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Research Article

Approximate solutions of the fractional Harry Dym equation

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ABSTRACT

In this paper, the approximate solutions of the time fractional Harry Dym equation with fractional derivative in the Caputo sense are obtained by using the Residual power series method (RPSM). This equation is a significant dynamical equation that occurs in a variety of physical systems. The suggested method provides good accuracy for the approximate solution when compared numerically with the exact solution. The effectiveness of the proposed method is also illustrated with the aid of numerical results. These results indicate that the RPSM is a power, useful, and applicable for determining the solutions of the time Hary Dym equation. Some of these results are illustrated by 2D and 3D graphics. Besides, the proposed method can be applied to many different differential equations due to its ease of use and reliability.

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INTRODUCTION

The Harry Dym equation is in the form

$$\frac{\partial u}{\partial t} = u^3 \frac{\partial^3 u}{\partial x^3}$$

was first studied by Kruskal and Moser and is referred to an unpublished work of Harry Dym. This equation is entirely integrable nonlinear evolution equation linked to the traditional string problems [1]. More detailed information about these problems can be seen in [2-5]. The Harry-Dym equation is also closely related to the Korteweg-de Vries equation [6]. In the literature, numerous methods have been utilized to solve this equation. The solution methods for the Harry Dym equation are moving frame [7], Adomian decomposition [8], He's variational iteration [8], direct integration [8], power series [8], residual power series [8], Bäcklund transform [9], new iterative method [10], haar wavelet [11], homotopy perturbation [12], reconstruction of variational iteration [12], Darboux transformation [13], and nonlinear steepest decent [14].

Recently, it has become very popular for scientists to obtain solutions of the fractional differential equations. These equations are widely used to model problems in viscoelasticity, turbulence, electrical networks, nonlinear biological systems, control theory, thermodynamics, fluid dynamics, signal processing, and so on [15-20]. The time fractional Harry Dym equation is one of the most important of them. So far, many researchers have used various analytical and numerical methods to obtain the time fractional Harry Dym equation. These methods are Adomian decomposition [21,22], homotopy perturbation Sumudu

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transform [22], Elzaki transform technique [23], Lie symmetry group analysis [24], similarity [25,26], homotopy analysis [27,28], Lie classical [29], homotopy perturbation [30], Mohand homotopy perturbation transform scheme [31], reduced differential transform [32], finite difference [33], q-homotopy analysis [34], and optimal system [35]. However, it is seen that the time fractional Harry Dym equation has not yet been solved with the RPSM.

The RPSM, proposed by Abu Arqub in 2013, is an efficient approach to obtain the approximate solutions of the different differential equations. These solutions are gained without the need for linearization, discretization, or perturbation. The RPSM does not require comparing the coefficients of the corresponding terms and does not need a recursion relation. By selecting an appropriate value for the initial guesses approximations, the proposed method can be also directly applied to the equations. Besides, with this method, high precision is achieved by utilizing less time and small calculations. Moreover, by minimizing the residual error, the suggested method provides an easy way to achieve the convergence of the series solution. Furthermore, the RPSM relies on derivation, which is more accurate and much easier than integration. This is the basis of most other solution methods. In addition to all these, the proposed method suggests obtaining infinite series solutions with iterated operations.

In the present paper, the RPSM is used to get the approximate solutions of the time fractional Harry Dym equation of the form

$$D_t^{\beta} u(x,t) = u^3(x,t) u_{xxx}(x,t), \quad 0 < \beta \le 1 \tag{1}$$

by the initial condition

$$u(x,0) = \left(4 - \tfrac{3}{2}x\right)^{\frac{2}{3}} \tag{2}$$

where $D_t^{\beta} u$ is the Caputo fractional derivative of order β with respect to the time variable t. When $\beta = 1$, Eq. (1) turns into the standard Harry Dym equation. The exact solution for the Harry Dym is

$$u(x,t) = \left(4 - \tfrac{3}{2}(x+t)\right)^{\frac{2}{3}}.$$

The plan of this paper is as follows. In Section 2, the definitions and theorems of the Caputo derivative and the fractional power series are mentioned. In Section 3, the basic idea of the RPSM is expressed. In Section 4, the RPS solutions for the time fractional Harry Dym equation are obtained by suggested method. Besides, the efficiency and the reliability of this method are demonstrated by table and figures. In Section 5, the Conclusions are given.

