DOI: 10.38061/idunas.1590039

IDUNAS	NATURAL & APPLIED SCIENCES JOURNAL	2024 Vol. 7 No. 2 (1-7)

Fourier Analysis of Inverse Coefficient Nonlinear Hyperbolic Equations under Periodic Boundary Conditions

Research Article

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DOI: 10.38061/idunas.1590039

Received: 22.11.2024; Accepted: 09.12.2024

Abstract

This study presents an analytical investigation of a one-dimensional inverse coefficient nonlinear hyperbolic equation with periodic boundary conditions. The analytical solution is derived from applying the Fourier method. An iterative approach is used to establish convergence and to assess the existence, uniqueness, and stability of the solution to the nonlinear problem.

Keywords: The hyperbolic problem, Fourier iterative method, Nonlocal conditions.

1. INTRODUCTION

The study of inverse coefficient problems in hyperbolic equations has gained significant attention recently due to its wide-ranging applications in physics, engineering, and medical imaging. These problems are centered around the challenge of identifying unknown coefficients or parameters from observed data, a task that can be quite difficult given their inherently ill-posed nature. In this context, Tekin [1] highlights the conditions necessary for the unique solvability of an inverse problem related to second-order hyperbolic equations, stressing the critical role that additional constraints can play in ensuring a well-defined solution.

The introduction of periodic boundary conditions adds another layer of complexity to these inverse problems, as they impose specific constraints on the solutions over a defined interval. Such periodic boundary conditions are vital in various physical scenarios, such as the vibrations of strings or the propagation of waves in bounded media, where system behavior exhibits periodicity over time [2]. Research into the uniqueness and stability of solutions under periodic conditions has shown that, despite the ill-posedness of these problems, unique solutions can still be obtained under certain circumstances [3-5].

One particularly effective approach to tackling these challenges is the Fourier method, which has been successfully employed in numerous studies to address inverse coefficient problems [6]. This method has demonstrated its capability to establish the existence, uniqueness, convergence, and stability of solutions for a variety of equations, including the Euler-Bernoulli, heat, Burger, and Klein-Gordon equations with periodic boundary conditions [7-9].

In this study, we consider an inverse coefficient nonlinear hyperbolic equation represented as

$$\nu_{tt} - \nu_{xx} = \theta(t) f(y,t,\nu), \quad (y,t) \in \Omega \tag{1}$$

with the initial condition

$$\nu(y,0) = \varphi(y)$$
$$\nu_t(y,t) = \psi(y), \tag{2}$$

the periodic boundary condition

$$\nu(0,t) = \nu(\pi,t)$$
$$\nu_y(0,t) = \nu_y(\pi,t) \tag{3}$$

and overdetermination data

$$E(t) = \int_0^\pi y\nu(y,t)dy \tag{4}$$

for a nonlinear source term represented by $f(y,t,\nu)$.

Here $\Omega := \{0 < y < \pi, 0 < t < T\}$, $\bar{\Omega} := \{0 \le y \le \pi, 0 \le t \le T\}$.
The functions φ, ψ, E and $f(y,t,\nu)$ are given functions.
Here $\phi(x) \in [0,\pi]$,$\psi(x) \in [0,\pi]$ and $f(y,t,\nu), \bar{\Omega} \times (-\infty,\infty)$, for $(y,t) \in \bar{\Omega}, \nu(y,t) \in (-\infty,\infty)$.

By using the Fourier method, the solution function, determined by $\{\theta,\nu\}$, can be found. The existence, uniqueness, and stability of the solutions to the inverse problems are proven using an iterative approach.

2. ANALYTICAL SOLUTION OF THE PROBLEM

Definition 1. The problem of finding the values of $\{\theta,\nu\}$ that satisfy (1)-(4) is known as the inverse problem.

Definition 2. If the set $\{\nu(t)\} = \{\nu_0(t), \nu_{sk}(t), \nu_{ck}(t), k = \overline{1,N}\}$ of continuous functions on [0, T] satisfies the norm condition $\|\nu(t)\| = \max_{0\le t\le T}|\nu_0(t)| + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|\nu_{ck}(t)| + \max_{0\le t\le T}|\nu_{sk}(t)|\right)$, then space B is called a Banach space.

