma+1)^2}-2x\left(\frac{vx}{v+\gamma+1}+\frac{\alpha}{v+\gamma+1}\right),$$

additionally taking into consideration that the above inequality holds its maximum at $x=1$ for every $n>1$, we obtain

$$\sum_{l=0}^{v}\left|x-\frac{l+\alpha}{v+\gamma+1}\right|p_{v,l}(x)\leq\frac{\sqrt{\gamma+1}}{\sqrt{v+\gamma+1}}.$$

Now, let us calculate the following integral to estimate E_1:

$$\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}|s-z_{v+1}(x)|ds$$
$$\leq\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\left|s-\frac{l+\alpha}{v+\gamma+1}\right|ds+\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\left|\frac{l+\alpha}{v+\gamma+1}-z_{v+1}(x)\right|ds$$
$$=\frac{1}{2(v+\gamma+1)^2}+\frac{1}{v+\gamma+1}\left|\frac{l+\alpha}{v+\gamma+1}-z_{v+1}(x)\right|.$$

Considering the following inequality, for $\lambda,\delta>0$,

$$\omega(f,\lambda\delta)\leq(1+\lambda)\omega(f,\delta),$$

we can estimate E_1as follows

$$E_1\leq\omega\left(exp_\mu^{-1}f,1/\sqrt{v+\gamma+1}\right)(v+\gamma+1)e^{\mu}\sum_{l=0}^{v}p_{v,l}\left(z_{v+1}(x)\right)$$
$$\times\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}\left(1+\sqrt{v+\gamma+1}|s-a_{v+1}(x)|\right)ds$$
$$\leq\omega\left(exp_\mu^{-1}f,1/\sqrt{v+\gamma+1}\right)$$
$$\times e^{\mu}\left(\sum_{l=0}^{v}p_{v,l}\left(z_{v+1}(x)\right)\right.$$
$$\left.+\sqrt{v+\gamma+1}(v+\gamma+1)\sum_{l=0}^{v}p_{v,l}\left(z_{v+1}(x)\right)\int_{\frac{l+\alpha}{v+\gamma+1}}^{\frac{l+\alpha+1}{v+\gamma+1}}|s-a_{v+1}(x)|ds\right)$$
$$\leq\omega\left(exp_\mu^{-1}f,1/\sqrt{v+\gamma+1}\right)e^{\mu}\left(1+\frac{\sqrt{v+\gamma+1}}{(v+\gamma+1)\sqrt{\gamma+1}}\sum_{l=0}^{v}p_{v,l}\left(z_{v+1}(x)\right)\right.$$
$$\left.+\sqrt{v+\gamma+1}\sum_{l=0}^{v}p_{v,l}\left(z_{v+1}(x)\right)\left|\frac{l+\alpha}{v+\gamma+1}-z_{v+1}(x)\right|\right)$$
$$\leq\omega\left(exp_\mu^{-1}f,1/\sqrt{v+\gamma+1}\right)e^{\mu}\left(1+\frac{1}{\sqrt{v+\gamma+1}\sqrt{\gamma+1}}+\sqrt{\gamma+1}\right).$$

On the other side, regarding the estimation of E_2, we have

$$E_2 \leq \omega(exp_{\mu}^{-1}f, \max_{x\in[0,1]}|z_{v+1}(x) - x|)e^{\mu}\sum_{l=0}^{v} p_{v,l}\big(z_{v+1}(x)\big)$$
$$= \omega(exp_{\mu}^{-1}f, \max_{x\in[0,1]}|z_{v+1}(x) - x|)e^{\mu}.$$

Therefore, we obtain the proof of the theorem.

3. Conclusion

For over a century, researchers have been interested in approaching functions because of their structure and the wide variety of fields that make use of them. Furthermore, there has been significant investigation into the method of continuous functions via sequences of linear positive operators, a subject with several non-mathematical applications in fields like engineering and physics. Bernstein provided the definition for the proof of the Weierstrass approximation theorem in 1912; it was subsequently cited by him (see in [1]). The literature has many articles with studies of various generalizations and modifications of Bernstein operators. Reproducing exponential functions, we presented a generalization of Bernstein-Stancu-Kantorovich operators in the article. The uniform and L^p convergences have been proven for appropriate function spaces. The positive approximation processes identified by Korovkin are significant and emerge naturally in various mathematical fields. In order to obtain the uniform convergence conclusion using the famous Korovkin Theorem, we demonstrated that the new operators are exponentially compatible and computed their analytical expressions evaluated on different powers of $e^{\mu x}$. So, the convergence theorem for the new operators has been proven. These operators move the weighted space $L_{\mu}^{p}([0,1])$ to itself. Consequently, we describe a generalization of Bernstein-Stancu-Kantorovich operators and give some important approximation results so that we can get a better estimate.

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