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Description of Maximally Dissipative Quasi-Differential Operators for First Order

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ABSTRACT

In this work, the general form of maximally dissipative extensions of the minimal operator generated by first order linear symmetric quasi-differential expression in the weighted Hilbert space of vector-functions at right semi-infinite interval has been found. Later on, the geometry of spectrum of these extensions is investigated.

Keywords: deficiency indices, space of boundary values, dissipative operator, spectrum

1. INTRODUCTION

It is known that a linear closed densely defined operator $T: D(T) \subset H \to H$ in Hilbert space H is said to be dissipative if and only if

$$Im(Tx, x) \geq 0$$

for all $x \in D(T)$, i.e. in other words its numerical range is contained in the upper complex plane. Moreover, it is called maximally dissipative if it has no non-trivial dissipative extension [1]. Maximally dissipative operators play a very important role in mathematics and physics. Dissipative operators have many interesting applications in physics like hydrodynamic, laser and nuclear scattering theories.

Note that the study of abstract extension problems for operators on Hilbert spaces goes at least back to J.von Neumann [2] such that in [2] a full characterization of all selfadjoint extensions of a given closed symmetric operator with equal defect indices was investigated.

The further investigations of M. I. Vishik and M. S. Birman devoted to characterization of all non-negative selfadjoint extensions of a positive closed symmetric operator (see [3]). And more general informations can be found in [3]. Class of dissipative operators is an important class of nonselfadjoint operators in the operator theory. Functional model theory of B. Sz.-Nagy and C. Foias [4] is a basic method for investigation the spectral properties of dissipative operators. Note that spectrum set of the dissipative operator lies in closed upper half-pane.

The maximal dissipative extensions and their spectral analysis of the minimal operator having equal deficiency indices generated by formally symmetric differential-operator expression in the Hilbert space of vector-functions defined in one finite or infinite interval case have been researched by V. I. Gorbachuk, M. I. Gorbachuk [1] and F. S.

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Rofe-Beketov, A. M. Kholkin [5] in terms of generalized boundary values.

In this work using the Calkin-Gorbachuk method representation of all maximally dissipative extensions of the minimal operator generated by first order linear symmetric quasi-differential expression with operator coefficient in the weighted Hilbert spaces of vector-functions defined in the right semi-infinite interval case is given (section 3). In section 4 the structure of spectrum of these type extensions is investigated.

2. STATEMENT OF THE PROBLEM

Let H be a separable Hilbert space and $a \in \mathbb{R}$. In addition assumed that $\alpha, w : (a,\infty) \to (0,\infty), \alpha, w \in C(a,\infty)$ and $\frac{w}{\alpha^2} \in L^1(a,\infty)$. In the weighted Hilbert space $L_w^2(H,(a,\infty))$ of vector-functions consider the following linear quasi-differential expression with operator coefficient for first order in form

$$l(u) = i\frac{\alpha(t)}{w(t)}(\alpha u)'(t) + Au(t),$$

where A is a linear bounded selfadjoint operator in H. And also the operator E will be indicated identify operator in corresponding spaces.

By a standard way the minimal L_0 corresponding to quasi-differential expression $l(.)$ in $L_w^2(H,(a,\infty))$ can be defined (see [6]). The operator $L = (L_0)^*$ is called maximal operator corresponding to $l(.)$ in $L_w^2(H,(a,\infty))$.

In this case it will be shown that the minimal operator is symmetric and it has non zero equal deficiency indices in $L_w^2(H,(a,\infty))$ (see section 3).

In this work, firstly the representation of all maximally dissipative extensions of the minimal operator L_0 in $L_w^2(H,(a,\infty))$ is investigated. Later on, structure of the spectrum of these extensions will be investigated.

3. DESCRIPTION OF MAXIMALLY DISSIPATIVE EXTENSIONS

In this section using the Calkin-Gorbachuk method will be investigated the general representation of all maximally dissipative extensions of the minimal operator L_0 in $L_w^2(H,(a,\infty))$.

