Sakarya University Journal of Science, 22 (6), 1532-1537, 2018.

SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE

e-ISSN: 2147-835X
Dergi sayfası: http://www.saujs.sakarya.edu.tr

Geliş/Received
Sep 27, 2017
Kabul/Accepted
Jan 12, 2018

Doi
10.16984/saufenbilder.340076

Some Characterizations Of Surfaces Generated by Two Curves In $Heis_3$

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ABSTRACT

In this paper, some characterizations of surfaces which are constructed by using the group operation in $Heis_3$ are given. Then, a new classification for minimal factorable surfaces in $Heis_3$ is obtained.

Keywords: Heisenberg group, Levi-Civita connection, mean curvature, Gaussian curvature.

$Heis_3$ de İki Eğri Tarafından Üretilen Yüzeylerin Bazı Karakterizasyonları

ÖZ

Bu makalede, 3- boyutlu Heisenberg grubunda, grup çarpımıyla elde edilen yüzeylerin bazı karakterizasyonları incelendi. Daha sonra, 3- boyutlu Heisenberg grubunda minimal factorable yüzeylerin yeni bir sınıflandırması elde edildi..

Anahtar Kelimeler: Heisenberg grup, Levi-Civita konneksiyonu, ortalama eğrilik, Gauss eğriliği .

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1. INTRODUCTION

Much of the modern global theory of complete minimal surfaces in three dimensional Euclidean space has been affected by the work of Osserman during the 1960's. Recently, many of the global questions arose in this classical subject. These questions deal with analytic and conformal properties, the geometry and asymptotic behavior, and the topology and classification of the images of certain injective minimal immersions $\varphi : M \rightarrow \mathbb{E}^3$ which are complete in the induced Riemannian metric, [1-6]. In [7], a Weierstrass representation formula for simply connected immersed minimal surfaces in Heisenberg group H^{2n+1} is studied.

A surface S in the Euclidean 3-space is denoted by

$$r(u,v)=\{x(u,v),y(u,v),z(u,v)\}.$$

A classification and some fundamental formulas is given for factorable surfaces which are parametrized as

$$z=f(x)g(y) \quad \text{or} \quad y=f(x)g(z) \quad \text{or} \quad x=f(y)g(z),$$

in the Euclidean space and in the Minkowski space, where f and g are smooth functions on some interval of $\mathbb{R}$, [8,9]. In [9], factorable surfaces in 3- dimensional Minkowski space is studied and some classification of such surfaces whose mean curvature and Gauss curvature satisfy certain conditions are given.

A factorable surface in $Heis_3$, which is given by a left invariant Riemannian metric, parametrized with group product for two curves. The purpose of this paper is to study and classify minimal surfaces and minimal factorable surfaces which are obtained with group operation in $Heis_3$.

2. PRELIMINARIES

The Heisenberg group $Heis_3$ is defined as $\mathbb{R}^3$ with the group operation

$$(x,y,z)*(x_1,y_1,z_1)=\left(x+x_1,y+y_1,z+z_1+\frac{1}{2}(xy_1-x_1y)\right). \quad (1)$$

The left invariant Riemannian metric given by

$$g=ds^2=dx^2+dy^2+\left(dz+\frac{1}{2}(ydx-xdy)\right)^2. \quad (2)$$

The following vector fields form a left invariant orthonormal frame on $Heis_3$, which is given the left invariant Riemann metric g:

$$e_1=\frac{\partial}{\partial x}-\frac{y}{2}\frac{\partial}{\partial z}, e_2=\frac{\partial}{\partial y}+\frac{x}{2}\frac{\partial}{\partial z}, e_3=\frac{\partial}{\partial z}. \quad (3)$$

These vector fields are dual to the coframe

$$w^1=dx, w^2=dy, w^3=dz+\frac{y}{2}dx-\frac{x}{2}dy. \quad (4)$$

We obtain

$$2\nabla_{e_i}e_j=\begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}, \quad (5)$$

also, we have the Heisenberg bracket relations.

