BEYKENT UNIVERSITY JOURNAL OF SCIENCE AND ENGINEERING
Volume 5(1-2), 2012, 109-129

ON THE STABILITY OF SECOND ORDER "NEUTRAL DELAY DIFFERANTIAL EQUATION"

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ABSTRACT

On the stability of second order neutral delay differential equation; In this paper, a basic theorem on the behavior of solutions of linear second order neutral delay differential equation is established. As a consequence of this theorem, a stability criterion is obtained.

Keywords: *"Neutral delay" differential equation, Characteristic equation, Stability, Trivial solution.*

ÖZET

İkinci mertebeden neutral delay differansiyel denklemlerin kararlılığı: Bu makalede sabit katsayılı doğrusal ikinci mertebeden neutral delay diferansiyel denklemlerin davranışları üzerinde temel bir teorem verilmiştir. Bu teoremin sonuçlarından yararlanarak kararlılık kriterleri elde edilmiştir.

Anahtar Kelimeler: *"Neutral delay" differansiyel denklemi, karakteristik denklem, kararlılık, aşikar çözüm*

1. INTRODUCTION

Let us consider initial value problem for second order delay differential equation

$$y''(t)+cy''(t-\tau)=p_1y'(t)+p_2y'(t-\tau)+q_1y(t)+q_2y(t-\tau),\ t\geq 0, \qquad (1)$$

$$y(t)=\phi(t)\ ,\ -\tau\leq t\leq 0, \qquad (2)$$

where c, p_1, p_2, q_1, q_2 are real numbers, τ is positive real number and $\phi(t)$ is a given continuously differentiable initial function on the interval $[-\tau,0]$.

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [1,4,5]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The theory of linear delay differential equations has been developed in the fundamental monographs [1], [4-6], [8].

The equation of form of (1) is of interest in biology in explaining self-balancing of the human body and in robotics in consructing biped robots (see [10], [12]). These are illustrations of inverted pendulum problems. A typical example is the balancing of a stick (see [13]). Equation of form of (1) can be used as test of equations for numerical methods (see [7], [14]).

In [3], it has been established the boundedness under the conditions $b>0$, $k>0$, $p>-k$ and $|q|+r|p|<b$, on $[0,\infty)$ of the solution of the second order equation

$$y''(t)+by'(t)+qy'(t-r)+ky(t)+py(t-r)=0,\ \text{for}\ t\geq 0,$$

where $r>0$, together with a given continuously differentiable initial function

$$y(t)=\phi(t)\ \text{on}\ [-r,0].$$

This equation Lyapunov function with the help of the stability criteria were obtained.

Recently, Cahlon and Schmidt et al. [2] have established the stability criteria for a second order delay differential equation of form (1) with $p_1 p_2 \geq 0$ and $q_1 q_2 < 0$. This equation is obtained the stability of second order delay differential equation using Pontryagin's theory for quasi-polynomials. However, we study the stability of the some problem using the method of characteristic roots.

This paper deals with the stability of the trivial solution for a second order linear neutral delay differential equation with constant delay. An estimate of the solutions is established. The sufficient conditions for the stability, the asymptotic stability and instability of the trivial solution are given. Our results are derived by the use of real roots (with an appropriate property) of the corresponding (in a sense) characteristic equations. The techniques applied in obtaining our results are originated in a combination of the methods used in [9] and [11].

As usual, a twice continuously differentiable real-valued function y defined on the interval $[-\tau, \infty)$ is said to be a solution of the initial value problem (1) and (2) if y satisfies (1) for all $t \geq 0$ and (2) for all $-\tau \leq t \leq 0$.

It is known that (see, for example, [4]), for any given initial function ϕ, there exists a unique solution of the initial problem (1)-(2) or, more briefly, the solution of (1)-(2).

Before closing this section, we will give two well-known definitions (see, for example, [5]). The trivial solution of (1) is said to be "stable" (at 0) if for every $\varepsilon > 0$, there exists a number $\ell = \ell(\varepsilon) > 0$ such that, for any initial fuction ϕ with

$$\|\phi\| \equiv \max_{-\tau \leq t \leq 0} |\phi(t)| < \ell ,$$

the solution y of (1)-(2) satisfies

$$| y(t) | < \varepsilon \ , \quad \text{for all } t \in [-\tau, \infty) .$$

Otherwise, the trivial solution of (1) is said to be "unstable" (at 0). Moreover, the trivial solution of (1) is called "asymptotically stable" (at 0) if it is stable in the above sense and in addition there exists a number $\ell_0 > 0$ such that, for any initial function ϕ with $\|\phi\| < \ell_0$, the solution y of (1)-(2) satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

2. STATEMENT OF THE MAİN RESULTS AND COMMENTS

If we look for a solution of (1) of the form $y(t) = e^{\lambda t}$ for $t \in IR$, we see that λ is a root of the first characteristic equation

$$\lambda^2(1+c) = p_1\lambda + p_2\lambda e^{-\lambda\tau} + q_1 + q_2 e^{-\lambda\tau}. \tag{3}$$