Before of all prove the following proposition.

Lemma 3.1 If $\widetilde{L}$ is any maximally dissipative extension of the minimal operator L_0 in $L_w^2(H,(a,\infty))$, then for every function $u \in D(\widetilde{L})$ exist the boundary values $(\alpha u)(a)$ and $(\alpha u)(\infty)$ in H.

Proof. In this case for any $u \in D(\widetilde{L})$ we have

$$\widetilde{L}\,u(t) = i\frac{\alpha(t)}{w(t)}(\alpha u)'(t) + Au(t) \in L_w^2(H,(a,\infty)).$$

From these relations it is implies that $\frac{\alpha}{w}(\alpha u)' \in L_w^2(H,(a,\infty))$.

Now before of all it will be shown that the integrals

$$\int_t^c (\alpha u)'(x)dx,\ \ a \le t < c$$

and

$$\int_c^t (\alpha u)'(x)dx,\ \ a < c < t \le \infty$$

exist. Indeed, the following expressions are implemented, respectively, for any $a \le t < c$

$$\int_t^c |(\alpha u)'|(x)dx = \int_t^c \left|\frac{\alpha}{w}(\alpha u)'\sqrt{w}\,\frac{\sqrt{w}}{\alpha}\right|(x)dx$$

$$\le \left(\int_t^c \left|\frac{\alpha}{w}(\alpha u)'\right|^2(x)w(x)dx\right)^{1/2}\left(\int_t^c \frac{w(x)}{\alpha^2(x)}dx\right)^{1/2}$$

$$\le \left(\int_a^\infty \left|\frac{\alpha}{w}(\alpha u)'\right|^2(x)w(x)dx\right)^{1/2}\left(\int_a^\infty \frac{w(x)}{\alpha^2(x)}dx\right)^{1/2} < \infty$$

and for $a < c < t \le \infty$

$$\int_c^t |(\alpha u)'|(x)dx = \int_c^t \left|\frac{\alpha}{w}(\alpha u)'\sqrt{w}\,\frac{\sqrt{w}}{\alpha}\right|(x)dx$$

$$\leq \left(\int_c^t \left| \frac{\alpha}{w} (\alpha u)' \right|^2 (x) w(x) dx \right)^{1/2} \left(\int_c^t \frac{w(x)}{\alpha^2(x)} dx \right)^{1/2}$$

$$\leq \left(\int_a^\infty \left| \frac{\alpha}{w} (\alpha u)' \right|^2 (x) w(x) dx \right)^{1/2} \left(\int_a^\infty \frac{w(x)}{\alpha^2(x)} dx \right)^{1/2}$$

$< \infty$.

Then for $a \leq t < c$ from the following equality

$$(\alpha u)(c) - (\alpha u)(t) = \int_t^c (\alpha u)'(x) dx$$

and the relation

$$\left| \int_t^c (\alpha u)'(x) dx - \int_a^c (\alpha u)'(x) dx \right|$$

$$= \left| \int_a^t (\alpha u)'(x) dx \right|$$

$$\leq \int_a^t |(\alpha u)'|(x) dx$$

$$\leq \left(\int_a^t \left| \frac{\alpha}{w} (\alpha u)' \right|^2 (x) w(x) dx \right)^{1/2} \left(\int_a^t \frac{w(x)}{\alpha^2(x)} dx \right)^{1/2}$$

$\to 0, t \to a + 0$

it is obtained that

$$\lim_{t \to a+0} \int_t^c (\alpha u)'(x) dx$$

exists. Then

$$\lim_{t \to a+0} (\alpha u)(t) = (\alpha u)(c) + \lim_{t \to a+0} \int_t^c (\alpha u)'(x) dx.$$

