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SOME CONSTRUCTION METHODS FOR IMPLICATION OPERATORS ON BOUNDED LATTICES

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Abstract. In this paper, we focus the construction methods of implications on bounded lattices. We introduce several methods to obtain implication on bounded lattices. We basically base on implication defined on subinterval such as $[a,1]$ or $[0,b]$ of the bounded lattice L which has $a,b \in L$ with $a \le b$ for these construction methods. We also use fuzzy logic operators such as t-norms, t-conorms, negations and implications on L in some construction methods. In addition, we give some remarks and examples to make the new construction methods.

1. Introduction

Fuzzy implications generalize the classical implications taking values from $\{0,1\}$ to the fuzzy logic, where the truth values belong to the unit interval $[0,1]$. Since fuzzy implications have been used in many areas such as fuzzy control, approximate reasoning, and decision support systems, fuzzy control and etc. [8, 9, 6, 11, 12], the construction methods of these operators are especially important for aplications of them and thus, fuzzy implications construction methods have attracted the attention of researchers. In [8], Baczyński and Jayaram introduced construction methods of implication which are obtained from fuzzy logic operators on unit real interval $[0,1]$.

In [10], Neres et al. proposed fuzzy implications construction methods, which is called as a new construction technique, from a pair of bivariate aggregation functions and a fuzzy negation on unit interval real $[0,1]$. In [7] Karaçal et al. introduced two construction methods to built implication operators on bounded lattices by means of t-norms, t-conorms and implications. In [3], Kesicioğlu et al. offered implication construction methods which is called the linear and $g-$convex combination for implications on bounded lattices, where they benefited from fuzzy logic operators. In [4], Karaçal et al. gived many construction methods for

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implications by means of an arbitrary element and basic logic connectives such as t-norms, t-conorms and negations on bounded lattices.

In this paper, our main aim is to obtain an implication from an implication on subinterval $[a, 1]$ ($[0, b]$) of the bounded lattice L where $a, b \in L$ with $a \leq b$. In addition, we give some other construction methods for implications on bounded lattices via some logic operators besides implications. Firstly, in the section 2 we remind some main definitions and results, which are useful for our paper. In the next section, we give construction methods to built implications on bounded lattices and we add various examples and results from these construction methods. Finally, we finish with concluding remarks .

2. Preliminaries

In this section, we list some basic notions and results which will be use in the paper.

Definition 2.1. [5] Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The subinterval $[a, b]$ is defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, $(a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$ can be defined.

Definition 2.2. [1, 2] Let $(L, \leq, 0, 1)$ be a bounded lattice. A function $T : L^2 \longrightarrow L$ is a t-norm if it satisfies the following conditions for any $x, y \in L$.

(T1) $T(x, y) = T(y, x)$ (commutavity).
(T2) $T(x, 1) = x$ (neutral element).
(T3) If $y \leq z$, then $T(x, y) \leq T(x, z)$ (monotonicity).
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

Definition 2.3. [1, 2] Let $(L, \leq, 0, 1)$ be a bounded lattice. A function $S : L^2 \longrightarrow L$ is a t-conorm if it satisfies the following conditions for any $x, y \in L$.

(S1) $S(x, y) = S(y, x)$ (commutavity).
(S2) $S(x, 0) = x$ (neutral element).
(S3) If $y \leq z$, then $S(x, y) \leq S(x, z)$ (monotonicity).
(S4) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity).

Example 2.4. Let $(L, \leq, 0, 1)$ be a bounded lattice. Two basic t-norms T_D and $T_\wedge$ on a bounded lattice L are respectively given by

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T_\wedge(x, y) = x \wedge y.$$

Two basic t-conorms S_D and $S_\vee$ on a bounded lattice L are respectively given as follows:

$$S_D(x,y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{if } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$S_\vee(x,y) = x \vee y.$$

Definition 2.5. [8, 3, 7] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$.