Let now y be the solution of (1)-(2). Define

$$x(t) = e^{-\lambda_0 t} y(t), \quad \text{for } t \in [-\tau, \infty),$$

where λ_0 is a real root of the characteristic equation (3). Then, for every $t \geq 0$, we have

$$x''(t) + 2\lambda_0 x'(t) + \lambda_0^2 x(t) + ce^{-\lambda_0\tau}\left(x''(t-\tau) + 2\lambda_0 x'(t-\tau) + \lambda_0^2 x(t-\tau)\right)$$
$$= p_1 x'(t) + p_1\lambda_0 x(t) + p_2 e^{-\lambda_0\tau} x'(t-\tau) + p_2\lambda_0 e^{-\lambda_0\tau} x(t-\tau) + q_1 x(t) + q_2 e^{-\lambda_0\tau} x(t-\tau)$$

or

$$\left[x'(t) + ce^{-\lambda_0\tau} x'(t-\tau) + (2\lambda_0 - p_1)x(t) + e^{-\lambda_0\tau}(2c\lambda_0 - p_2)x(t-\tau)\right]'$$
$$= (p_1\lambda_0 + q_1 - \lambda_0^2)x(t) + (p_2\lambda_0 + q_2 - c\lambda_0^2)e^{-\lambda_0\tau} x(t-\tau). \tag{4}$$

Moreover, the initial condition (2) can be equivalently written

$$x(t) = e^{-\lambda_0 t}\phi(t), \quad \text{for } t \in [-\tau, 0] \tag{5}$$

Furthermore, by using the fact that λ_0 is a root of (3) and taking into account (5), we can verify that (4) is equivalent to

$$x'(t) + ce^{-\lambda_0\tau} x'(t-\tau) + (2\lambda_0 - p_1)x(t) + e^{-\lambda_0\tau}(2c\lambda_0 - p_2)x(t-\tau)$$

$$= (p_1\lambda_0 + q_1 - \lambda_0^2)\int_0^t x(s)ds + (p_2\lambda_0 + q_2 - c\lambda_0^2)e^{-\lambda_0\tau}\int_0^t x(s-\tau)ds$$

$$+x'(0)+ce^{-\lambda_0\tau}x'(-\tau)+(2\lambda_0-p_1)x(0)+e^{-\lambda_0\tau}(2c\lambda_0-p_2)x(-\tau),$$

$$x'(t)+ce^{-\lambda_0\tau}x'(t-\tau)=(p_1-2\lambda_0)x(t)+e^{-\lambda_0\tau}(p_2-2c\lambda_0)x(t-\tau)$$

$$+(p_1\lambda_0+q_1-\lambda_0^2)\int_0^t x(s)ds+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_{-\tau}^{t-\tau}x(s)ds$$

$$+\phi'(0)-\lambda_0\phi(0)+c(\phi'(-\tau)-\lambda_0\phi(-\tau))+(2\lambda_0-p_1)\phi(0)+(2c\lambda_0-p_2)\phi(-\tau),$$

$$x'(t)+ce^{-\lambda_0\tau}x'(t-\tau)=(p_1-2\lambda_0)x(t)+e^{-\lambda_0\tau}(p_2-2c\lambda_0)x(t-\tau)$$

$$+(p_1\lambda_0+q_1-\lambda_0^2)\int_0^t x(s)ds+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_0^{t-\tau}x(s)ds+L(\lambda_0;\phi),$$

$$x'(t)+ce^{-\lambda_0\tau}x'(t-\tau)=(p_1-2\lambda_0)x(t)+e^{-\lambda_0\tau}(p_2-2c\lambda_0)x(t-\tau)$$

$$-(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_0^t x(s)ds+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_0^{t-\tau}x(s)ds+L(\lambda_0;\phi),$$

$$x'(t)+ce^{-\lambda_0\tau}x'(t-\tau)=(p_1-2\lambda_0)x(t)+e^{-\lambda_0\tau}(p_2-2c\lambda_0)x(t-\tau)$$

$$+(c\lambda_0^2-p_2\lambda_0-q_2)e^{-\lambda_0\tau}\int_{t-\tau}^t x(s)ds+L(\lambda_0;\phi), \tag{6}$$

where

$$L(\lambda_0;\phi)=\phi'(0)-\lambda_0\phi(0)+c(\phi'(-\tau)-\lambda_0\phi(-\tau))+(2\lambda_0-p_1)\phi(0)+(2c\lambda_0-p_2)\phi(-\tau)$$

$$+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_{-\tau}^0 e^{-\lambda_0 s}\phi(s)ds. \tag{7}$$

Let

$$\beta_{\lambda_0}\equiv(p_2\lambda_0+q_2-c\lambda_0^2)\tau e^{-\lambda_0\tau}+2\lambda_0-p_1+(2c\lambda_0-p_2)e^{-\lambda_0\tau}\neq 0$$

and we define

$$z(t) = x(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \text{ , for } \quad t \geq -\tau \, .$$

Then we can see that (6) reduces to the following equivalent equation

$$z'(t) + ce^{-\lambda_0\tau} z'(t-\tau) = (p_1 - 2\lambda_0) z(t) + e^{-\lambda_0\tau}(p_2 - 2c\lambda_0) z(t-\tau)$$

$$+ (c\lambda_0^2 - p_2\lambda_0 - q_2) e^{-\lambda_0\tau} \int_{t-\tau}^{t} z(s)ds \, . \qquad (8)$$

If we look for a solution of (8) of the form $z(t) = e^{\delta t}$ for $t \in IR$, we see that δ is a root of the second characteristic equation

$$\delta\left(1 + ce^{-\tau(\lambda_0+\delta)}\right) = p_1 - 2\lambda_0 + (p_2 - 2c\lambda_0) e^{-\tau(\lambda_0+\delta)}$$

$$+ \delta^{-1}(1 - e^{-\delta\tau})(c\lambda_0^2 - p_2\lambda_0 - q_2) e^{-\lambda_0\tau} \text{.”} \qquad (9)$$