In the similar way for $a < c < t \leq \infty$ from equality

$$(\alpha u)(t) - (\alpha u)(c) = \int_c^t (\alpha u)'(x) dx$$

and

$$\left| \int_c^t (\alpha u)'(x) dx - \int_c^\infty (\alpha u)'(x) dx \right|$$

$$\leq \int_t^\infty |(\alpha u)'|(x) dx$$

$$\leq \left(\int_t^\infty \left| \frac{\alpha}{w} (\alpha u)' \right|^2 (x) w(x) dx \right)^{1/2} \left(\int_t^\infty \frac{w(x)}{\alpha^2(x)} dx \right)^{1/2}$$

$\to 0, t \to \infty$

we have existence of the following limit

$$\lim_{t \to \infty} \int_c^t (\alpha u)'(x) dx.$$

Consequently for any $c \in (a, \infty)$,

$$\lim_{t \to \infty} (\alpha u)(t) = (\alpha u)(c) + \lim_{t \to \infty} \int_a^t (\alpha u)'(x) dx .$$

Hence the proof of theorem is completed.

Lemma 3.2 The deficiency indices of the minimal operator L_0 in $L_w^2(H, (a, \infty))$ are in form

$$(n_+(L_0), n_-(L_0)) = (\dim H, \dim H).$$

Proof. For the simplicity of calculations it will be taken $A = 0$. It is clear that the general solutions of differential equations

$$i \frac{\alpha(t)}{w(t)} (\alpha u_\pm)'(t) \pm i u_\pm(t) = 0, t > a$$

in $L_w^2(H, (a, \infty))$ are in form

$$u_\pm(t) = \frac{1}{\alpha(t)} exp \left(\mp \int_a^t \frac{w(s)}{\alpha^2(s)} ds \right) f,$$

$$f \in H, t > a.$$

From these representations we have

$$\|u_+\|^2_{L^2_w(H,(a,\infty))} = \int_a^\infty w(t)\,\|u_+(t)\|^2_H dt$$

$$= \int_a^\infty w(t) \left\| \frac{1}{\alpha(t)} exp\left(-\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) f \right\|^2_H dt$$

$$= \int_a^\infty \frac{w(t)}{\alpha^2(t)} exp\left(-2\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) dt \|f\|^2_H$$

$$= \int_a^\infty exp\left(-2\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) d\left(\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) \|f\|^2_H$$

$$= \frac{1}{2}\left(1 - exp\left(-2\int_a^\infty \frac{w(s)}{\alpha^2(s)} ds\right)\right) \|f\|^2_H < \infty.$$

Hence

$$n_+(L_0) = \dim\ker(L + iE) = \dim H.$$

On the other hand it is clear that

$$\|u_-\|^2_{L^2_w(H,(a,\infty))} = \int_a^\infty w(t)\,\|u_-(t)\|^2_H dt$$

$$= \int_a^\infty \frac{w(t)}{\alpha^2(t)} exp\left(2\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) dt \|f\|^2_H$$

$$= \int_a^\infty exp\left(2\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) d\left(\int_a^t \frac{w(s)}{\alpha^2(s)} ds\right) \|f\|^2_H$$

$$= \frac{1}{2}\left(exp\left(2\int_a^\infty \frac{w(s)}{\alpha^2(s)} ds\right) - 1\right) \|f\|^2_H < \infty.$$

Then we obtain

$$n_-(L_0) = \dim\ker(L - iE) = \dim H.$$

This completes the proof.

Consequently, the minimal operator has a maximally dissipative extension (see [1]).

In order to describe these extensions we need to obtain the space of boundary values.

Definition 3.3 [1] Let $\mathcal{H}$ be any Hilbert space and $S: D(S) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$ having equal finite or infinite deficiency indices. A triplet $(\mathbf{H}, \gamma_1, \gamma_2)$ where $\mathbf{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathbf{H}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$(S^*f, g)_{\mathcal{H}} - (f, S^*g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\boldsymbol{H}} - (\gamma_2(f), \gamma_1(g))_{\boldsymbol{H}}$$

while for any $F_1, F_2 \in \boldsymbol{H}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 3.4 The triplet (H, γ_1, γ_2),

$$\gamma_1: D(L) \to H, \gamma_1(u) = \frac{1}{\sqrt{2}}\big((\alpha u)(\infty) - (\alpha u)(a)\big)$$

$$\gamma_2: D(L) \to H, \gamma_2(u) = \frac{1}{i\sqrt{2}}\big((\alpha u)(\infty) + (\alpha u)(a)\big), u \in D(L)$$

is a space of boundary values of the minimal operator L_0 in $L^2_w(H, (\mathrm{a}, \infty))$.