Let us assume the following conditions on the data for the problem (1)-(4)
(C1) $E(t) \in C^2[0,T], \theta(t) \in C[0,T]$.
(C2) $\varphi(y) \in C^1[0,\pi], \psi(y) \in C[0,\pi]$.
(C3) Let $f(y,t,\nu)$ be a function that is continuous in all arguments $\Omega \times (-\infty,\infty)$ and satisfies the following conditions:

1) $\left|\frac{\partial^{(k)} f(y,t,\nu)}{\partial y^{(k)}} - \frac{\partial^{(k)} f(y,t,\tilde{\nu})}{\partial y^{(k)}}\right| \le b(y,t)|\nu - \tilde{\nu}|, k = \overline{0,2}$, where $b(y,t) \in L_2(\Omega), b(y,t) \ge 0$.
2) $f(y,t,\nu) \in C[0,\pi], t \in [0,T], |f(y,t,\nu)| \le M,$
3) $\int_0^\pi f(y,t,\nu)dy \ne 0, \forall t \in [0,T]$.

By applying the Fourier method, the solution to (1)-(3) is obtained as follows:

$$\nu(y,t)=\frac{1}{2}\left(\varphi_0+\psi_0 t+\int_0^t (t-\tau)\theta(\tau)f_0(\tau)d\tau\right)$$
$$+\sum_{k=1}^{\infty}\left(\varphi_{ck}\cos 2kt+\frac{\psi_{ck}}{2k}\sin 2kt\right)\cos 2ky+\sum_{k=1}^{\infty}\left(\frac{1}{2k}\int_0^t \theta(\tau)f_{ck}(\tau)\sin 2k(t-\tau)d\tau\right)\cos 2ky$$
$$+\sum_{k=1}^{\infty}\left(\varphi_{sk}\cos 2kt+\frac{\psi_{sk}}{2k}\sin 2kt\right)\sin 2ky+\sum_{k=1}^{\infty}\left(\frac{1}{2k}\int_0^t \theta(\tau)f_{sk}(\tau)\sin 2k(t-\tau)d\tau\right)\sin 2ky. \tag{5}$$

Differentiating the overdetermination condition under conditions (C1)–(C3), we obtain:

$$E''(t)=\int_0^\pi y\nu_{tt}dy. \tag{6}$$

Equations (5), (6) yield

$$\theta(t)=\frac{E''(t)}{\int_0^\pi yf(y,t,\nu)dy}-\frac{\pi\sum_{k=1}^{\infty}(2k)\left(\varphi_{sk}\cos 2kt+\frac{\psi_{sk}}{2k}\sin 2kt\right)}{\int_0^\pi yf(y,t,\nu)dy}+\frac{\pi\sum_{k=1}^{\infty}\left(\int_0^t \theta(\tau)f_{sk}(\tau)\sin 2k(t-\tau)d\tau\right)}{\int_0^\pi yf(y,t,\nu)dy}. \tag{7}$$

3. THE EXISTENCE AND UNIQUENESS OF SOLUTIONS

Theorem 1. If conditions (C1)–(C3) are satisfied, then the problem (1)–(4) has a unique solution.

Proof. Let us provide an iteration for (5) and the inverse coefficient as follows:

$$\nu_0^{(N+1)}(t)=\varphi_0+\psi_0 t+\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\theta^{(N)}(\tau)f(\xi,\tau,\nu^{(N)})d\xi d\tau,$$
$$\nu_{ck}^{(N+1)}(t)=\varphi_{ck}\cos 2kt+\frac{\psi_{ck}}{2k}\sin 2kt+\frac{1}{\pi k}\int_0^t\int_0^\pi \theta^{(N)}(\tau)f(\xi,\tau,\nu^{(N)})\cos 2k\xi\sin 2k(t-\tau)d\xi d\tau,$$
$$\nu_{sk}^{(N+1)}(t)=\varphi_{sk}\cos 2kt+\frac{\psi_{sk}}{2k}\sin 2kt+\frac{1}{\pi k}\int_0^t\int_0^\pi \theta^{(N)}(\tau)f(\xi,\tau,\nu^{(N)})\sin 2k\xi\sin 2k(t-\tau)d\xi d\tau, \tag{8}$$