On the other hand, the initial condition (5) can be equivalently written

$$z(t) = \phi(t) e^{-\lambda_0 t} - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \, , \quad t \in [-\tau, 0] \, . \qquad (10)$$

Let $F(\delta)$ denote the characteristic function of (9), i.e.,

$$F(\delta) = \delta\left(1 + ce^{-\tau(\lambda_0+\delta)}\right) - p_1 + 2\lambda_0 + (2c\lambda_0 - p_2) e^{-\tau(\lambda_0+\delta)}$$

$$+ \delta^{-1}(e^{-\delta\tau} - 1)(c\lambda_0^2 - p_2\lambda_0 - q_2) e^{-\lambda_0\tau}$$

Since $\delta = 0$ is a removable singularity of $F(\delta)$, we can regard $F(\delta)$ as a entire function with

$$F(0) = 2\lambda_0 - p_1 + (2c\lambda_0 - p_2) e^{-\lambda_0\tau} - (c\lambda_0^2 - p_2\lambda_0 - q_2)\tau e^{-\lambda_0\tau} \equiv \beta_{\lambda_0}$$

But, by the definition of $\beta_{\lambda_0} \neq 0$, a root of the characteristic equation (9) must become $\delta_0 \neq 0$.

Let now z be the solution of (8)-(10) and δ_0 be a real root of the characteristic equation (9). Define for $\delta_0 \neq 0$

$$v(t) = e^{-\delta_0 t} z(t) \ , \quad \text{for all } t \in [-\tau, \infty).$$

Then, for every $t \geq 0$, we have

$$v'(t) + ce^{-\tau(\lambda_0+\delta_0)} v'(t-\tau) = (p_1 - 2\lambda_0 - \delta_0) v(t) + (p_2 - 2c\lambda_0 - c\delta_0) e^{-(\lambda_0+\delta_0)\tau} v(t-\tau)$$

$$+ (c\lambda_0^2 - p_2\lambda_0 - q_2) e^{-\lambda_0 \tau} \int_{t-\tau}^{t} e^{-\delta_0 s} v(t-s) ds . \qquad (11)$$

Moreover, the initial condition (10) can be equivalently written

$$v(t) = \phi(t) e^{-(\lambda_0+\delta_0)t} - e^{-\delta_0 t} \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}} \ , \quad \text{for } t \in [-\tau, 0]. \qquad (12)$$

Furthermore, by using the fact that $\delta_0 \neq 0$ is a real root of (9) and taking into account (12), we can verify that (11) is equivalent to

$$v(t) + ce^{-\tau(\lambda_0+\delta_0)} v(t-\tau) =$$

$$(p_1 - 2\lambda_0 - \delta_0) \int_0^t v(s) ds + (p_2 - 2c\lambda_0 - c\delta_0) e^{-(\lambda_0+\delta_0)\tau} \int_0^t v(s-\tau) ds$$

$$+ (c\lambda_0^2 - p_2\lambda_0 - q_2) e^{-\lambda_0 \tau}$$

$$\int_0^\tau e^{-\delta_0 s} \left\{ \int_0^t v(u-s) du \right\} ds + v(0) + ce^{-\tau(\lambda_0+\delta_0)} v(-\tau)$$

$$v(t) + ce^{-\tau(\lambda_0+\delta_0)} v(t-\tau) =$$

$$(p_1 - 2\lambda_0 - \delta_0) \int_0^t v(s) ds + (p_2 - 2c\lambda_0 - c\delta_0) e^{-(\lambda_0+\delta_0)\tau} \int_{-\tau}^{t-\tau} v(s) ds$$

$$+(c\lambda_0^2 - p_2\lambda_0 - q_2)e^{-\lambda_0\tau}$$

$$\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{t-s} v(u)du\right\}ds + v(0) + ce^{-\tau(\lambda_0+\delta_0)}v(-\tau),$$

$$v(t) + ce^{-\tau(\lambda_0+\delta_0)}v(t-\tau) = (p_1 - 2\lambda_0 - \delta_0)\int_0^t v(s)ds$$

$$+(p_2 - 2c\lambda_0 - c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\left\{\int_{-\tau}^{0} v(s)ds + \int_0^{t-\tau} v(s)ds\right\}$$

$$+(c\lambda_0^2 - p_2\lambda_0 - q_2)e^{-\lambda_0\tau}$$

$$\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} v(u)du + \int_0^{t-s} v(u)du\right\}ds + v(0) + ce^{-\tau(\lambda_0+\delta_0)}v(-\tau)$$

$$v(t) + ce^{-\tau(\lambda_0+\delta_0)}v(t-\tau) = (p_1 - 2\lambda_0 - \delta_0)$$

$$\int_0^t v(s)ds + (p_2 - 2c\lambda_0 - c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\int_0^{t-\tau} v(s)ds$$

$$+(c\lambda_0^2 - p_2\lambda_0 - q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{t-s} v(u)du\right\}ds + R(\lambda_0,\delta_0;\phi),$$

$$v(t) + ce^{-\tau(\lambda_0+\delta_0)}v(t-\tau) = (c\delta_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\delta_0)\tau}\int_0^t v(s)ds$$