Preliminaries

There are numerous definitions of fractional operators, such as Grunwald-Letnikov, Caputo, Riemann-Liouville, Hadamard, Wely, and Marchaud in the literature. In this part, Caputo's definition is utilized since the derivative of a constant is zero and the initial conditions for the fractional differential equations with Caputo derivative take the familiar manner of integer order differential equations. The definition of Caputo derivative is defined as follows:

Definition 1. [36] The time fractional derivative of $u(x, t)$ in Caputo form is described as

$$D_t^{\beta} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t (t-\tau)^{m-1-\beta} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & m-1 < \beta < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & m = \beta \in \mathbb{N}. \end{cases}$$

The definition and theorems for the fractional power series are given below. Details of them can be found in [37].

Definition 2. [37] A power series expansion of the manner

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\beta} = c_0 + c_1 (t-t_0)^{\beta} + c_2 (t-t_0)^{2\beta} + \cdots,$$
$$0 \le m-1 < \beta \le m, \quad t \ge t_0,$$

is called the fractional power series about t_0. Here, t is a variable and the c_m's are constants.

Theorem 1. [37] Assume that g is a fractional power series representation at t_0 of the manner

$$g(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\beta}, \quad 0 \le m-1 < \beta \le m,$$
$$t_0 \le t < t_0 + R.$$

If $D^{m\beta} g(t)$ are continuous on $(t_0, t_0 + R)$, then coefficients c_m are expressed as

$$c_m = \frac{D^{m\beta} g(t_0)}{\Gamma(m\beta + 1)}, \qquad m = 0,1,2,\ldots,$$

where R is the radius of convergence and $D^{m\beta} = D^{\beta}.D^{\beta}\ldots D^{\beta}$.

Theorem 2. [37] Assume that $u(x, t)$ has a multiple fractional power series representation at t_0 of the manner

$$u(x,t) = \sum_{m=0}^{\infty} g_m(x) (t-t_0)^{m\beta}, \quad x \in I,$$
$$0 \le m-1 < \beta \le m, \quad t_0 \le t < t_0 + R.$$

If $D_t^{m\beta} u(x,t)$ are continuous on $I \times (t_0, t_0 + R)$, then $g_m(x)$ are expressed as

$$g_m(x) = \frac{D_t^{m\beta} u(x,t_0)}{\Gamma(m\beta + 1)}, \quad m = 0,1,2,\ldots.$$

Here, $D_t^{m\beta} = \frac{\partial^{m\beta}}{\partial t^{m\beta}} = \frac{\partial^\beta}{\partial t^\beta} \cdot \frac{\partial^\beta}{\partial t^\beta} \cdots \frac{\partial^\beta}{\partial t^\beta}$, and $R = \min_{c \in I} R_c$ that R_c is radius of convergence of the fractional power series $\sum_{m=0}^{\infty} g_m(c)(t-t_0)^{m\beta}$.