$$[e_1,e_2]=e_3,[e_3,e_1]=[e_2,e_3]=0. \quad (6)$$

Let $\varphi : M \rightarrow Heis_3$ be an orientable surface, isometrically immersed in $Heis_3$. Denote the Levi-Civita connections of M and $Heis_3$ by $\tilde{\nabla}$ and ∇, respectively. Let X and Y denote vector fields tangent to M and let $\mathbf{N}$ be a normal vector field. Then the Gauss and Weingarten formulas are given, respectively,

$$\nabla_X Y=\tilde{\nabla}_X Y+h(X,Y)\mathbf{N}, \quad (7)$$
$$\nabla_X Y=-AX,$$

where h and A are the second fundamental form and the shape operator. It is well known that the second fundamental form h and the shape operator A are related by

$$h(X,Y)=g(AX,Y). \quad (8)$$

At each tangent plane T_PM, $\{\varphi_u,\varphi_v\}$ is a basis, where u, v are local coordinates on M. Denote by E, F, G the coefficients of the first fundamental form on M:

$$E=g(\varphi_u,\varphi_u), F=g(\varphi_u,\varphi_v), G=g(\varphi_v,\varphi_v). \quad (9)$$

Let $\{\varphi_u,\varphi_v\}$ form an arbitrary basis on the surface M. We know that A is a self-adjoint endomorphism with respect to the metric on M, that is, $g(A(u),v)=g(u,A(v))$, u, $v\in T_PM$. Also,

$$-g(\nabla_X\mathbf{N},Y)=g(\nabla_X Y,\mathbf{N}). \quad (10)$$

At each tangent plane T_PM it can be taken a basis $\{\varphi_u,\varphi_v\}$ and

$$A(\varphi_u)=-\nabla_{\varphi_u}N=h_{11}\varphi_u+h_{12}\varphi_v, \quad (11)$$
$$A(\varphi_v)=-\nabla_{\varphi_v}N=h_{21}\varphi_u+h_{22}\varphi_v.$$

So, the mean curvature of the surface is

$$H=\frac{Gh_{11}-Fh_{12}-Fh_{21}+Eh_{22}}{2\left(EG-F^2\right)}. \qquad (12)$$

It is known that, minimal surface is obtain with $H=0$. So, M is a minimal surface if and only if

$$Gh_{11}-Fh_{12}-Fh_{21}+Eh_{22}=0. \qquad (13)$$

The Gaussian curvature of the surface is

$$K=\frac{h_{11}h_{22}-h_{12}h_{21}}{\left(EG-F^2\right)}. \qquad (14)$$

3. MINIMAL SURFACES IN $Heis_3$

In this section, a new type of surfaces which are defined by two curves in $Heis_3$ is obtained. Also, some characterizations of this surface is given.

Theorem 3.1. Let $\alpha(x)=(\alpha_1(x),\alpha_2(x),\alpha_3(x))$ and $\beta(y)=(\beta_1(y),\beta_2(y),\beta_3(y))$ be differentiable nongeodesic curves in $Heis_3$ which is endowed with left invariant Riemannian metric g. Then, with the group operation in the equality (2.1), there is a $\varphi(x,y)=\alpha(x)*\beta(y)$ surface in $(Heis_3,g)$. The mean curvature of the surface $\varphi(x,y)$ is