$$+(p_2\lambda_0 + q_2 - c\lambda_0^2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}ds\int_0^t v(s)ds + (p_2 - 2c\lambda_0 - c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\int_0^{t-\tau} v(s)ds$$

$$+(c\lambda_0^2 - p_2\lambda_0 - q_2)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_0^{t-s} v(u)du\right\}ds + R(\lambda_0,\delta_0;\phi),$$

$$v(t)+ce^{-\tau(\lambda_0+\delta_0)}v(t-\tau)=(c\delta_0+2c\lambda_0-p_2)e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t}v(s)ds$$

$$+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{t-s}^{t}v(u)du\right\}ds+R(\lambda_0,\delta_0;\phi), \qquad (13)$$

where

$$R(\lambda_0,\delta_0;\phi)=\phi(0)+c\phi(-\tau)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\left(1+ce^{-\lambda_0\tau}\right)$$

$$+(p_2-2c\lambda_0-c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0}e^{-\delta_0 s}\left(e^{-\lambda_0 s}\phi(s)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)ds$$

$$+(c\lambda_0^2-p_2\lambda_0-q_2)e^{-\lambda_0\tau}\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{-s}^{0}e^{-\delta_0 u}\left(e^{-\lambda_0 u}\phi(u)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}\right)du\right\}ds \quad (14)$$

Next, we define

$$w(t)=v(t)-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}, \text{ for } t\geq-\tau, \qquad (15)$$

where

$$\eta_{\lambda_0,\delta_0}\equiv 1+e^{-(\lambda_0+\delta_0)\tau}+(p_2-2c\lambda_0-c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\tau\,.$$

$$+\delta_0^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)\left(c\lambda_0^2-p_2\lambda_0-q_2\right)e^{-\lambda_0\tau}. \qquad (16)$$

Then we can see that (13) reduces to the following equivalent equation

$$w(t)+ce^{-\tau(\lambda_0+\delta_0)}w(t-\tau)=(c\delta_0+2c\lambda_0-p_2)e^{-(\lambda_0+\delta_0)\tau}\int_{t-\tau}^{t}w(s)ds$$

$$+(p_2\lambda_0+q_2-c\lambda_0^2)e^{-\lambda_0\tau}\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{t-s}^{t}w(u)du\right\}ds,\ \ t\geq 0. \qquad (17)$$

On the other hand, the initial condition (12) can be equivalently written

$$w(t)=\phi(t)e^{-(\lambda_0+\delta_0)t}-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}, \text{ for } t\in[-\tau,0]. \quad (18)$$

We have the following basic theorem.

Theorem 1. Let λ_0 and δ_0 $(\delta_0 \neq 0)$ be real roots of the characteristic equations (3) and (9). Assume that the roots λ_0 and δ_0 have the following property

$$\mu_{\lambda_0,\delta_0} \equiv \left(|c|+|c\delta_0+2c\lambda_0-p_2|\tau\right)e^{-(\lambda_0+\delta_0)\tau}$$

$$+\delta_0^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)|p_2\lambda_0+q_2-c\lambda_0^2|e^{-\lambda_0\tau}<1 \quad (19)$$

and

$$\beta_{\lambda_0}\equiv(p_2\lambda_0+q_2-c\lambda_0^2)\tau e^{-\lambda_0\tau}+2\lambda_0-p_1+(2c\lambda_0-p_2)e^{-\lambda_0\tau}\neq 0.$$

Then, for any $\phi\in C^1([-\tau,0],IR)$, the solution y of (1)-(2) satisfies

$$\left|e^{-(\lambda_0+\delta_0)t}y(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right|\leq M(\lambda_0,\delta_0;\phi)\mu_{\lambda_0,\delta_0},$$

for all $t\geq 0$, (20) where $L(\lambda_0;\phi)$, $R(\lambda_0,\delta_0;\phi)$ and $\eta_{\lambda_0,\delta_0}$ were given in (7), (14) and (16), respectively and

$$M(\lambda_0,\delta_0;\phi)=\max_{-\tau\leq t\leq 0}\left|e^{-(\lambda_0+\delta_0)t}\phi(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}-\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}\right|. \quad (21)$$

Proof. It easy to see that property (19) guarantees that $\eta_{\lambda_0,\delta_0}>0$. Applying the definitions of x, z, v and w we can obtain that (20) is equivalent to

$$|w(t)|\leq M(\lambda_0,\delta_0;\phi)\mu_{\lambda_0,\delta_0}, \quad \forall t\geq 0. \quad (22)$$

So, we will prove (22).

From (18) and (21) it follows that

$$|w(t)| \le M(\lambda_0, \delta_0; \phi) \ , \quad \text{for.} \ t \in [-\tau, 0] \tag{23}$$

We will show that $M(\lambda_0, \delta_0; \phi)$ is a bound of w on the whole interval $[-\tau, \infty)$. Namely

$$|w(t)| \le M(\lambda_0, \delta_0; \phi) \ , \quad \text{for all} \ t \in [-\tau, \infty). \tag{24}$$

To this end, let us consider an arbitrary number $\varepsilon > 0$. We claim that

$$|w(t)| < M(\lambda_0, \delta_0; \phi) + \varepsilon \ , \quad \text{for every} \ t \in [-\tau, \infty). \tag{25}$$