Proof. For any $u, v \in D(L)$,

$$(Lu, v)_{L^2_w(H,(a,\infty))} - (u, Lv)_{L^2_w(H,(a,\infty))}$$

$$= \left(i\frac{\alpha}{w}(\alpha u)' + Au, v\right)_{L^2_w(H,(a,\infty))} - \left(u, i\frac{\alpha}{w}(\alpha v)' + Av\right)_{L^2_w(H,(a,\infty))}$$

$$= \left(i\frac{\alpha}{w}(\alpha u)', v\right)_{L^2_w(H,(a,\infty))} - \left(u, i\frac{\alpha}{w}(\alpha v)'\right)_{L^2_w(H,(a,\infty))}$$

$$= \int_a^\infty \left(i\frac{\alpha(t)}{w(t)}(\alpha u)'(t), v(t)\right)_H w(t)dt - \int_a^\infty \left(u(t), i\frac{\alpha(t)}{w(t)}(\alpha v)'(t)\right)_H w(t)dt$$

$$= i\left[\int_a^\infty \big((\alpha u)'(t), (\alpha v)(t)\big)_H dt + \int_a^\infty \big((\alpha u)(t), (\alpha v)'(t)\big)_H dt\right]$$

$$= i\int_a^\infty ((\alpha u)(t),(\alpha v)(t))'_H dt$$

$$= i\left[((\alpha u)(\infty),(\alpha v)(\infty))_H - ((\alpha u)(a),(\alpha v)(a))_H\right]$$

$$= (\gamma_1(u),\gamma_2(v))_H - (\gamma_2(u),\gamma_1(v))_{\boldsymbol{H}}.$$

Now for any given element $f,g \in H$ find the function $u \in D(L)$ such that

$$\gamma_1(u) = \frac{1}{\sqrt{2}}\big((\alpha u)(\infty) - (\alpha u)(a)\big) = f$$

and

$$\gamma_2(u) = \frac{1}{i\sqrt{2}}\big((\alpha u)(\infty) + (\alpha u)(a)\big) = g.$$

From this it is obtained that

$$(\alpha u)(\infty) = (ig+f)/\sqrt{2}$$

and

$$(\alpha u)(a) = (ig-f)/\sqrt{2}.$$

If we choose the function $u(.)$ in following form

$$u(t) = \frac{1}{\alpha(t)}(1-e^{a-t})\ (ig+f)/\sqrt{2} + \frac{1}{\alpha(t)}e^{a-t}\,(ig-f)/\sqrt{2},$$

then it is clear that $u \in D(L)$ and $\gamma_1(u) = f, \gamma_2(u) = g$.

Hence the lemma is proof.

By using the method in [1] it can be established the following result.

Theorem 3.5 If $\tilde{L}$ is a maximally dissipative extension of the minimal operator L_0 in $L_w^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(.)$ and boundary condition

$$(\alpha u)(a) = K(\alpha u)(\infty),$$

where $K: H \to H$ is contraction operator. Moreover, the contraction operator K in H is determined uniquely by the extension $\tilde{L}$, i.e. $\tilde{L} = L_K$ and vice versa.