$$\theta^{(N+1)}(t)=\frac{E''(t)}{\int_0^\pi yf(y,t,\nu^{(N)})dy}-\frac{\pi\sum_{k=1}^{\infty}(2k)\left(\varphi_{sk}\cos 2kt+\frac{\psi_{sk}}{2k}\sin 2kt\right)}{\int_0^\pi yf(y,t,\nu^{(N)})dy}+\frac{\pi\sum_{k=1}^{\infty}\left(\int_0^t \theta^{(N)}(\tau)f_{sk}(\tau)\sin 2k(t-\tau)d\tau\right)}{\int_0^\pi yf(y,t,\nu^{(N)})dy}. \tag{9}$$

$\nu^{(0)}(t)\in B, t\in[0,T]$ is from the conditions of the theorem.

For $N=0$, by adding and subtracting $\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau$ in (8) and applying Cauchy inequality, we obtain

$$\left|\nu_0^{(1)}(t)\right|=\left|\varphi_0+\psi_0 t\right|+\left(\int_0^t (t-\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\left|\frac{2}{\pi}\int_0^\pi \theta^{(0)}(\tau)\left[f(\xi,\tau,\nu^{(0)})-f(\xi,\tau,0)\right]d\xi\right|\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+\left(\int_0^t (t-\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{2}{\pi}\left|\int_0^\pi \theta^{(0)}(\tau)f(\xi,\tau,0)d\xi\right|\right\}^2 d\tau\right)^2,$$

$$\left|\nu_{ck}^{(1)}(t)\right|\le\left|\varphi_{ck}\right|+\left|\frac{\psi_{ck}}{2k}\right|+\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{1}{\pi k}\int_0^\pi\left|\theta^{(0)}(\tau)\left[f(\xi,\tau,\nu^{(0)})-f(\xi,\tau,0)\right]\cos 2k\xi\sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\frac{1}{\pi k}\int_0^\pi\left|\theta^{(0)}(\tau)f(\xi,\tau,0)\cos 2k\xi\sin 2k(t-\tau)\right|d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|\nu_{sk}^{(1)}(t)\right| \le \left|\varphi_{sk}\right| + \left|\frac{\psi_{sk}}{2k}\right| + \left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau)\left[f(\xi,\tau,\nu^{(0)}) - f(\xi,\tau,0)\right]\sin 2k\xi \sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}$$

$$+\left(\int_0^t d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\sin 2k\xi \sin 2k(t-\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

After using Lipschitz condition, we have

$$\left|\nu_0^{(1)}(t)\right| \le \left|\varphi_0 + \psi_0 t\right| + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\right| d\xi\right\}^2 d\tau\right),$$

$$\left|\nu_{ck}^{(1)}(t)\right| \le \left|\varphi_{ck}\right| + \left|\frac{\psi_{ck}}{2k}\right| + \sqrt{t}\left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}(\xi,\tau)\cos 2k\xi \sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t}\left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\cos 2k\xi \sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}},$$

$$\left|\nu_{sk}^{(1)}(t)\right| \le \left|\varphi_{sk}\right| + \left|\frac{\psi_{sk}}{2k}\right| + \sqrt{t}\left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}(\xi,\tau)\sin 2k\xi \sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}} +$$

$$\sqrt{t}\left(\int_0^t \left\{\frac{1}{\pi k}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\sin 2k\xi \sin 2k(t-\tau)\right|\right\}^2 d\xi d\tau\right)^{\frac{1}{2}}.$$

By applying Hölder inequality,

$$\left|\nu_0^{(1)}(t)\right| \le \left|\varphi_0 + \psi_0 t\right| + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\right| d\xi\right\}^2 d\tau\right),$$

$$\left|\nu_{ck}^{(1)}(t)\right| \le \sum_{k=1}^\infty \left|\varphi_{ck}\right| + \frac{1}{2}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \left|\psi_{ck}\right|^2\right)^{\frac{1}{2}} + \sqrt{t}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \int_0^t \left\{\frac{1}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\cos 2k\xi \sin 2k(t-\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{t}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \int_0^t \left\{\frac{1}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\cos 2k\xi \sin 2k(t-\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|\nu_{sk}^{(1)}(t)\right| \le \sum_{k=1}^\infty \left|\varphi_{sk}\right| + \frac{1}{2}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \left|\psi_{sk}\right|^2\right)^{\frac{1}{2}} + \sqrt{t}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \int_0^t \left\{\frac{1}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\sin 2k\xi \sin 2k(t-\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{t}\left(\sum_{k=1}^\infty \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty \int_0^t \left\{\frac{1}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\sin 2k\xi \sin 2k(t-\tau)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