Definition 2.6. [8, 3, 7] A function $I : L^2 \to L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:

(I1) I is a decreasing operation on the first variable, that is, for every $x, z \in L$ with $x \leq z$, $I(z,y) \leq I(x,y)$ for all $y \in L$.

(I2) I is an increasing operation on the second variable, that is, for every $y, z \in L$ with $y \leq z$, $I(x,y) \leq I(x,z)$ for all $x \in L$.

(I3) $I(0,0) = 1$.

(I4) $I(1,1) = 1$.

(I5) $I(1,0) = 0$.

Theorem 2.7. [8] *Let $S : [0,1]^2 \to L$ be a t-conorm and $N : [0,1] \to [0,1]$ be a negation. Then the function $I : [0,1]^2 \to [0,1]$ defined by, for all $x, y \in L$,*

$$I(x,y) = S(N(x), y) \tag{2.1}$$

is an implication.

Theorem 2.8. [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L$. Then the function $I_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I_a(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x > y, \\ a & \textit{otherwise,} \end{cases} \tag{2.2}$$

is an implication.

Theorem 2.9. [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm and $N : L \to L$ be a negation. Then the function $I : L^2 \to L$ defined by, for all $x, y \in L$,*

$$I(x,y) = \begin{cases} 1 & x \leq y, \\ y & x > y, \\ S(N(x), y) & \textit{otherwise,} \end{cases} \tag{2.3}$$

is an implication.

Theorem 2.10. [4] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $N : L \to L$ be a negation and $J_1, J_2, J_3 : L^2 \to L$ be implications. Then the function $I : L^2 \to L$ defined by*

$$I(x,y) = J_3(N(J_1(x,y)), J_2(x,y)) \tag{2.4}$$

is an implication.

Theorem 2.11. [3] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $TS_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$TS_a(x,y) = T(S(a, I(x,y)), J(x,y)) \tag{2.5}$$

is an implication.

Theorem 2.12. [3] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications and $a \in L$. The function $ST_a : L^2 \to L$ defined by, for all $x, y \in L$,*

$$ST_a(x,y) = S(T(a, I(x,y)), J(x,y)) \tag{2.6}$$

is an implication.

Theorem 2.13. [7] *Let $(L, \leq, 0, 1)$ be a bounded lattice, $S : L^2 \to L$ be a t-conorm, $T : L^2 \to L$ be a t-norm, $I, J : L^2 \to L$ be implications, $N : L \to L$ be a negation and $a \in L$. The function $K^{I,J}_{a,T,S,N} : L^2 \to L$ defined by, for all $x, y \in L$,*

$$K^{I,J}_{a,T,S,N} = S(T(a, I(x,y)), T(N(a), J(x,y)))) \tag{2.7}$$

is an implication if and only if $S(a, N(a)) = 1$.

3. Some construction methods of implication on L

In this section, we offer many construction methods of implication operators. In Theorem 3.1 (3.4) focus on extension of an implication on the subinterval $[a, 1]$ ($[0, b]$) to bounded lattice L, where $a, b \in L$ such as $a \leq b$. In the following construction methods, we give some different construction methods for implications on bounded lattices considering some logic operators such as t-norms, t-conorms, negations as well as implications. Also we illustrate the new construction methods with the several examples.

Theorem 3.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$ and $J : [a, 1]^2 \longrightarrow [a, 1]$ be an implication. Then, the function $I_1 : L^2 \longrightarrow L$ defined by,*

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1) \text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ b & \text{if } x \notin [a,1] \text{ and } y \notin [a,1], \end{cases} \tag{3.1}$$

is an implication on L.

Proof. I3, I4 and I5 are obtained directly from the definition of I_1.