Proof. It is known that each maximally dissipative extension $\tilde{L}$ of the minimal operator L_0 is described by differential-operator expression $l(.)$ with boundary condition

$$(V-E)\gamma_1(u) + i(V+E)\gamma_2(u) = 0,$$

where $V: H \to H$ is a contraction operator. Therefore from Lemma 3.4 we obtain that

$$(V-E)\big((\alpha u)(\infty) - (\alpha u)(a)\big) + (V+E)\big((\alpha u)(\infty) + (\alpha u)(a)\big) = 0,$$

$$u \in D(\tilde{L}).$$

From this it is implies that

$$(\alpha u)(a) = -V(\alpha u)(\infty).$$

Choosing $K = -V$ in last boundary condition we have

$$(\alpha u)(a) = K(\alpha u)(\infty).$$

4. THE SPECTRUM OF THE MAXIMALLY DISSIPATIVE EXTENSIONS

In this section the structure of the spectrum of the maximally dissipative extensions of the minimal operator L_0 in $L_w^2(H,(a,\infty))$ will be investigated.

Theorem 4.1 The spectrum of any maximally dissipative extension L_K is in the form

$$\sigma(L_K) = \left\{\lambda \in \mathbb{C}: \lambda = \left(\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)^{-1} \times (iln|\mu|^{-1} + arg\mu + 2n\pi)\right.$$
$$\left. \mu \in \sigma\left(Kexp\left(iA\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right), n \in \mathbb{Z}\right\}$$

Proof. Consider the following problem to spectrum for the extension L_K , i.e.

$$L_K(u) = \lambda u + f, f \in L_w^2(H,(a,\infty)),$$
$$\lambda \in \mathbb{C}, \lambda_i = Im\lambda \geq 0.$$

Then we have

$$i\frac{\alpha(t)}{w(t)}(\alpha u)'(t) + Au(t) = \lambda u(t) + f(t), t > a,$$
$$(\alpha u)(a) = K(\alpha u)(\infty).$$

The general solution of the last differential equation, i.e.,

$$(\alpha u)'(t) = i\frac{w(t)}{\alpha^2(t)}(A-\lambda E)(\alpha u)(t) - i\frac{w(t)}{\alpha(t)}f(t)$$

is in form

$$u(t;\lambda)=\frac{1}{\alpha(t)}exp\left(i(A-\lambda E)\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)f_\lambda$$
$$+\frac{i}{\alpha(t)}\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)$$
$$\times\frac{w(s)}{\alpha(s)}f(s)ds,\ f_\lambda\in H, t>a.$$

In this case

$$\left\|\frac{1}{\alpha(t)}exp\left(i(A-\lambda E)\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)f_\lambda\right\|^2_{L^2_w(H,(a,\infty))}$$
$$=\int_a^\infty\left\|\frac{1}{\alpha(t)}exp\left(i(A-\lambda E)\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)f_\lambda\right\|^2_H w(t)dt$$
$$=\int_a^\infty\frac{w(t)}{\alpha^2(t)}exp\left(2\lambda_i\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)dt\|f_\lambda\|^2_H$$
$$=\int_a^\infty exp\left(2\lambda_i\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)d\left(\int_a^t\frac{w(s)}{\alpha^2(s)}ds\right)\|f_\lambda\|^2_H$$
$$=\frac{1}{2\lambda_i}\left(exp\left(2\lambda_i\int_a^\infty\frac{w(s)}{\alpha^2(s)}ds\right)-1\right)\|f_\lambda\|^2_H$$
$$<\infty$$