Otherwise, by (23), there exists a $t^* > 0$ such that $|w(t)| < M(\lambda_0, \delta_0; \phi) + \varepsilon$, for $t < t^*$ and $|w(t^*)| = M(\lambda_0, \delta_0; \phi) + \varepsilon$. Then using (17), we obtain

$$M(\lambda_0, \delta_0; \phi) + \varepsilon = |w(t^*)| \le |c| e^{-\tau(\lambda_0+\delta_0)} |w(t^* - \tau)|$$

$$+ |c\delta_0 + 2c\lambda_0 - p_2| e^{-(\lambda_0+\delta_0)\tau} \int_{t^*-\tau}^{t^*} |w(s)| ds$$

$$+ |p_2\lambda_0 + q_2 - c\lambda_0^2| e^{-\lambda_0\tau} \int_0^{\tau} e^{-\delta_0 s} \left\{ \int_{t^*-s}^{t^*} |w(u)| du \right\} ds$$

$$\le \Big\{ \big(|c| + |c\delta_0 + 2c\lambda_0 - p_2| \tau \big) e^{-(\lambda_0+\delta_0)\tau}$$

$$+ \delta_0^{-2} \left(1 - e^{-\delta_0\tau} - \delta_0 \tau e^{-\delta_0\tau} \right) |p_2\lambda_0 + q_2 - c\lambda_0^2| e^{-\lambda_0\tau} \Big\} [M(\lambda_0, \delta_0; \phi) + \varepsilon]$$

$$< [M(\lambda_0, \delta_0; \phi) + \varepsilon],$$

which, in view of (19), leads to a contradiction. So, our claim is true. Since (25) holds for every $\varepsilon > 0$, it follows that (24) is always satisfied. By using (24) and (17), we derive

$$|w(t)| \le |c| e^{-\tau(\lambda_0+\delta_0)} w(t-\tau) + |c\delta_0 + 2c\lambda_0 - p_2| e^{-(\lambda_0+\delta_0)\tau} \int_{t-\tau}^{t} |w(s)| ds$$

$$+|p_2\lambda_0+q_2-c\lambda_0^2|e^{-\lambda_0\tau}\int_0^{\tau}e^{-\delta_0 s}\left\{\int_{t-s}^{t}|w(u)|du\right\}ds$$

$$\le\left\{\left(|c|+|c\delta_0+2c\lambda_0-p_2|\tau\right)e^{-(\lambda_0+\delta_0)\tau}\right.$$

$$\left.+\delta_0^{-2}\left(1-e^{-\delta_0\tau}-\delta_0\tau e^{-\delta_0\tau}\right)|p_2\lambda_0+q_2-c\lambda_0^2|e^{-\lambda_0\tau}\right\}M(\lambda_0,\delta_0;\phi)$$

$$=M(\lambda_0,\delta_0;\phi)\mu_{\lambda_0,\delta_0},$$

for all $t\ge 0$. That means (22) holds.

Theorem 2. Let λ_0 and δ_0 $(\delta_0\ne 0)$ be real roots of the characteristic equations (3) and (9). Consider β_{λ_0} as in Theorem 1. Then, for any $\phi\in C^1([-\tau,0],IR)$, the solution y of (1)-(2) satisfies

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t)-\frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\}=\frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}},$$

where $L(\lambda_0;\phi)$, $R(\lambda_0,\delta_0;\phi)$ and $\eta_{\lambda_0,\delta_0}$ were given in (7), (14) and (16), respectively.

Proof. By the definitions of x, z, v and w, we have to prove that

$$\lim_{t\to\infty}w(t)=0. \tag{26}$$

In the end of the proof we will establish (26). By using (17) and taking into account (22) and (24), one can show, by an easy induction, that w satisfies

$$|w(t)|\le\left(\mu_{\lambda_0,\delta_0}\right)^n M(\lambda_0,\delta_0;\phi),\ \text{for all}\ \ t\ge n\tau-\tau\ ,\ (n=0,1,...). \tag{27}$$

But, (19) guarantees that $0<\mu_{\lambda_0,\delta_0}<1$. Thus, from (27) it follows immediately that w tends to zero as $t\to\infty$, i.e. (26) holds.

The proof of the Theorem 2 is completed.

Theorem 3. Let λ_0 and δ_0 $(\delta_0\ne 0)$ be real roots of the characteristic equations (3) and (9) and also the conditions in

Theorem 1 β_{λ_0} and μ_{λ_0,δ_0} be provided. Then, for any $\phi \in C^1([-\tau,0], IR)$, the solution y of (1)-(2) satisfies for all $t \geq 0$

$$|y(t)| \leq \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} N(\lambda_0,\delta_0;\phi)e^{\lambda_0 t}$$

$$+\left[\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} + \left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right)\mu_{\lambda_0,\delta_0}\right] N(\lambda_0,\delta_0;\phi)e^{(\lambda_0+\delta_0)t}, \quad (28)$$

where

$\eta_{\lambda_0,\delta_0}$ was given in (16),

$$k_{\lambda_0} = 1+|\lambda_0|+|c|(1+|\lambda_0|)+|2\lambda_0 - p_1| + |2c\lambda_0 - p_2|e^{-\lambda_0\tau},$$

$$+|p_2\lambda_0 + q_2 - c\lambda_0^2|e^{-\lambda_0\tau}\tau \quad (29)$$

$$h_{\lambda_0,\delta_0} = 1+|c| + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\left(1+|c|e^{-\lambda_0\tau}\right)$$

$$+\delta_0^{-1}(e^{-\delta_0\tau}-1)|p_2 - 2c\lambda_0 - c\delta_0|e^{-(\lambda_0+\delta_0)\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)$$