(I1) Let us show that I_1 is a decreasing function on the first variable. Then it should be $I_1(x_2, y) \leq I_1(x_1, y)$ for every elements $x_1, x_2, y \in L$ with $x_1 \leq x_2$. If $x_1 = 0$ or $(x_2, y) = (1, 0)$ or $y = 1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1, y) \in [a, 1]^2$.

$$I_1(x_2, y) = J(x_2, y) \leq J(x_1, y) = I_1(x_1, y).$$

2. Let $x_1 \in [a,1]$ and $y \notin [a,1]$.

$$I_1(x_2,y) = a \leq a = I_1(x_1,y).$$

3. Let $x_1 \notin [a,1]$ and $y \in [a,1]$.
3.1. If $x_2 \in [a,1]$,

$$I_1(x_2,y) = J(x_2,y) \leq 1 = I_1(x_1,y).$$

3.2. If $x_2 \notin [a,1]$,

$$I_1(x_2,y) = 1 \leq 1 = I_1(x_1,y).$$

4. Let $x_1 \notin [a,1]$ and $y \notin [a,1]$.
4.1. If $x_2 \in [a,1]$,

$$I_1(x_2,y) = a \leq b = I_1(x_1,y).$$

4.2. If $x_2 \notin [a,1]$,

$$I_1(x_2,y) = b \leq b = I_1(x_1,y).$$

(I2) Let us show that I_1 is an increasing function on the second variable. Then it should be $I_1(x,y_1) \leq I_1(x,y_2)$ for every elements $x, y_1, y_2 \in L$ with $y_1 \leq y_2$. If $x = 0$ or $y_2 = 1$ or $(x,y_1) = (1,0)$, the proof is immediate. The proof can be split into all possible cases.

1. Let $(x,y_1) \in [a,1]^2$.

$$I_1(x,y_1) = J(x,y_1) \leq J(x,y_2) = I_1(x,y_2).$$

2. Let $x \in [a,1]$ and $y_1 \notin [a,1]$.
2.1. If $y_2 \in [a,1]$,

$$I_1(x,y_1) = a \leq J(x,y_2) = I_1(x,y_2).$$

2.2. If $y_2 \notin [a,1]$,

$$I_1(x,y_1) = a \leq a = I_1(x,y_2).$$

3. Let $x \notin [a,1]$ and $y_1 \in [a,1]$.

$$I_1(x,y_1) = 1 \leq 1 = I_1(x,y_2).$$

4. Let $x \notin [a,1]$ and $y_1 \notin [a,1]$.
4.1. If $y_2 \in [a,1]$,

$$I_1(x,y_1) = b \leq 1 = I_1(x,y_2).$$

4.2. If $y_2 \notin [a,1]$,

$$I_1(x,y_1) = b \leq b = I_1(x,y_2).$$

□

Remark 3.2. (i) If $b = 1$, then the implication I_1 given by the formula (3.1) can be rewritten as follows:

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1)\text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ 1 & \text{if } x \notin [a,1] \text{ and } y \notin [a,1]. \end{cases} \tag{3.2}$$

(ii) If $a = b$, then the implication I_1 given by the formula (3.5) can be rewritten as follows:

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x = 0 \text{ or } y = 1)\text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{otherwise.} \end{cases} \tag{3.3}$$

In order to apply the formula (3.1), we include the following example.

Example 3.3. Consider the bounded lattice $(L = \{0, t_1, t_2, t_3, t_4, t_5, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Fig. 1.

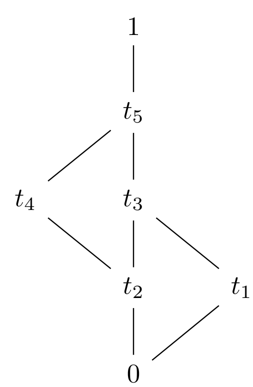

Figure 1. The lattice L.

Let use take the implication $J : [t_2, 1]^2 \to [t_2, 1]$ as in Table 1:

J	t_2	t_3	t_4	t_5	1
t_2	1	1	1	1	1
t_3	t_4	t_5	t_4	t_5	1
t_4	t_3	t_3	t_5	t_5	1
t_5	t_2	t_3	t_4	t_5	1
1	t_2	t_3	t_4	t_5	1

Table 1. The implication J on $[t_2, 1]$.