and

$$\left\|\frac{i}{\alpha(t)}\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds\right\|^2_{L^2_w(H,(a,\infty))}$$
$$=\int_a^\infty\left\|\frac{i}{\alpha(t)}\int_t^\infty exp\left(i(A-\lambda E)\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds\right\|^2_H w(t)dt$$
$$\le\int_a^\infty\frac{w(t)}{\alpha^2(t)}\left(\int_t^\infty exp\left(\lambda_i\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\times\frac{w(s)}{\alpha(s)}\|f(s)\|_H ds\right)^2dt$$
$$\le\int_a^\infty\frac{w(t)}{\alpha^2(t)}\left(\int_t^\infty\frac{\sqrt{w(s)}}{\alpha(s)}exp\left(\lambda_i\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\times\left\|\sqrt{w(s)}f(s)\right\|_H ds\right)^2dt$$
$$\le\int_a^\infty\frac{w(t)}{\alpha^2(t)}\left(\int_t^\infty\frac{w(s)}{\alpha^2(s)}exp\left(2\lambda_i\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)ds\right)\times\left(\int_t^\infty w(s)\|f(s)\|^2_H\right)dt$$
$$\le\int_a^\infty\frac{w(t)}{\alpha^2(t)}\left(\int_a^\infty\frac{w(s)}{\alpha^2(s)}exp\left(2\lambda_i\int_s^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)ds\right)dt\times\|f\|^2_{L^2_w(H,(a,\infty))}$$
$$=\frac{1}{2\lambda_i}\int_a^\infty\frac{w(t)}{\alpha^2(t)}\left(exp\left(2\lambda_i\int_a^t\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)-exp\left(-2\lambda_i\int_t^\infty\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\right)dt\|f\|^2_{L^2_w(H,(a,\infty))}$$
$$=\frac{1}{4{\lambda_i}^2}\left(exp\left(2\lambda_i\int_a^\infty\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)+exp\left(-2\lambda_i\int_a^\infty\frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)-2\right)\|f(s)\|^2_{L^2_w(H,(a,\infty))}<\infty.$$

Hence, for $\lambda\in\mathbb{C},\lambda_i=Im\lambda\ge 0$, $u(.,\lambda)\in L^2_w(H,(a,\infty))$.

From this and boundary condition we have

$$\left(E - Kexp\left(i(A-\lambda E)\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right)f_\lambda$$

$$= -i\int_a^\infty exp\left(i(A-\lambda E)\int_s^a \frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)$$

$$\times \frac{w(s)}{\alpha(s)}f(s)ds$$

that is,

$$\left(exp\left(i\lambda\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)E\right.$$

$$\left.-Kexp\left(iA\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right)f_\lambda$$

$$= -iexp\left(i\lambda\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)$$

$$\times\int_a^\infty exp\left(i(A-\lambda E)\int_s^a \frac{w(\tau)}{\alpha^2(\tau)}d\tau\right)\frac{w(s)}{\alpha(s)}f(s)ds.$$

Therefore in order to $\lambda \in \sigma(L_K)$ the necessary and sufficient condition is

$$exp\left(i\lambda\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right) = \mu$$

$$\in \sigma\left(Kexp\left(iA\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right).$$

Consequently,

$$\lambda\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds = iln|\mu|^{-1} + arg\mu + 2n\pi, n \in \mathbb{Z}.$$

On the other hand since $\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds > 0$, then

$$\lambda = \left(\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)^{-1}(iln|\mu|^{-1} + arg\mu + 2n\pi),$$

$$\mu \in \sigma\left(Kexp\left(iA\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right), n \in \mathbb{Z}.$$

This completes proof of theorem.

Corollary 4.2 If L_K is any maximally dissipative extension of the minimal operator L_0 and

$$\sigma\left(Kexp\left(iA\int_a^\infty \frac{w(s)}{\alpha^2(s)}ds\right)\right) = \{0\},$$

then $\sigma(L_K) = \emptyset$.

Example 4.3 All maximally dissipative extensions L_r of the minimal operator L_0 generated by the following first order linear symmetric singular quasi-differential expression

$$l(u) = it^{\gamma-\tau}\left(t^\gamma u(t)\right)' + au(t), \gamma, \tau, a \in \mathbb{R} \text{ and } \tau - 2\gamma + 1 > 0$$

in the Hilbert space $L^2_{t^\tau}(1,\infty)$ are described by the boundary condition

$$(t^\gamma u)(1) = r(t^\gamma u)(\infty),$$

where $r \in \mathbb{C}$ and $|r| \leq 1$.

Moreover, in case when $r \neq 0$ the spectrum of maximally dissipative extension L_r is in the form

$$\sigma(L_r) = (\tau - 2\gamma + 1)\left(iln|r|^{-1} + \frac{a}{\tau - 2\gamma + 1} + 2n\pi\right), n \in \mathbb{Z}.$$

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