$$\begin{aligned}H=&\frac{1}{\|n\|(EG-F^2)}\{(\alpha_2'Q-P\beta_2')[G(\alpha_1'\alpha_1''+\alpha_2'P)\\&+E(\beta_2'\beta_1''+\beta_2'Q)-F(\alpha_2'\beta_1''+\beta_1'\alpha_1'']\\&+\frac{1}{2}(\alpha_2'Q+\beta_2'P+\beta_2'P+\alpha_2'Q)\\&+(\alpha_1'Q-\beta_1'P)[G(\alpha_1'\alpha_2''-\alpha_1'P)\\&+E(\beta_2'\beta_2''-\beta_1'Q)-F(\alpha_2'\beta_2''+\beta_1'\alpha_2''\\&-\frac{1}{2}(\beta_1'P+\alpha_1'Q+\alpha_1'Q+\beta_1'P)]\\&+(\alpha_1'\beta_2'-\alpha_2'\beta_1')[G((\alpha_1'P_x+\alpha_2'P_y)(\alpha_1'\beta_2'-\alpha_2'\beta_1'))\\&+E((\beta_1'Q_x+\beta_2'Q_y)(\alpha_1'\beta_2'-\alpha_2'\beta_1'))\\&-F(\alpha_1'Q_x+\alpha_2'Q_y+\beta_1'P_x+\beta_2'P_y\\&+\frac{1}{2}(\alpha_1'\beta_2'+\alpha_2'\beta_1'-\alpha_2'\beta_1'-\alpha_1'\beta_2'))]\},\end{aligned} \qquad (15)$$

where

$$P=\frac{1}{2}(\alpha_1'(\alpha_2+2\beta_2)-\alpha_2'(\alpha_1+2\beta_1)+2\alpha_3'), \qquad (16)$$

$$Q=\frac{1}{2}(\beta_1'\beta_2-\beta_2'\beta_1+2\beta_3'), \qquad (17)$$

E, F and G the are coefficients of the first fundamental form.

Proof. From derivatives of the surface

$$\varphi(x,y)=\alpha(x)*\beta(y) \qquad (18)$$

depend to x and y, we have

$$\begin{aligned}\varphi_x(x,y)=&\alpha_1'e_1+\alpha_2'e_2+\frac{1}{2}(\alpha_1'(\alpha_2+2\beta_2)\\&-\alpha_2'(\alpha_1+2\beta_1)+2\alpha_3')e_3,\end{aligned} \qquad (19)$$

$$\varphi_y(x,y)=\beta_1'e_1+\beta_2'e_2+\frac{1}{2}(\beta_1'\beta_2-\beta_2'\beta_1+2\beta_3')e_3. \qquad (20)$$

From equations (19) and (20), coefficients of the first fundamental form are

$$E=g(\varphi_x,\varphi_x)=\alpha_1'^2+\alpha_2'^2+P^2, \qquad (21)$$

$$F=g(\varphi_x,\varphi_y)=\alpha_1'\beta_1'+\alpha_2'\beta_2'+PQ, \qquad (22)$$

$$G=g(\varphi_y,\varphi_y)=\beta_1'^2+\beta_2'^2+Q^2, \qquad (23)$$

On the other hand, if (5), (19) and (20) are thought together, Levi- Civita connections obtained as

$$\begin{aligned}\nabla_{\varphi_x}\varphi_x=&(\alpha_1'\alpha_1''+\alpha_2'P)e_1+(\alpha_1'\alpha_2''-\alpha_1'P)e_2\\&+(\alpha_1'P_x+\alpha_2'P_y)e_3,\end{aligned} \qquad (24)$$

$$\begin{aligned}\nabla_{\varphi_x}\varphi_y=&(\alpha_2'\beta_1''+\frac{1}{2}(\alpha_2'Q+\beta_2'P))e_1\\&+(\alpha_2'\beta_2''-\frac{1}{2}(\beta_1'P+\alpha_1'Q))e_2\\&+(\alpha_1'Q_x+\alpha_2'Q_y+\frac{1}{2}(\alpha_1'\beta_2'-\alpha_2'\beta_1'))e_3,\end{aligned} \qquad (25)$$

$$\begin{aligned}\nabla_{\varphi_y}\varphi_x=&(\beta_1'\alpha_1''+\frac{1}{2}(\beta_2'P+\alpha_2'Q))e_1\\&+(\beta_1'\alpha_2''-\frac{1}{2}(\alpha_1'Q+\beta_1'P))e_2\\&+(\beta_1'P_x+\beta_2'P_y+\frac{1}{2}(\alpha_2'\beta_1'-\alpha_1'\beta_2')))e_3,\end{aligned} \qquad (26)$$

$$\begin{aligned}\nabla_{\varphi_y}\varphi_y=&(\beta_2'\beta_1''+\beta_2'Q)e_1+(\beta_2'\beta_2''-\beta_1'Q)e_2\\&+(\beta_1'Q_x+\beta_2'Q_y)e_3.\end{aligned} \qquad (27)$$