By applying the formula (3.1) in Theorem 3.1 with $a = t_2$ and $b = t_3$, the implication I_1 can be obtained as in Table 2.

I_1	0	t_1	t_2	t_3	t_4	t_5	1
0	1	1	1	1	1	1	1
t_1	t_3	t_3	1	1	1	1	1
t_2	t_2	t_2	1	1	1	1	1
t_3	t_2	t_2	t_4	t_5	t_4	t_5	1
t_4	t_2	t_2	t_3	t_3	t_5	t_5	1
t_5	t_2	t_2	t_2	t_3	t_4	t_5	1
1	0	t_2	t_2	t_3	t_4	t_5	1

Table 2. The implication I_1 on L.

Theorem 3.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$ and $J : [0, b]^2 \longrightarrow [0, b]$ be an implication. Then, the function $I_1^* : L^2 \longrightarrow L$ defined by,*

$$I_1^*(x, y) = \begin{cases} 1 & \textit{if } x = 0 \textit{ or } y = 1, \\ 0 & \textit{if } (x, y) = (1, 0) \textit{ or } (x \notin [0, b] \textit{ and } y \in [0, b]), \\ J(x, y) & \textit{if } (x, y) \in [0, b]^2, \\ b & \textit{if } x \in [0, b] \textit{ and } y \notin [0, b], \\ a & \textit{if } x \notin [0, b] \textit{ and } y \notin [0, b], \end{cases} \tag{3.4}$$

is an implication on L.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

We present another construction method for implication operators. For this construction method, we use some logic operators on a bounded lattice L, an implication on the subinterval $[a, 1]$ of the bounded lattice L and $a, b \in L$.

Theorem 3.5. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $T : L^2 \to L$ be a t-norm, $N : L \to L$ be a negation and $J : [a, 1]^2 \longrightarrow [a, 1]$ be an implication. Then, the function $I_2 : L^2 \longrightarrow L$ defined by,*

$$I_2(x, y) = \begin{cases} 1 & \textit{if } (x = 0 \textit{ or } y = 1) \textit{or } (x \notin [a, 1] \textit{ and } y \in [a, 1]), \\ 0 & \textit{if } (x, y) = (1, 0), \\ J(x, y) & \textit{if } (x, y) \in [a, 1]^2, \\ T(N(x), a) & \textit{if } x \in [a, 1] \textit{ and } y \notin [a, 1], \\ T(N(x), b) & \textit{if } x \notin [a, 1] \textit{ and } y \notin [a, 1], \end{cases} \tag{3.5}$$

is an implication on L.

Proof. I3, I4 and I5 are obtained directly from the definition of I_2.

(I1) Let us show that I_2 is a decreasing function on the first variable. Then it should be $I_2(x_2, y) \leq I_2(x_1, y)$ for every elements $x_1, x_2, y \in L$ with $x_1 \leq x_2$. If $x_1 = 0$ or $(x_2, y) = (1, 0)$ or $y = 1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1, y) \in [a,1]^2$.

$$I_2(x_2, y) = J(x_2, y) \leq J(x_1, y) = I_2(x_1, y).$$

2. Let $x_1 \in [a,1]$ and $y \notin [a,1]$.

$$I_2(x_2, y) = T(N(x_2), a) \leq T(N(x_1), a) = I_2(x_1, y).$$

3. Let $x_1 \notin [a,1]$ and $y \in [a,1]$.
3.1. If $x_2 \in [a,1]$,

$$I_2(x_2, y) = J(x_2, y) \leq 1 = I_2(x_1, y).$$

3.2. If $x_2 \notin [a,1]$,

$$I_2(x_2, y) = 1 \leq 1 = I_2(x_1, y).$$

4. Let $x_1 \notin [a,1]$ and $y \notin [a,1]$
4.1. If $x_2 \in [a,1]$,

$$I_2(x_2, y) = T(N(x_2), a) \leq T(N(x_1), a) \leq T(N(x_1), b) = I_2(x_1, y).$$

4.2. If $x_2 \notin [a,1]$,

$$I_2(x_2, y) = T(N(x_2), b) \leq T(N(x_1), b) = I_2(x_1, y).$$

(I2) Let us show that I_2 is an increasing function on the second variable. Then it should be $I_2(x, y_1) \leq I_2(x, y_2)$ for every elements $x, y_1, y_2 \in L$ with $y_1 \leq y_2$. If $x = 0$ or $y_2 = 1$ or $(x, y_1) = (1, 0)$, the proof is immediate. The proof can be split into all possible cases.