Basic Idea of the RPSM

In this section, to demonstrate the basic idea of the RPSM, we examine a general nonlinear fractional differential equation by the initial condition of the manner

$$D_t^\beta u(x,t) = N(u) + R(u), \quad 0 < \beta \leq 1, \quad t > 0, \qquad u(x,0) = g(x), \tag{3}$$

where D_t^β represents the fractional derivative in the Caputo sense, N is nonlinear differential operator and R is linear differential operator. This method suggests the solution for Eq. (3) as a fractional power series for $t = 0$. Assume the solution takes the following form:

$$u(x,t) = \sum_{m=0}^{\infty} g_m(x) \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \quad x \in I, \qquad 0 < \beta \leq 1, \quad 0 \leq t < R.$$

The $u_l(x, t)$ is also expressed as

$$u_l(x,t) = \sum_{m=0}^{l} g_m(x) \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \quad x \in I, \qquad 0 < \beta \leq 1, \quad 0 \leq t < R. \tag{4}$$

Then, the 0-th RPS approximate solution of $u(x, t)$ is given as

$$u_0 = g_0(x) = u(x,0) = g(x).$$

Eq. (4) can be written as

$$u_l(x,t) = g(x) + \sum_{m=1}^{l} g_m(x) \frac{t^{m\beta}}{\Gamma(m\beta+1)}, \quad x \in I, \qquad 0 < \beta \leq 1, \ 0 \leq t < R, \quad l = 1,2,\ldots \tag{5}$$

The residual function for Eq. (3) is expressed as

$$Res_u(x,t) = D_t^\beta u(x,t) - N(u) - R(u).$$

Therefore, $Res_{u,l}$ is stated as

$$Res_{u,l}(x,t) = D_t^\beta u_l(x,t) - N(u_l) - R(u_l). \tag{6}$$

Some significant relations of the suggested method are as follows and it can be seen in [38-42].

$$Res_u(x,t) = 0,$$
$$\lim_{l\to\infty} Res_{u,l}(x,t) = Res_u(x,t) \text{ with } t \geq 0 \text{ and } x \in I, \tag{7}$$
$$D_t^{m\beta} Res_u(x,0) = D_t^{m\beta} Res_{u,l}(x,0) = 0, \quad m = 0,1,\ldots,l.$$

Substituting the $u_l(x, t)$ in Eq. (6) and calculating the $D_t^{(l-1)\beta}$ of $Res_{u,l}(x,t)$ for $l = 1, 2\ldots$, the suggested method is clearly expressed. Then, applying the relation (7), the following equation

$$D_t^{(l-1)\beta} Res_{u,l}(x,0) = 0, \quad 0 < \beta \leq 1, \qquad 0 \leq t < R, \quad t = 0, \quad l = 1,2,\ldots. \tag{8}$$

is solved to obtain the $g_m(x)$ with $m = 1, 2\ldots, l$ in Eq. (5).

Approximate Solutions of the Fractional Harry Dym Equation By RPSM

In this segment of the study, we utilize the RPSM to gain the RPS solutions for Eq. (1) by the initial condition (2).

Let us consider the residual function for Eq. (1) as

$$Res_u(x,t) = D_t^\beta u(x,t) - u^3(x,t) \frac{\partial^3}{\partial x^3} u(x,t).$$

Therefore, $Res_{u,l}(x,t)$ is written as

$$Res_{u,l}(x,t) = D_t^\beta u_l(x,t) - u_l^3(x,t) \frac{\partial^3}{\partial x^3} u_l(x,t). \tag{9}$$

To determine the $g_1(x)$, we write $l = 1$ in Eq. (9) and we have

$$Res_{u,1}(x,t) = D_t^\beta u_1(x,t) - u_1^3(x,t) \frac{\partial^3}{\partial x^3} u_1(x,t).$$

From Eq. (5) for $l = 1$, we get

$$u_1(x,t) = g(x) + g_1(x) \frac{t^\beta}{\Gamma(\beta+1)}.$$

Hence,

$$Res_{u,1}(x,t) = g_1(x) - \left(g(x) + g_1(x) \frac{t^\beta}{\Gamma(\beta+1)}\right)^3 \text{x} \left(g'''(x) + g_1'''(x) \frac{t^\beta}{\Gamma(\beta+1)}\right).$$