$$+\delta_0^{-2}\left(\delta_0\tau + e^{-\delta_0\tau} - 1\right)|c\lambda_0^2 - p_2\lambda_0 - q_2|e^{-\lambda_0\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right), \quad (30)$$

$$e_{\delta_0} = \max_{-\tau\leq t\leq 0}\left\{e^{-\delta_0 t}\right\} \quad (31)$$

and

$$N(\lambda_0,\delta_0;\phi) = \max\left\{\max_{-\tau\leq t\leq 0}\left|e^{-\lambda_0 t}\phi(t)\right|, \max_{-\tau\leq t\leq 0}\left|e^{-(\lambda_0+\delta_0)t}\phi(t)\right|, \max_{-\tau\leq t\leq 0}\left|\phi'(t)\right|, \max_{-\tau\leq t\leq 0}\left|\phi(t)\right|\right\}$$

(32)

Moreover, the trivial solution of (1) is stable if $\lambda_0 \le 0$, $\lambda_0 + \delta_0 \le 0$, it is asymptotically stable if $\lambda_0 < 0$, $\lambda_0 + \delta_0 < 0$ and it is unstable if $\delta_0 > 0$, $\lambda_0 + \delta_0 > 0$.

Proof. By Theorem 1, (20) is satisfied, where $L(\lambda_0;\phi)$ and $M(\lambda_0,\delta_0;\phi)$ are defined by (7) and (21), respectively. From (20) it follows that

$$e^{-(\lambda_0+\delta_0)t}\left|y(t)\right| \le \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}e^{-\delta_0 t} + \frac{|R(\lambda_0,\delta_0;\phi)|}{\eta_{\lambda_0,\delta_0}} + M(\lambda_0,\delta_0;\phi)\mu_{\lambda_0,\delta_0}. \quad (33)$$

Furthermore, by using (29), (30), (31) and (32), from (7), (14) and (21), we obtain

$$|L(\lambda_0;\phi)| \le |\phi'(0)| + |\lambda_0||\phi(0)|$$
$$+|c|\left(|\phi'(-\tau)| + |\lambda_0||\phi(-\tau)|\right) + |2\lambda_0 - p_1||\phi(0)|$$

$$+|2c\lambda_0 - p_2||\phi(-\tau)| + |p_2\lambda_0 + q_2 - c\lambda_0^2|e^{-\lambda_0\tau}\int_{-\tau}^{0} e^{-\lambda_0 s}|\phi(s)|ds$$

$$\le \Big(1 + |\lambda_0| + |c|(1+|\lambda_0|) + |2\lambda_0 - p_1| + |2c\lambda_0 - p_2|e^{-\lambda_0\tau}$$

$$+|p_2\lambda_0 + q_2 - c\lambda_0^2|e^{-\lambda_0\tau}\tau\Big)N(\lambda_0,\delta_0;\phi)$$

$$= k_{\lambda_0}N(\lambda_0,\delta_0;\phi),$$

$$|R(\lambda_0,\delta_0;\phi)| \le |\phi(0)| + |c||\phi(-\tau)| + \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\left(1 + |c|e^{-\lambda_0\tau}\right)$$

$$+|p_2 - 2c\lambda_0 - c\delta_0|e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{-\delta_0 s}\left(e^{-\lambda_0 s}|\phi(s)| + \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\right)ds$$

$$+|c\lambda_0^2 - p_2\lambda_0 - q_2|e^{-\lambda_0\tau}\int_{0}^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{-\delta_0 u}\left(e^{-\lambda_0 u}|\phi(u)| + \frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\right)du\right\}ds$$

$$\leq \left[1+|c|+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\left(1+|c|e^{-\lambda_0\tau}\right)+\delta_0^{-1}(e^{-\delta_0\tau}-1)|p_2-2c\lambda_0-c\delta_0|e^{-(\lambda_0+\delta_0)\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)\right.$$

$$\left. +\delta_0^{-2}\left(\delta_0\tau+e^{-\delta_0\tau}-1\right)|c\lambda_0^2-p_2\lambda_0-q_2|e^{-\lambda_0\tau}\left(1+\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}\right)\right]N(\lambda_0,\delta_0;\phi)$$

$$=h_{\lambda_0,\delta_0}N(\lambda_0,\delta_0;\phi),$$

$$M(\lambda_0,\delta_0;\phi)\leq \max_{-\tau\leq t\leq 0}\left\{e^{-(\lambda_0+\delta_0)t}|\phi(t)|\right\}+\frac{|L(\lambda_0;\phi)|}{|\beta_{\lambda_0}|}\max_{-\tau\leq t\leq 0}\left\{e^{-\delta_0 t}\right\}+\frac{|R(\lambda_0,\delta_0;\phi)|}{\eta_{\lambda_0,\delta_0}}$$

$$\leq\left\{1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right\}N(\lambda_0,\delta_0;\phi)$$

Hence, from (33), we conclude that for all $t\geq 0$,

$$e^{-(\lambda_0+\delta_0)t}|y(t)|\leq\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}N(\lambda_0,\delta_0;\phi)e^{-\delta_0 t}+\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}N(\lambda_0,\delta_0;\phi)$$

$$+\left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}+\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right)N(\lambda_0,\delta_0;\phi)\mu_{\lambda_0,\delta_0},\qquad(34)$$

and consequently, (28) holds.