The unit normal vector field of the surface $\varphi(x,y)$ is

$$\mathbf{N}=\frac{1}{\|n\|}((\alpha_2'Q-P\beta_2')e_1-(\alpha_1'Q-\beta_1'P)e_2+(\alpha_1'\beta_2'-\alpha_2'\beta_1')e_3 \qquad (28)$$

where

$$\|n\| = \sqrt{(\alpha_2' Q - P\beta_2')^2 + (\alpha_1' Q - \beta_1' P)^2 + (\alpha_1'\beta_2' - \alpha_2'\beta_1')^2}.$$

From (24) - (28) coefficients of the second fundamental form are

$$h_{11} = \frac{1}{\|n\|}[(\alpha_2' Q - P\beta_2')(\alpha_1'\alpha_1'' + \alpha_2' P)$$

$$-(\alpha_1' Q - \beta_1' P)(\alpha_1'\alpha_2'' - \alpha_1' P) \tag{29}$$

$$+(\alpha_1'\beta_2' - \alpha_2'\beta_1')(\alpha_1' P_x + \alpha_2' P_y)],$$

$$h_{12} = \frac{1}{\|n\|}[(\alpha_2' Q - P\beta_2')(\alpha_2'\beta_1'' + \frac{1}{2}(\alpha_2' Q + \beta_2' P)$$

$$-(\alpha_1' Q - \beta_1' P)(\alpha_2'\beta_2'' - \frac{1}{2}(\beta_1' P + \alpha_1' Q)) \tag{30}$$

$$+(\alpha_1'\beta_2' - \alpha_2'\beta_1')(\alpha_1' Q_x + \alpha_2' Q_y + \frac{1}{2}(\alpha_1'\beta_2' - \alpha_2'\beta_1'))],$$

$$h_{21} = \frac{1}{\|n\|}[(\alpha_2' Q - P\beta_2')(\beta_1'\alpha_1'' + \frac{1}{2}(\beta_2' P + \alpha_2' Q))$$

$$-(\alpha_1' Q - \beta_1' P)(\beta_1'\alpha_2'' - \frac{1}{2}(\alpha_1' Q + \beta_1' P)) \tag{31}$$

$$+(\alpha_1'\beta_2' - \alpha_2'\beta_1')(\beta_1' P_x + \beta_2' P_y + \frac{1}{2}(\alpha_2'\beta_1' - \alpha_1'\beta_2')))],$$

$$h_{22} = \frac{1}{\|n\|}[(\alpha_2' Q - P\beta_2')(\beta_2'\beta_1'' + \beta_2' Q)$$
$$-(\alpha_1' Q - \beta_1' P)(\beta_2'\beta_2'' - \beta_1' Q) \tag{32}$$
$$+(\alpha_1'\beta_2' - \alpha_2'\beta_1')(\beta_1' Q_x + \beta_2' Q_y)].$$

Then, if (21)- (23), (29)- (32) are written in the equation (12), the mean curvature of the surface $\varphi(x,y)$ is (15).

Corollary 3.2. Let $\varphi(x,y)$ be a surface in $(Heis_3, g)$. If $\varphi(x,y)$ is a minimal surface, then

$$(\alpha_2' Q - P\beta_2')[G(\alpha_1'\alpha_1'' + \alpha_2' P)$$

$$+E(\beta_2'\beta_1'' + \beta_2' Q) - F(\alpha_2'\beta_1'' + \beta_1'\alpha_1''$$