From Eq. (8), we find the $Res_{u,1}(x,0) = 0$, and therefore

$$g_1(x) = -\frac{1}{\left(4 - \frac{3}{2}x\right)^{\frac{1}{3}}}.$$

Thus, we get

$$u_1(x,t) = \left(4 - \frac{3}{2}x\right)^{\frac{2}{3}} - \frac{1}{\left(4 - \frac{3}{2}x\right)^{\frac{1}{3}}} \frac{t^\beta}{\Gamma(\beta+1)}.$$

To determine $g_2(x)$, we write $l = 2$ in Eq. (9) and we have

$$Res_{u,2}(x,t) = D_t^\beta u_2(x,t) - u_2^3(x,t) \frac{\partial^3}{\partial x^3} u_2(x,t).$$

From Eq. (5) at $l = 2$, we get

$$u_2(x,t) = g(x) + g_1(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}.$$

Hence,

$$Res_{u,2}(x,t) = g_1(x) + g_2(x)\frac{t^{\beta}}{\Gamma(\beta+1)} - \left(g(x) + g_1(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right)^3 \times \left(g'''(x) + g_1'''(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2'''(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right).$$

From Eq. (8), we find $D_t^{\beta} Res_{u,2}(x,0) = 0$, and therefore

$$g_2(x) = -\frac{1}{2\left(4-\frac{3}{2}x\right)^{\frac{4}{3}}}.$$

Thus,

$$u_2(x,t) = \left(4-\frac{3}{2}x\right)^{\frac{2}{3}} - \frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{1}{3}}}\frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{1}{2\left(4-\frac{3}{2}x\right)^{\frac{4}{3}}}\frac{t^{2\beta}}{\Gamma(2\beta+1)}.$$

To determine $g_3(x)$, we write $l = 3$ in Eq. (9) and we get

$$Res_{u,3}(x,t) = D_t^{\beta} u_3(x,t) - u_3^3(x,t)\frac{\partial^3}{\partial x^3}u_3(x,t).$$

From Eq. (5) at $l = 3$, we have

$$u_3(x,t) = g(x) + g_1(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g_3(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}.$$

Thus,

$$Res_{u,3}(x,t) = g_1(x) + g_2(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_3(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \left(g(x) + g_1(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g_3(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}\right)^3 \left(g'''(x) + g_1'''(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2'''(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g_3'''(x)\frac{t^{3\beta}}{\Gamma(3\beta+1)}\right).$$

From Eq. (8), we gain $D_t^{2\beta} Res_{u,3}(x,0) = 0$, and therefore

$$g_3(x) = -\frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{7}{3}}}.$$

Hence,

$$u_3(x,t) = \left(4-\frac{3}{2}x\right)^{\frac{2}{3}} - \frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{1}{3}}}\frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{1}{2\left(4-\frac{3}{2}x\right)^{\frac{4}{3}}}\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{7}{3}}}\frac{t^{3\beta}}{\Gamma(3\beta+1)}.$$

Using the same operation for $l = 4$, we get

$$g_4(x) = -\frac{7}{2\left(4-\frac{3}{2}x\right)^{\frac{10}{3}}},$$

$$u_4(x,t) = \left(4-\frac{3}{2}x\right)^{\frac{2}{3}} - \frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{1}{3}}}\frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{1}{2\left(4-\frac{3}{2}x\right)^{\frac{4}{3}}}\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{1}{\left(4-\frac{3}{2}x\right)^{\frac{7}{3}}}\frac{t^{3\beta}}{\Gamma(3\beta+1)} - \frac{7}{2\left(4-\frac{3}{2}x\right)^{\frac{10}{3}}}\frac{t^{4\beta}}{\Gamma(4\beta+1)}.$$

In Table 1, the $u_4(x, t)$ solution is gained for $\beta = 0.25$, $\beta = 0.50$, $\beta = 0.75$, and $\beta = 1$ with the different values of t and x. Besides, the exact solution is compared with the $u_4(x, t)$ solution for $\beta = 1$ in this table. From Table 1, it can be seen that the absolute error gets smaller as the value of t decreases.