Now, let us assume that $\lambda_0\leq 0$ and $\lambda_0+\delta_0\leq 0$. Define $\|\phi\|\equiv\max_{-\tau\leq t\leq 0}|\phi(t)|$. It follows that $\|\phi\|\leq N(\lambda_0,\delta_0;\phi)$. From (34), it follows that

$$|y(t)|\leq\left\{\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|}+\left(1+\frac{k_{\lambda_0}e_{\delta_0}}{|\beta_{\lambda_0}|}\right)\mu_{\lambda_0,\delta_0}+\left(1+\mu_{\lambda_0,\delta_0}\right)\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right\}N(\lambda_0,\delta_0;\phi),$$

for every $t \geq 0$. Since $\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} > 1$, by taking into account the

fact that $\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left(1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|}\right)\mu_{\lambda_0,\delta_0} + \left(1 + \mu_{\lambda_0,\delta_0}\right)\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}} > 1$, we have

$$|y(t)| \leq \left\{\frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left(1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|}\right)\mu_{\lambda_0,\delta_0} + \left(1 + \mu_{\lambda_0,\delta_0}\right)\frac{h_{\lambda_0,\delta_0}}{\eta_{\lambda_0,\delta_0}}\right\} N(\lambda_0, \delta_0; \phi),$$

for all $t \in [-\tau, \infty)$, which means that the trivial solution of (1) is stable (at 0).

Next, if $\lambda_0 < 0$ and $\lambda_0 + \delta_0 < 0$, then (28) guarantees that $\lim_{t \to \infty} y(t) = 0$ and so the trivial solution of (1) is asymptotically stable (at 0).

Finally, if $\delta_0 > 0$, $\lambda_0 + \delta_0 > 0$, then the trivial solution of (1) is unstable (at 0). Otherwise, there exists a number $\ell \equiv \ell(1) > 0$ such that, for any $\phi \in C^1([-\tau, 0], IR)$ with $\|\phi\| < \ell$, the solution y of problem (1)-(2) satisfies

$$|y(t)| < 1 \quad \text{for all} \quad t \geq -\tau. \tag{35}$$

Define

$$\phi_0(t) = e^{(\lambda_0 + \delta_0)t} - e^{\lambda_0 t} \quad \text{for} \quad t \in [-\tau, 0].$$

Furthermore, by the definition of $L(\lambda_0; \phi)$ and $R(\lambda_0, \delta_0; \phi)$, by using (9), we have

$$L(\lambda_0; \phi_0) = \delta_0 + c\delta_0 e^{-(\lambda_0 + \delta_0)\tau} + (2c\lambda_0 - p_2)\left(e^{-(\lambda_0 + \delta_0)\tau} - e^{-\lambda_0 \tau}\right)$$

$$+ \left(p_2\lambda_0 + q_2 - c\lambda_0^2\right)e^{-\lambda_0 \tau}\left(\int_{-\tau}^{0} e^{\delta_0 s} ds - \tau\right)$$

$$= p_1 - 2\lambda_0 - (2c\lambda_0 - p_2)e^{-\lambda_0 \tau} - \left(p_2\lambda_0 + q_2 - c\lambda_0^2\right)e^{-\lambda_0 \tau}\tau$$

$$\equiv -\beta_{\lambda_0},$$

$$R(\lambda_0,\delta_0;\phi_0) = 1 + ce^{-(\lambda_0+\delta_0)\tau} + (p_2 - 2c\lambda_0 - c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\int_{-\tau}^{0} e^{-\delta_0 s}\left(e^{-\lambda_0 s}\left(e^{(\lambda_0+\delta_0)s} - e^{\lambda_0 s}\right)+1\right)ds$$

$$+\left(c\lambda_0^2 - p_2\lambda_0 - q_2\right)e^{-\lambda_0\tau}\int_0^{\tau} e^{-\delta_0 s}\left\{\int_{-s}^{0} e^{-\delta_0 u}\left(e^{-\lambda_0 u}\left(e^{(\lambda_0+\delta_0)u} - e^{\lambda_0 u}\right)+1\right)du\right\}ds$$

$$= 1 + e^{-(\lambda_0+\delta_0)\tau} + (p_2 - 2c\lambda_0 - c\delta_0)e^{-(\lambda_0+\delta_0)\tau}\tau$$

$$+\delta_0^{-2}\left(1 - e^{-\delta_0\tau} - \delta_0\tau e^{-\delta_0\tau}\right)\left(c\lambda_0^2 - p_2\lambda_0 - q_2\right)e^{-\lambda_0\tau}$$

$$\equiv \eta_{\lambda_0,\delta_0} > 0.$$

Let $\phi \in C^1([-\tau,0], IR)$ be defined by

$$\phi = \frac{\ell_1}{\|\phi_0\|}\phi_0,$$

where ℓ_1 is a number with $0 < \ell_1 < \ell$. Moreover, let y be the solution of (1)-(2). From Theorem 2 it follows that y satisfies

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\} = \lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t) + \frac{\ell_1}{\|\phi_0\|}e^{-\delta_0 t}\right\}$$

$$= \frac{R(\lambda_0,\delta_0;\phi)}{\eta_{\lambda_0,\delta_0}}$$

$$= \frac{\left(\ell_1/\|\phi_0\|\right)R(\lambda_0,\delta_0;\phi_0)}{\eta_{\lambda_0,\delta_0}} = \frac{\ell_1}{\|\phi_0\|} > 0.$$

But, we have $\|\phi\| = \ell_1 < \ell$ and hence from (35) and conditions $\delta_0 > 0$, $\lambda_0 + \delta_0 > 0$ it follows that

$$\lim_{t\to\infty}\left\{e^{-(\lambda_0+\delta_0)t}y(t) - \frac{L(\lambda_0;\phi)}{\beta_{\lambda_0}}e^{-\delta_0 t}\right\} = 0.$$

This is a contradiction. The proof of Theorem 3 is completed.