$$+\frac{1}{2}(\alpha_2' Q + \beta_2' P + \beta_2' P + \alpha_2' Q)]$$

$$+(\alpha_1' Q - \beta_1' P)[G(\alpha_1'\alpha_2'' - \alpha_1' P)$$

$$+E(\beta_2'\beta_2'' - \beta_1' Q) - F(\alpha_2'\beta_2'' + \beta_1'\alpha_2'' \tag{33}$$

$$-\frac{1}{2}(\beta_1' P + \alpha_1' Q + \alpha_1' Q + \beta_1' P)]$$

$$+(\alpha_1'\beta_2' - \alpha_2'\beta_1')[G((\alpha_1' P_x + \alpha_2' P_y)(\alpha_1'\beta_2' - \alpha_2'\beta_1'))$$

$$+E((\beta_1' Q_x + \beta_2' Q_y)(\alpha_1'\beta_2' - \alpha_2'\beta_1'))$$

$$-F(\alpha_1' Q_x + \alpha_2' Q_y + \beta_1' P_x + \beta_2' P_y$$

$$+\frac{1}{2}(\alpha_1'\beta_2' + \alpha_2'\beta_1' - \alpha_2'\beta_1' - \alpha_1'\beta_2'))] = 0.$$

4. FACTORABLE SURFACES IN $Heis_3$

In this section we deduce new types of factorable surfaces in $Heis_3$. Moreover we obtain some characterizations of these surfaces. Then, some comperations are given with tables for new types of factorable surfaces.

4.1 Surfaces of Type 1

Let α and β be curves in $Heis_3$, which are given by $\alpha(x) = (u_1(x), 0, c)$, $\beta(y) = (0, v_2(y), -c)$. The factorable surface $\varphi(x,y) = \alpha(x) * \beta(y)$ of type 1 is can be parmetrized as

$$\varphi(x,y) = (u_1(x), 0, c) * (0, v_2(y), -c)$$
$$= \left(u_1(x), v_2(y), \frac{1}{2}u_1(x)v_2(y)\right), \tag{34}$$

where c is a nonzero constant.

Theorem 4.1. Let $\varphi(x,y)$ be factorable surface of type 1 in $Heis_3$ which is endowed Riemannian metric. $\varphi(x,y)$ is a minimal surface in $Heis_3$.

Proof. From (34), it can be easily obtain,

$$\varphi_x = u_1'(x)(e_1 + v_2 e_3), \tag{35}$$

$$\varphi_y = v_2'(y)e_2. \tag{36}$$

An orthogonal vector at each point is

$$\mathbf{N} = \frac{1}{\sqrt{1 + v_2^2(y)}}(-v_2 e_1 + e_3). \tag{37}$$

The coefficients of the first fundamental form are

$$E = g(\varphi_x, \varphi_x) = u_1'^2(x)(1 + v_2^2), \tag{38}$$

$$F = g(\varphi_x,\varphi_y) = g(\varphi_y,\varphi_x) = 0, \qquad (39)$$

$$G = g(\varphi_y,\varphi_y) = v_2'^2. \qquad (40)$$

On the other hand, from (5),

$$\nabla_{\varphi_x}\varphi_x = u_1''(x)u_1'(x)e_1 - u_1'^2(x)v_2(y)e_2 + u_1''(x)u_1'(x)v_2(y)e_3,$$
(41)

$$\nabla_{\varphi_x}\varphi_y = \frac{1}{2}v_2'(y)u_1'(x)(v_2(y)e_1 + e_3), \qquad (42)$$

$$\nabla_{\varphi_y}\varphi_x = \frac{1}{2}v_2'(y)u_1'(x)(v_2(y)e_1 + e_3), \qquad (43)$$

$$\nabla_{\varphi_y}\varphi_y = v_2''(y)v_2'(y)e_2. \qquad (44)$$

So, the coefficients of second fundamental form are

$$h_{11} = 0, \qquad (45)$$

$$h_{12} = \frac{u_1'(x)v_2'(y)}{2\sqrt{1+v_2^2(y)}}(1 - v_2^2(y)), \qquad (46)$$

$$h_{21} = \frac{u_1'(x)v_2'(y)}{2\sqrt{1+v_2^2(y)}}(1 - v_2^2(y)), \qquad (47)$$

$$h_{22} = 0. \qquad (48)$$

Then, from equations (45)- (48) and (12), we have the mean curvature of the surface φ is $H = 0$. So, the surface $\varphi(x,y)$ is a minimal surface.