For $0 \leq t \leq 1$ and $-30 \leq x \leq 0$ at $\beta = 1$, the comparison of the $u_4(x, t)$ and the exact solution is illustrated in Figure 1. When equal parameters are used, it is seen that the $u_4(x, t)$ solution has nearly the same shape as the exact solution in this figure.

In Figure 2, the geometrical behavior of the $u_4(x, t)$ with 3D plot for $0 \leq t \leq 5$, $0 \leq x \leq 1$, and the different values

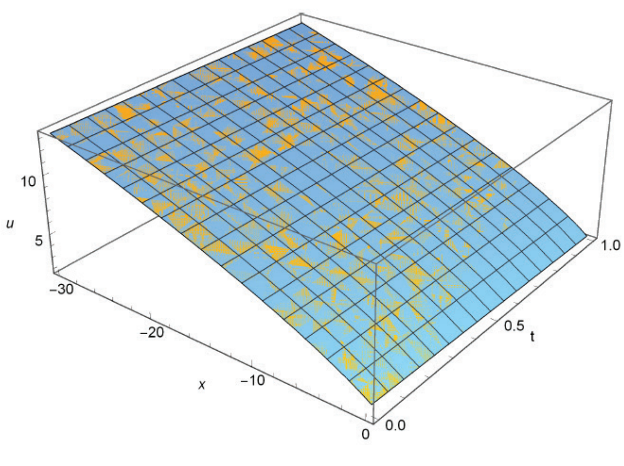

Figure 1. The plot of the $u_4(x, t)$ and exact solution.

Table 1. Comparing the $u_4(x, t)$ solution and the exact solution with the different values of t and x.

		$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.75$	$\beta = 1$		
x	t	$u_4(x, t)$	$u_4(x, t)$	$u_4(x, t)$	$u_4(x, t)$	Exact solution	Absolute error
-10	0.2	6.83852	6.92921	6.99774	7.04522	7.04522	1.35745x10^{-10}
	0.4	6.78388	6.84877	6.91334	6.96966	6.96966	4.39464x10^{-9}
	0.6	6.74698	6.78650	6.83880	6.89370	6.89370	3.37672x10^{-8}
	0.8	6.71828	6.73363	6.76987	6.81731	6.81731	1.44003x10^{-7}
	1	6.69446	6.68676	6.70475	6.7405	6.7405	4.44810x10^{-7}
-5	0.2	4.75697	4.86719	4.94936	5.00586	5.00586	1.20481x10^{-9}
	0.4	4.69016	4.77026	4.84856	4.91607	4.91607	3.93112x10^{-8}
	0.6	4.64474	4.69472	4.75904	4.82544	4.82544	3.04515x10^{-7}
	0.8	4.60926	4.63019	4.67584	4.73396	4.73396	1.30959x10^{-6}
	1	4.57968	4.57266	4.59684	4.64159	4.64159	4.08059x10^{-6}
0	0.2	1.99562	2.18286	2.30909	2.39222	2.39222	1.21435x10^{-7}
	0.4	1.87560	2.02852	2.15777	2.26110	2.26110	4.12452x10^{-6}
	0.6	1.78945	1.90202	2.01889	2.12609	2.12605	3.33870x10^{-5}
	0.8	1.71927	1.78884	1.88536	1.98673	1.98658	1.50729x10^{-4}
	1	1.65869	1.68342	1.75398	1.84251	1.84202	4.95747x10^{-4}
5	0.2	2.75042	2.62131	2.51370	2.43513	2.43513	1.92919x10^{-7}
	0.4	2.82387	2.74355	2.64987	2.56166	2.56167	5.83621x10^{-6}
	0.6	2.87013	2.83355	2.76562	2.68511	2.68515	4.20345x10^{-5}
	0.8	2.90395	2.90641	2.86895	2.80569	2.80586	1.68489x10^{-4}
	1	2.93043	2.96788	2.96320	2.92353	2.92402	4.90352x10^{-4}
10	0.2	5.26849	5.16913	5.09107	5.03561	5.03561	1.40554x10^{-9}
	0.4	5.32758	5.25941	5.18845	5.12435	5.12435	4.41275x10^{-8}
	0.6	5.36687	5.32792	5.27289	5.21232	5.21232	3.28892x10^{-7}
	0.8	5.39709	5.38516	5.34972	5.29956	5.29956	1.36081x10^{-6}
	1	5.42196	5.43522	5.42122	5.38608	5.38609	4.07902x10^{-6}