3. EXAMPLES

Example 1. Consider

$$y''(t)-\frac{1}{6}y''\left(t-\frac{1}{2}\right)=-3y'(t)+\frac{1}{2}y'\left(t-\frac{1}{2}\right)+\frac{\sqrt{e}-13}{6}y(t)+\frac{1}{3}y\left(t-\frac{1}{2}\right)$$
$$,t\geq 0, \qquad (36)$$

$$y(t)=\phi(t), \quad -\frac{1}{2}\leq t\leq 0,$$

where $\phi(t)$ is an arbitrary continuously differentiable initial function on the interval $[-\frac{1}{2},0]$. In this example we apply the characteristic equations (3) and (9). That is, the characteristic equation (3) is

$$\lambda^2\left(1-\frac{1}{6}\right)=-3\lambda+\frac{1}{2}\lambda e^{-\frac{\lambda}{2}}+\frac{\sqrt{e}-13}{6}+\frac{1}{3}e^{-\frac{\lambda}{2}}, \qquad (37)$$

and we see that $\lambda=-1$ is a root of (37). Then, for $\lambda_0=-1$ the characteristic equation (9) is

$$\delta\left(1-\frac{1}{6}e^{-\frac{1}{2}(\delta-1)}\right)=-1+\frac{1}{6}e^{-\frac{1}{2}(\delta-1)}.$$

Therefore, $\delta=\delta_0=-1$ is a root, and the conditions of Theorem 3 are satisfied. That is,

$$\mu_{\lambda_0,\delta_0}=\mu_{-1,-1}=\frac{e}{6}<1 \quad \text{and} \quad \beta_{\lambda_0}=\beta_{-1}=1-\frac{\sqrt{e}}{6}\neq 0.$$

Since $\lambda_0=-1<0$ and $\lambda_0+\delta_0=-2<0$, the zero solution of (36) is asymptotically stable.

Example 2. Consider

$$y''(t)+\frac{1}{2e}y''(t-1)=-\frac{1}{e}y'(t)-\frac{1}{2e}y'(t-1)-\frac{1}{e}y(t)+\frac{1}{e}y(t-1),$$

$$t\geq 0, \qquad (38)$$

$$y(t)=\phi(t),-1\leq t\leq 0,$$

where $\phi(t)$ is an arbitrary continuously differentiable initial function on $[-1,0]$. The characteristic equation (3) is

$$\lambda^2\left(1+\frac{1}{2e}\right)=-\frac{1}{e}\lambda-\frac{1}{2e}\lambda e^{-\lambda}-\frac{1}{e}+\frac{1}{e}e^{-\lambda}, \qquad (39)$$

and we see easily that $\lambda=0$ is a root of (39). Taking $\lambda_0=0$, the characteristic equation (9) is

$$\delta\left(1+\frac{1}{2e}e^{-\delta}\right)=-\frac{1}{e}-\frac{1}{2e}e^{-\delta}-\frac{1}{e}\int_0^1 e^{-\delta s}ds \qquad .$$

Therefore, we find that $\delta=\delta_0=-1$ is a root. Corresponding to the roots $\lambda_0=0$ and $\delta_0=-1$, the conditions of Theorem 3 are satisfied. That is,

$$\mu_{\lambda_0,\delta_0}=\mu_{0,-1}=\frac{1}{2}+\frac{1}{e}<1 \qquad \text{and} \qquad \beta_{\lambda_0}=\beta_0=\frac{5}{2e}\neq 0.$$

Since $\lambda_0=0$ and $\lambda_0+\delta_0<0$, the zero solution of (38) is stable.

Example 3. Consider

$$y''(t)+y''\left(t-\frac{\pi}{2}\right)=3y'(t)+3y'(t-\frac{\pi}{2})-\left(1+e^{-\frac{\pi}{2}}\right)y(t)-2y(t-\frac{\pi}{2}),$$

$$t\geq 0, \qquad (40)$$

$$y(t)=\phi(t), \qquad -\frac{\pi}{2}\le t\le 0,$$

where $\phi(t)$ is an arbitrary continuously differentiable initial function on $[-\frac{\pi}{2},0]$. The characteristic equation (3) is

$$2\lambda^2=3\lambda+3\lambda e^{-\frac{\lambda\pi}{2}}-\left(1+e^{-\frac{\pi}{2}}\right)-2e^{-\frac{\lambda\pi}{2}}, \tag{41}$$

and we see easily that $\lambda=1$ is a root of (41). Taking $\lambda_0=1$, the characteristic equation (9) is

$$\delta\left(1+e^{-\frac{\pi}{2}(\delta+1)}\right)=1+e^{-\frac{\pi}{2}(\delta+1)}.$$

Therefore, we find that $\delta=\delta_0=1$ is a root. Corresponding to the roots $\lambda_0=1$ and $\delta_0=1$, the conditions of Theorem 3 are satisfied. Since $\delta_0>0$ and $\lambda_0+\delta_0>0$, the zero solution of (40) is unstable.

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