1. Let $(x, y_1) \in [a,1]^2$.

$$I_2(x, y_1) = J(x, y_1) \leq J(x, y_2) = I_2(x, y_2).$$

2. Let $x \in [a,1]$ and $y_1 \notin [a,1]$.
2.1. If $y_2 \in [a,1]$,

$$I_2(x, y_1) = T(N(x), a) \leq a \leq J(x, y_2) = I_2(x, y_2).$$

2.2. If $y_2 \notin [a,1]$,

$$I_2(x, y_1) = T(N(x), a) \leq T(N(x), a) = I_2(x, y_2).$$

3. Let $x \notin [a,1]$ and $y_1 \in [a,1]$.

$$I_2(x, y_1) = 1 \leq 1 = I_2(x, y_2).$$

4. Let $x \notin [a,1]$ and $y_1 \notin [a,1]$.
4.1. If $y_2 \in [a,1]$,

$$I_2(x, y_1) = T(N(x), b) \leq 1 = I_2(x, y_2).$$

4.2. If $y_2 \notin [a,1]$,

$$I_2(x, y_1) = T(N(x), b) \leq T(N(x), b) = I_2(x, y_2).$$

□

Remark 3.6. Let T be the t-norm $T_\wedge$ in Theorem 3.5.
(i) The implication I_2 given by the formula (3.5) can be rewritten

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ N(x)\wedge b & \text{if } x\notin[a,1] \text{ and } y\notin[a,1],\end{cases} \tag{3.6}$$

(ii) If $b=1$, then the implication I_2 given by the formula (3.5) can be rewritten as follows:

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ N(x) & \text{if } x\notin[a,1] \text{ and } y\notin[a,1],\end{cases} \tag{3.7}$$

(iii) If $a=b$, then the implication I_2 given by the formula (3.5) can be rewritten as follows:

$$I_2(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ N(x)\wedge a & \text{otherwise.}\end{cases} \tag{3.8}$$

Now, let us illustrate the application of Theorem 3.5 with the following example.

Example 3.7. Consider the lattice $(L=\{0,t_1,t_2,t_3,t_4,t_5,1\},\leq,0,1)$ as given in Fig. 1, the t-norm $T:L^2\to L$ as in $T_\wedge$ and the implication $J:[t_2,1]^2\to[t_2,1]$ as given in Table 1. Let the negation $N:L\to L$ be as in formula 3.9.

$$N(x)=\begin{cases}1 & \text{if } x=0,\\ t_3 & \text{if } x\in\{t_1,t_3\},\\ t_5 & \text{if } x=t_2,\\ t_4 & \text{if } x=t_4,\\ t_2 & \text{if } x=t_5,\\ 0 & \text{if } x=1.\end{cases} \tag{3.9}$$

By applying the formula (3.5) in Theorem 3.5 with $a=t_2$ and $b=t_3$, the implication I_2 can be obtained as in Table 3.

I_2	0	t_1	t_2	t_3	t_4	t_5	1
0	1	1	1	1	1	1	1
t_1	t_3	t_3	1	1	1	1	1
t_2	t_2	t_2	1	1	1	1	1
t_3	t_2	t_2	t_4	t_5	t_4	t_5	1
t_4	t_2	t_2	t_3	t_3	t_5	t_5	1
t_5	t_2	t_2	t_2	t_3	t_4	t_5	1
1	0	0	t_2	t_3	t_4	t_5	1

Table 3. The implication I_2 on L.