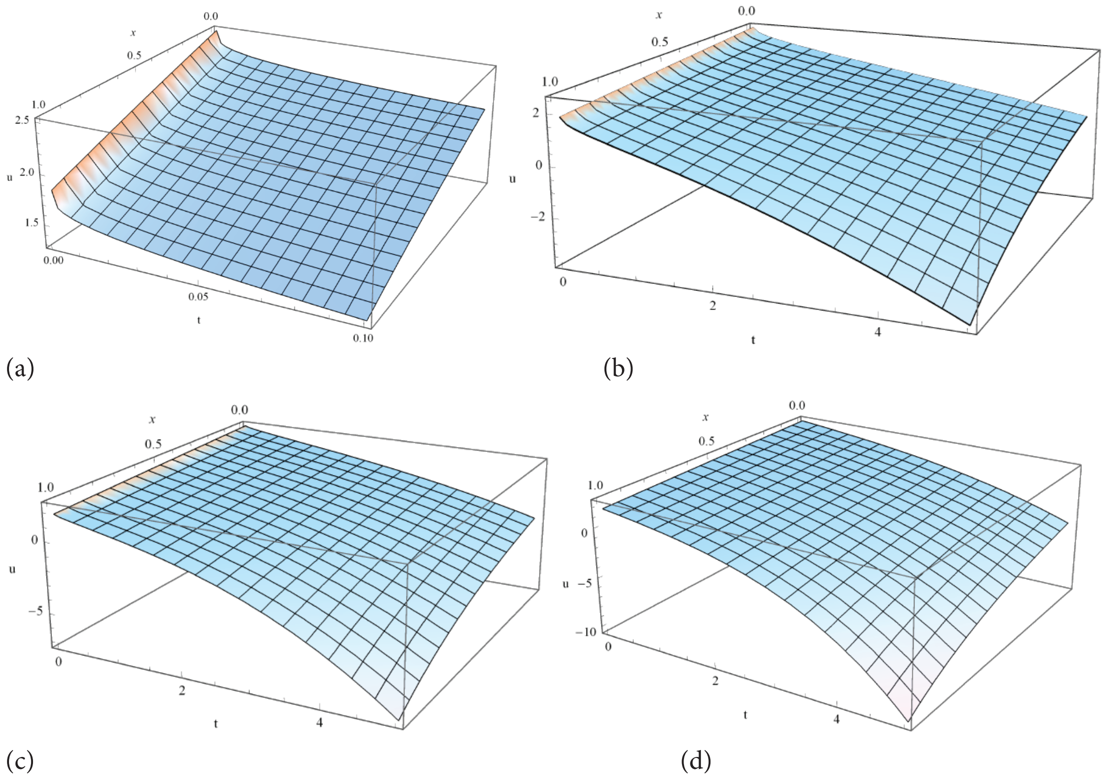

Figure 2. 3D plot of the $u_4(x, t)$: (a) $u_4(x, t)$ for $\beta = 0.25$, (b) $u_4(x, t)$ for $\beta = 0.50$, (c) $u_4(x, t)$ for $\beta = 0.75$, (d) $u_4(x, t)$ for $\beta = 1$.