and applying Bessel's inequality, we have

$$\left|\nu_0^{(1)}(t)\right| \le \left|\varphi_0 + \psi_0 t\right| + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{\frac{t^3}{3}} \left(\int_0^t \left\{\frac{2}{\pi}\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\right| d\xi\right\}^2 d\tau\right),$$

$$\left|\nu_{ck}^{(1)}(t)\right| \le \sum_{k=1}^\infty \left|\varphi_{ck}\right| + \frac{1}{2}\sqrt{\frac{\pi^2}{6}} \left(\sum_{k=1}^\infty \left|\psi_{ck}\right|^2\right)^{\frac{1}{2}} + \sqrt{\frac{t}{6}}\left(\int_0^t \left\{\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{\frac{t}{6}}\left(\int_0^t \left\{\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}},$$

$$\left|\nu_{sk}^{(1)}(t)\right| \le \sum_{k=1}^\infty \left|\varphi_{sk}\right| + \frac{1}{2}\sqrt{\frac{\pi^2}{6}} \left(\sum_{k=1}^\infty \left|\psi_{sk}\right|^2\right)^{\frac{1}{2}} + \sqrt{\frac{t}{6}}\left(\int_0^t \left\{\int_0^\pi \left|\theta^{(0)}(\tau) b(\xi,\tau)\nu^{(0)}\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} + \sqrt{\frac{t}{6}}\left(\int_0^t \left\{\int_0^\pi \left|\theta^{(0)}(\tau) f(\xi,\tau,0)\right| d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

Finally, we get

$$\left\|\nu^{(1)}(t)\right\| = \max_{0\le t\le T}\frac{\left|\nu_0^{(1)}(t)\right|}{2} + \sum_{k=1}^{\infty}\left[\max_{0\le t\le T}\left|\nu_{ck}^{(1)}(t)\right| + \max_{0\le t\le T}\left|\nu_{sk}^{(1)}(t)\right|\right] \le$$
$$\frac{\|\varphi_0\|+\|\psi_0\||T|}{2} + \sum_{k=1}^{\infty}\|\varphi_{ck}\|+\|\varphi_{sk}\| + \frac{\pi}{2\sqrt{6}}\sum_{k=1}^{\infty}\|\psi_{ck}\|+\|\psi_{sk}\| + A\left\|\theta^{(0)}(t)\right\|\|b(y,t)\|\left\|\nu^{(0)}(t)\right\| + A\left\|\theta^{(0)}(t)\right\|M,$$

where $A = \left(\frac{\sqrt{3}T\sqrt{T}(T+\sqrt{2}\pi)}{3}\right)$.

Hence from the conditions of the theorem $\nu^{(1)}(t) \in B, t \in [0,T]$.

If considered $2k\int_0^{\pi}\varphi(y)\,sin\,2\,kydy = \int_0^{\pi}\varphi'(y)\,cos\,2\,kydy$ for (7), we have

$$\theta^{(1)}(t) = \frac{E''(t) - \pi\sum_{k=1}^{\infty}\varphi'_{ck}\cos 2kt - \pi\sum_{k=1}^{\infty}\psi_{sk}\sin 2kt}{\int_0^{\pi} yf(y,t,\nu^{(0)})dy} - \frac{2\sum_{k=1}^{\infty}\int_0^t\int_0^{\pi}\theta^{(0)}(\tau)f(\tau)\sin 2k\xi\sin 2k(t-\tau)d\xi d\tau}{\int_0^{\pi} yf(y,t,\nu^{(0)})dy}.$$

applying the Cauchy and Bessel inequalities, along with the Lipschitz condition, we obtain:

$$\left\|\theta^{(1)}(t)\right\| \le \frac{2}{\pi^2 M_0}\left(E''(t) + \pi\|\varphi'_{ck}\| + \pi\|\psi_{sk}\|\right) + \frac{4T\sqrt{T}}{\pi M_0}\left(\left\|\theta^{(0)}(t)\right\|\|b(y,t)\|\left\|\nu^{(0)}(t)\right\| + \left\|\theta^{(0)}(t)\right\|M\right).$$