Theorem 3.8. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm, $N : L \to L$ be a negation and $J : [0, b]^2 \longrightarrow [0, b]$ be an implication. Then, the function $I_2^* : L^2 \longrightarrow L$ defined by,*

$$I_2^*(x,y) = \begin{cases} 1 & \textit{if } x = 0 \textit{ or } y = 1, \\ 0 & \textit{if } (x,y) = (1,0) \textit{ or } (x \notin [0,b] \textit{ and } y \in [0,b]), \\ J(x,y) & \textit{if } (x,y) \in [0,b]^2, \\ S(N(x),b) & \textit{if } x \in [0,b] \textit{ and } y \notin [0,b], \\ S(N(x),a) & \textit{if } x \notin [0,b] \textit{ and } y \notin [0,b], \end{cases} \tag{3.10}$$

is an implication on L.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

In the following theorem, we present a method to construct implication operators. To do that, we use a t-norm T on L, an implication on a subinterval of L and arbitrary fix elements of L.

Theorem 3.9. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $T : L^2 \to L$ be a t-norm and $J : [a, 1]^2 \longrightarrow [a, 1]$. Then, the function $I_3 : L^2 \longrightarrow L$ defined by,*

$$I_3(x,y) = \begin{cases} 1 & \textit{if } (x = 0 \textit{ or } y = 1) \textit{or } (x \notin [a,1] \textit{ and } y \in [a,1]), \\ 0 & \textit{if } (x,y) = (1,0), \\ J(x,y) & \textit{if } (x,y) \in [a,1]^2, \\ T(a,y) & \textit{if } x \in [a,1] \textit{ and } y \notin [a,1], \\ T(b,y) & \textit{if } x \notin [a,1] \textit{ and } y \notin [a,1], \end{cases} \tag{3.11}$$

is an implication on L.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

Theorem 3.10. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm and $J : [0, b]^2 \longrightarrow [0, b]$. Then, the function $I_3^* : L^2 \longrightarrow L$ defined by,*

$$I_3^*(x,y)=\begin{cases}1 & \textit{if } x=0 \textit{ or } y=1,\\ 0 & \textit{if } (x,y)=(1,0) \textit{ or } (x\notin[0,b] \textit{ and } y\in[0,b]),\\ J(x,y) & \textit{if } (x,y)\in[0,b]^2,\\ S(b,y) & \textit{if } x\in[0,b] \textit{ and } y\notin[0,b],\\ S(a,y) & \textit{if } x\notin[0,b] \textit{ and } y\notin[0,b],\end{cases} \tag{3.12}$$

is an implication on L.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.1. Therefore, we omit it. □

In the following Theorem 3.11, a construction method for implication operators is presented considering some logic operators on a bounded lattice L or on a subinterval of the bounded lattice L and $a,b\in L$.

Theorem 3.11. *Let $(L,\leq,0,1)$ be a bounded lattice, $a,b\in L$ with $a\leq b$, $T:L^2\to L$ be a t-norm, $K:L^2\to L$ be an implication and $J:[a,1]^2\longrightarrow[a,1]$ be an implication. Then, the function $I_4:L^2\longrightarrow L$ defined by,*

$$I_4(x,y)=\begin{cases}1 & \textit{if } (x=0 \textit{ or } y=1) \textit{or } (x\notin[a,1] \textit{ and } y\in[a,1]),\\ 0 & \textit{if } (x,y)=(1,0),\\ J(x,y) & \textit{if } (x,y)\in[a,1]^2,\\ T(K(x,y),a) & \textit{if } x\in[a,1] \textit{ and } y\notin[a,1],\\ T(K(x,y),b) & \textit{if } x\notin[a,1] \textit{ and } y\notin[a,1],\end{cases} \tag{3.13}$$

is an implication on L.

Proof. I3, I4 and I5 are obtained directly from the definition of I_4.