Corollary 4.2. Let $\varphi(x,y)$ be factorable surface of type 1 $\varphi(x,y)$ in $Heis_3$ which is endowed Riemannian metric. The Gaussian curvature of $\varphi(x,y)$ is $K \leq 0$ for all points in $Heis_3$.

Proof. From equations (45)- (48) , we have

$$K = -\frac{(1-v_2^2(y))^2}{4(1+v_2^2(y))^2} \qquad (49)$$

Example4.3. Let $\varphi(x,y) = \left(\sin x, \cos y, \frac{1}{2}\sin x \cos y\right)$ is a factorable surface Type 1 in $(Heis_3, g)$. The mean curvature of $\varphi(x,y)$ is, $H = 0$. Then, $\varphi(x,y)$ is a minimal surface in $(Heis_3, g)$.

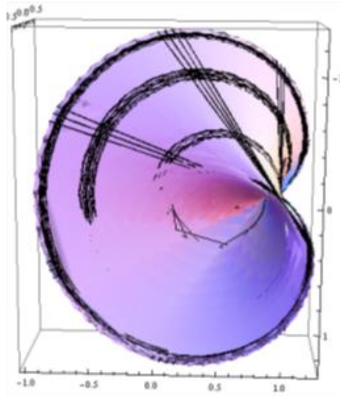

Figure 1. Minimal factorable surface Type1 in $(Heis_3, g)$.

4.2 Surfaces of Type 2

Let α and β be curves in $Heis_3$, which are given by $\alpha(x) = (0, u_2(x), d)$, $\beta(y) = (v_1(y), 0, -d)$. The factorable surface $M(\alpha,\beta) = \alpha(x) * \beta(y)$ of type 1 is can be parmetrized as

$$\psi(x,y) = (0, u_2(x), d) * (v_1(y), 0, -d)$$
$$= \left(v_1(y), u_2(x), -\frac{1}{2}v_1(y)u_2(x)\right), \qquad (50)$$

where d is a nonzero constant.

Theorem 4.4. Let $\psi(x,y)$ is factorable surface of type 2 in the $Heis_3$. $\psi(x,y)$ is a minimal surface if and only if $v_1(y) = \text{constant}$.

Proof. The mean curvature of the surface ψ is can be obtain as

$$H = \frac{v_1'}{2(1+v_1^2(y))^{3/2}}. \qquad (51)$$

So, if

$$v_1' = 0,$$

the surface $\psi(x,y)$ is a minimal surface.

Corollary 4.5 The Gaussian curvature of factorable surface of type 2 $\psi(x,y)$ in $Heis_3$ is

$$K = \frac{(v_1^2(y)+u_2^2(x))(1-v_1^2(y))}{4(1+v_1^2(y))^2}. \qquad (52)$$

Example4.6. Let $a \in \mathrm{R}$, $\psi(x,y) = \left(a, \cos x, \frac{1}{2}a\cos x\right)$ is a factorable surface Type 2 in $(Heis_3, g)$. The mean curvature of $\psi(x, y)$ is $H = 0$. Then, $\varphi(x, y)$ is a minimal surface in $(Heis_3, g)$.

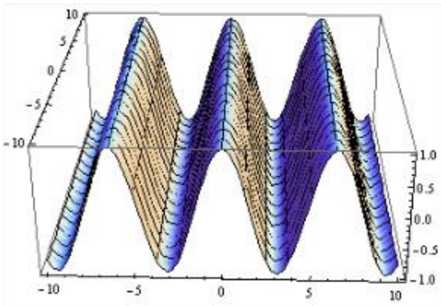

Figure 2. Minimal factorable surface Type2 in $(Heis_3, g)$.

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