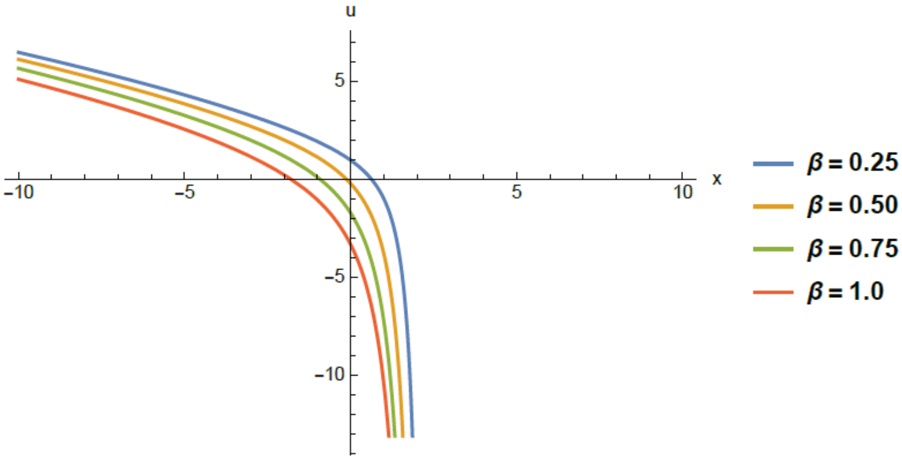

Figure 3. 2D plot of the $u_4(x, 5)$ for the different values of β.

Table 2. Comparison of HPSTM, ADM, RPSM, and exact solution for $\beta = 1$.

x	t	RPSM	HPSTM [22]	ADM [22]	Exact Solution
0	1	1.843946953	1.843946953	1.843946953	1.842015749
0.2	1	1.694117376	1.694117377	1.694117377	1.691538112
0.4	1	1.5337581542	1.537581542	1.537581542	1.534036644
0.6	1	1.373028020	1.373028020	1.373028020	1.367980757
0.8	1	1.198654865	1.198654865	1.198654865	1.91138425
1	1	1.011880652	1.011880649	1.011880649	1.000000000

of β is illustrated by suggested method. Besides, the same solution with 2D plot for $t = 5$ and $-10 \le x \le 10$ is demonstrated in Figure 3. The solution at $\beta = 0.25$ is showed with the blue line, the solution at $\beta = 0.50$ is showed with the orange line, the solution at $\beta = 0.75$ is showed with the green line, and the solution at $\beta = 1$ is showed with the red line in this figure. All plots in figures are illustrated by the aid of Mathematica 11.3.

For $\beta = 1$, the third order term solution $u_3(x, t)$ of the RPSM, homotopy perturbation Sumudu transform method (HPSTM) [22], Adomian decomposition method (ADM) [22], and exact solution are compared in Table 2. It is observed from this table that the RPSM solution performed a high accuracy agreement with the ADM and HPSM solution. It is also seen that the accuracy increases as the order of the approximation increases.

CONCLUSION

In this study, the RPSM was utilized for obtaining the approximate solutions of Eq. (1). These solutions were illustrated by numerically and graphically for the different values of β, t and x. By comparing the approximate solution and the exact solution, the accuracy and efficiency of the suggested method were demonstrated. When equal parameters were selected, it was observed that the approximate solution had almost the same shape as the exact solution. The proposed method was compared numerically with the HPSTM and the ADM by table. It was seen from this table that the RPSM made a good agreement with this methods. It is seen from the approximate solutions that only a few iterates were used by the proposed method. With these iterates, an infinite series solutions can be found. The accuracy of the RPSM increases as the order of these solutions increases. Besides, this method does not need a lot of time and computer memory. The RPSM indicates strong performance with less computation than other methods in the literature. Moreover, the RPSM does not require transformation, linearization, discretization, or perturbation. Furthermore, the suggested method can be used to get approximate solutions of different kinds of fractional partial differential equations.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw

data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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