(I1) We need to show that I_4 is a decreasing function on the first variable. Then it should be $I_4(x_2,y)\leq I_4(x_1,y)$ for $x_1,x_2,y\in L$ and $x_1\leq x_2$. If $x_1=0$ or $(x_2,y)=(1,0)$ or $y=1$, the proof is trivial. The proof can be split into all possible cases.

1. Let $(x_1,y)\in[a,1]^2$.

$$I_4(x_2,y)=J(x_2,y)\leq J(x_1,y)=I_4(x_1,y).$$

2. Let $x_1\in[a,1]$ and $y\notin[a,1]$.

$$I_4(x_2,y)=T(K(x_2,y),a)\leq T(K(x_1,y),a)=I_4(x_1,y).$$

3. Let $x_1\notin[a,1]$ and $y\in[a,1]$.

3.1. If $x_2\in[a,1]$,

$$I_4(x_2,y)=J(x_2,y)\leq 1=I_4(x_1,y).$$

3.2. If $x_2\notin[a,1]$,

$$I_4(x_2,y)=1\leq 1=I_4(x_1,y).$$

4. Let $x_1\notin[a,1]$ and $y\notin[a,1]$

4.1. If $x_2\in[a,1]$,

$$I_4(x_2,y) = T(K(x_2,y),a) \le T(K(x_1,y),a) \le T(K(x_1,y),b) = I_4(x_1,y).$$

4.2. If $x_2 \notin [a,1]$,

$$I_4(x_2,y) = T(K(x_2,y),b) \le T(K(x_1,y),b) = I_4(x_1,y).$$

(I2) We need to show that I_4 is an increasing function on the second variable. Then it should be $I_4(x,y_1) \le I_4(x,y_2)$ for $x,y_1,y_2 \in L$ and $y_1 \le y_2$. If $x = 0$ or $y_2 = 1$ or $(x,y_1) = (1,0)$, the proof is immediate. The proof can be split into all possible cases.

1. Let $(x,y_1) \in [a,1]^2$.

$$I_4(x,y_1) = J(x,y_1) \le J(x,y_2) = I_4(x,y_2).$$

2. Let $x \in [a,1]$ and $y_1 \notin [a,1]$.

2.1. If $y_2 \in [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),a) \le a \le J(x,y_2) = I_4(x,y_2).$$

2.2. If $y_2 \notin [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),a) \le T(K(x,y_2),a) = I_4(x,y_2).$$

3. Let $x \notin [a,1]$ and $y_1 \in [a,1]$.

$$I_4(x,y_1) = 1 \le 1 = I_4(x,y_2).$$

4. Let $x \notin [a,1]$ and $y_1 \notin [a,1]$.

4.1. If $y_2 \in [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),b) \le 1 = I_4(x,y_2).$$

4.2. If $y_2 \notin [a,1]$,

$$I_4(x,y_1) = T(K(x,y_1),b) \le T(K(x,y_2),b) = I_4(x,y_2).$$

□

Remark 3.12. Let T be the t-norm $T_\wedge$ in Theorem 3.11.

(i) The implication I_4 given by the formula (3.13) can be rewritten as

$$I_4(x,y) = \begin{cases} 1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x \notin [a,1] \text{ and } y \in [a,1]), \\ 0 & \text{if } (x,y) = (1,0), \\ J(x,y) & \text{if } (x,y) \in [a,1]^2, \\ K(x,y) \wedge a & \text{if } x \in [a,1] \text{ and } y \notin [a,1], \\ K(x,y) \wedge b & \text{if } x \notin [a,1] \text{ and } y \notin [a,1]. \end{cases} \tag{3.14}$$

(ii) If $b = 1$, then the implication I_4 given by the formula (3.13) can be rewritten as follows:

$$(3.15)\qquad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ K(x,y)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ K(x,y) & \text{if } x\notin[a,1] \text{ and } y\notin[a,1].\end{cases}$$

(iii) If $a=b$, then the implication I_4 given by the formula (3.13) can be rewritten as follows:

$$(3.16)\qquad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ K(x,y)\wedge a & \text{otherwise.}\end{cases}$$