The same estimations apply for step N, we obtain:

$$\left\|\nu^{(N+1)}(t)\right\| = \max_{0\le t\le T}\frac{\left|\nu_0^{(N)}(t)\right|}{2} + \sum_{k=1}^{\infty}\left[\max_{0\le t\le T}\left|\nu_{ck}^{(N)}(t)\right| + \max_{0\le t\le T}\left|\nu_{sk}^{(N)}(t)\right|\right]$$
$$\le \frac{\|\varphi_0\|+\|\psi_0\||T|}{2} + \sum_{k=1}^{\infty}\|\varphi_{ck}\|+\|\varphi_{sk}\| + \frac{\pi}{2\sqrt{6}}\sum_{k=1}^{\infty}\|\psi_{ck}\|+\|\psi_{sk}\| + A\left\|\theta^{(N)}(t)\right\|\|b(y,t)\|\left\|\nu^{(N)}(t)\right\| + A\left\|\theta^{(N)}(t)\right\|M,$$
$$\left\|\theta^{(N+1)}(t)\right\| \le \frac{2}{\pi^2 M_0}\left(E''(t) + \pi\|\varphi'_{ck}\| + \pi\|\psi_{sk}\|\right) + \frac{4T\sqrt{T}}{\pi M_0}\left(\left\|\theta^{(N)}(t)\right\|\|b(y,t)\|\left\|\nu^{(N)}(t)\right\| + \left\|\theta^{(N)}(t)\right\|M\right).$$

We obtain $\nu^{(N+1)}(t) \in B, t \in [0,T]$ because of $\nu^{(N)}(t) \in B, t \in [0,T]$.

By applying same methods for convergence, we get:

$$\left\|\nu^{(N+1)}(t) - \nu^{(N)}(t)\right\| \le (A+SM)^N \times \left\|\theta^{(N)}(t)\right\|\ldots\left\|\theta^{(1)}(t)\right\|C\frac{1}{\sqrt{N!}}\left(\int_0^t\int_0^{\pi} b^2(\xi,\tau)d\xi d\tau\right)^{\frac{N}{2}}.$$

Here $S = \frac{2\sqrt{T}}{M_* - 2\sqrt{T}M}$ and $C = \left\|\nu^{(0)} - \nu^{(1)}\right\|$.

As $N \to \infty, \nu^{(N+1)} \to \nu^{(N)}$ and $\theta^{(N+1)} \to \theta^{(N)}$.

Let us show that
$\lim_{N\to\infty}\nu^{(N+1)}(t) = \nu(t), \lim_{N\to\infty}\theta^{(N+1)}(t) = \theta(t)$.

Let's take the difference between the exact and approximate inverse coefficients:

$$\theta(t) - \theta^{(N+1)}(t) = \frac{\pi\sum_{k=1}^{\infty}\int_0^t\int_0^{\pi}\theta^{(N+1)}(\tau)f(\xi,\tau,\nu^{(N+1)})\sin 2k\xi\sin 2k(t-\tau)d\xi d\tau}{\int_0^{\pi} yf(y,t,\nu^{(N+1)})dy} - \frac{\pi\sum_{k=1}^{\infty}\int_0^t\int_0^{\pi}\theta(\tau)f(\xi,\tau,\nu)\sin 2k\xi\sin 2k(t-\tau)d\xi d\tau}{\int_0^{\pi} yf(y,t,\nu)dy}.$$

Then add and subtract $\int_0^t\int_0^{\pi}\theta(\tau)f(\xi,\tau,\nu^{(N+1)})d\xi d\tau$, and applying consecutively Cauchy, Bessel inequalities and Lipschitz condition, we have

$$\left\|\theta(t)-\theta^{(N+1)}(t)\right\| \le S\|\theta(t)\|\|b(y,t)\|\left\|\nu(t)-\nu^{(N+1)}(t)\right\|.$$

By applying the same methods for $\nu(t)-\nu^{(N+1)}(t)$, we find

$$\left\|\nu(t)-\nu^{(N+1)}(t)\right\| \le A\left(A+SM\right)^{N}\left\|\theta^{(N)}(t)\right\|\left\|\theta^{(N)}(t)\right\|\ldots\left\|\theta^{(1)}(t)\right\|C\times\frac{1}{\sqrt{N!}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^{\frac{N}{2}}\exp\left(2A+SM\right)\|\theta(t)\|\|b(y,t)\|.$$

As $N\to\infty$, $\nu^{(N+1)}(t)\to\nu(t)$ and $\theta^{(N+1)}(t)\to\theta(t)$.

Let consider we have two solutions (ν,θ) and (ω,ρ) of (1)-(4). We obtain the following, applying the same methods:

$$\|\theta(t)-\rho(t)\| \le S\|\theta(t)\|\|b(y,t)\|\|\nu(t)-\omega(t)\|. \tag{10}$$

$$\|\nu(t)-\omega(t)\| \le A\|\theta(t)\|\|b(y,t)\|\|\nu(t)-\omega(t)\| + A\|\theta(t)-\rho(t)\|M. \tag{11}$$

By using (10) in (11), we get $\|\nu(t)-\omega(t)\| \le (A+SM)\|\theta(t)\|\|b(y,t)\|\|\nu(t)-\omega(t)\|$.

Finally, applying Gronwall inequality to the last inequality, we have:

$$\|\nu(t)-\omega(t)\| \le 0\times exp(A+SM)\left(\int_0^t\int_0^\pi \theta^2(\tau)b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}.$$

Then $\nu(t)=\omega(t)$, therefore $\theta(t)=\rho(t)$.

The proof is completed.

4. STABILITY OF THE SOLUTION (ν,θ)

Theorem 2. If the assumptions (C1)–(C3) hold, then the solution pair (ν,θ) of the problem (1)-(4) depends continuously on the input data ϕ, ψ and E.

Proof. Let sets $\|\Phi\|=\{\varphi,\psi,E\}$ and $\|\bar{\Phi}\|\le\{\bar{\varphi},\bar{\psi},\bar{E}\}$ denote two data groups that conform to the assumptions (C1)–(C3). Assume there are positive constants M_i that ensure the following inequalities are satisfied:

$$\|\varphi\|\le M_1, \|\psi\|\le M_2, \|E\|\le M_3.$$

Let us denote

$$\|\Phi\|\le\|\varphi\|+\|\psi\|+\|E\|,\quad \|\bar{\Phi}\|\le\|\bar{\varphi}\|+\|\bar{\psi}\|+\|\bar{E}\|.$$

Let (ν,θ) and $(\bar{\nu},\bar{\theta})$ be the solutions of the problem (1)-(4) according to the data Φ and $\bar{\Phi}$. By applying

$$\left\|\theta(t)-\bar{\theta}(t)\right\| \le D\left(\left(E''(t)-\bar{E}''(t)\right)+\pi\sum_{k=1}^{\infty}\left\|\varphi'_{ck}-\bar{\varphi}'_{ck}\right\|\right)B\left(\sum_{k=1}^{\infty}\left\|\psi_{sk}-\bar{\psi}_{sk}\right\|+2\sqrt{T}\|\theta(t)\|\|b(y,t)\|\|\nu-\bar{\nu}\|\right),$$

$$\|\nu-\bar{\nu}\|^2 \le 2\left\|\Phi-\bar{\Phi}\right\|^2\times\exp 2L^2\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right).$$

Here $D=\frac{2}{M_0\pi^2-4M_0M\sqrt{T}}$..

For $\Phi\to\bar{\Phi}$ then $\nu\to\bar{\nu}$. Hence $\theta\to\bar{\theta}$.

5. CONCLUSION

An analytical investigation of a one-dimensional inverse coefficient nonlinear hyperbolic equation with periodic boundary conditions is conducted in this study. The analytical solution is obtained using the generalized Fourier method. Furthermore, an iterative approach is implemented to demonstrate convergence and to examine the existence, uniqueness, and stability of the solution for the nonlinear problem.

6. ACKNOWLEDGMENTS

We would like to thank the reviewer.

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