(iv) If $K(x,y)=N(x)\vee y$, then the implication I_4 given by the formula (3.13) can be rewritten as follows:

$$(3.17)\quad I_4(x,y)=\begin{cases}1 & \text{if } (x=0 \text{ or } y=1)\text{or } (x\notin[a,1] \text{ and } y\in[a,1]),\\ 0 & \text{if } (x,y)=(1,0),\\ J(x,y) & \text{if } (x,y)\in[a,1]^2,\\ (N(x)\vee y)\wedge a & \text{if } x\in[a,1] \text{ and } y\notin[a,1],\\ (N(x)\vee y)\wedge b & \text{if } x\notin[a,1] \text{ and } y\notin[a,1].\end{cases}$$

We illustrate a example for Theorem 3.11.

Example 3.13. Consider the lattice $(L=\{0,t_1,t_2,t_3,t_4,t_5,1\},\leq,0,1)$ as given in Fig. 1 and the implication $J:[t_2,1]^2\to[t_2,1]$ as given in Table 1 and the implication K be as in Table 4.

K	0	t_1	t_2	t_3	t_4	t_5	1
0	1	1	1	1	1	1	1
t_1	0	1	t_3	1	t_5	1	1
t_2	0	t_5	1	1	1	1	1
t_3	0	t_1	t_2	1	t_5	1	1
t_4	0	t_5	t_2	t_5	1	1	1
t_5	0	t_1	t_2	t_3	t_4	1	1
1	0	t_1	t_2	t_3	t_4	t_5	1

Table 4. The implication K on L.

By applying the formula (3.13) in Theorem 3.11 with $a=t_2$ and $b=t_3$, the implication I_4 can be obtained as in Table 5.

I_4	0	t_1	t_2	t_3	t_4	t_5	1
0	1	1	1	1	1	1	1
t_1	0	t_3	1	1	1	1	1
t_2	0	t_2	1	1	1	1	1
t_3	0	0	t_4	t_5	t_4	t_5	1
t_4	0	t_2	t_3	t_3	t_5	t_5	1
t_5	0	0	t_2	t_3	t_4	t_5	1
1	0	0	t_2	t_3	t_4	t_5	1

Table 5. The implication I_4 on L.

Theorem 3.14. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b \in L$ with $a \leq b$, $S : L^2 \to L$ be a t-conorm, $K : L^2 \to L$ be an implication and $J : [0,b]^2 \longrightarrow [0,b]$ be an implication. Then, the function $I_4^* : L^2 \longrightarrow L$ defined by,*

$$I_4^*(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{if } (x,y) = (1,0) \text{ or } (x \notin [0,b] \text{ and } y \in [0,b]), \\ J(x,y) & \text{if } (x,y) \in [0,b]^2, \\ S(K(x,y),b) & \text{if } x \in [0,b] \text{ and } y \notin [0,b], \\ S(K(x,y),a) & \text{if } x \notin [0,b] \text{ and } y \notin [0,b], \end{cases} \tag{3.18}$$

is an implication on L.

Proof. The proof can be done in a similar fashion as the proof of Theorem 3.11. Therefore, we omit it. □

Remark 3.15. (i) If we take the restriction of the implication operations in Theorems 3.1, 3.5, 3.9 and 3.11 on $[a,1]$, it is obtained that $I_1 = I_3 = I_5 = I_7 = J$.
(ii) If we take the restriction of the implication operations in Theorems 3.4, 3.8, 3.10 and 3.14 on $[0,b]$, it is obtained that $I_2 = I_4 = I_6 = I_8 = J$.

4. Conclusion

In this study, construction methods for implications on bounded lattices have been investigated by means of a implication operator which is defined on the subinterval $[a,1]$ ($[0,b]$) of the bounded lattice L having $a, b \in L$ with $a \leq b$. We also have benefited from some fuzzy logic operators in some of the methods. In addition we, the construction methods are clarified with the examples and corollaries.

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