

Applications of differential transformation method to solve systems of ordinary and partial differential equations

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Abstract

In this study, the numerical solutions of some systems of ordinary and partial differential equations have been analyzed by using the Differential Transformation Method (DTM) and compared with solutions of other numerical methods. This method can be used to solve some ordinary and partial differential equations in a short time by using very simple computer commands and codes. The first chapter, the basic definition of its Differential Transformation Method and properties are given. In the last chapter, examples are solved by using the differential transformation method and are compared with other solutions of numerical methods.

Keywords: *Differential transformation method, systems of ODEs and PDEs.*

Adi ve kısmi diferansiyel denklem sistemlerinin çözümü için diferansiyel dönüşüm yönteminin uygulamaları

Özet

Bu çalışmada Diferansiyel Dönüşüm Yöntemi ile bazı adi ve kısmi diferansiyel denklem sistemleri incelenmiş ve diğer çözüm yöntemleri ile yapılmış sonuçlarla karşılaştırılmıştır. Basit bilgisayar kodları ile bu yöntem birçok adi ve kısmi diferansiyel denklem için kısa sürede çözüme ulaşabilen bir yöntemdir. Giriş kısmında

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Diferansiyel Dönüşüm Yöntemi ile ilgili temel tanım ve ifadeler verilmiş olup son başlıkta uygulamalara yer verilmiştir.

Anahtar kelimeler: *Diferansiyel dönüşüm yöntemi, adi ve kısmi diferansiyel denklem sistemleri.*

1. Introduction

The Differential Transformation Method is a method aiming to find the Taylor expansion of differential equations. Since many functions can be written in the polynomial form, polynomial solutions of differential equations can be obtained with this method. It is also highly compatible with computer languages, such as MAPLE, MATLAB.

The Differential Transformation Method is first introduced by Zhou in 1986 [1]. In this article, linear and nonlinear initial boundary value problems on electrical circuits are examined. After Zhou's definition, Chen and colleagues have described a new structure of DTM that was similar to Zhou's definition and used this new structure to solve eigenvalue problems [2-4]. With Zhou's and Chen work, the known structure of DTM has begun to appear. In 2003, Ayaz is generalized DTM to two-variable partial differential equations and obtained new transformations about two variable functions [5]. Furthermore, Ayaz is used to DTM to obtain the exact solution of different type of differential equations [6-8]. In 2005, Arikoglu and Ozkol used to new form of DTM about solution of boundary value problems for integro-differential equations [9]. In 2006, they used to DTM for solving differential-difference equations [10]. They also developed new DTM form for the solutions of fractional differential equations [11, 12]. All these studies have shown that the differential transformation method can be used different kind of differential equations and some other equations. As well as a method other than DTM for numerical solutions; Padé Approximation [13]. Also new solution techniques continue to be developed in the literature [14-17]. Further information related to DTM and new solution techniques can be found in the literature.

In this paper, solutions of some systems of ordinary and partial differential equations have been analyzed by using the differential transformation method and are compared with other solutions of Finite Element methods.

All $k, h, m, n, p, q, r, s \in \mathbb{N}$ for transform function steps in the whole study and the abbreviations are as follows; DTM; (Differential Transform Method) [1, 2, 4], VIM; (Variational Iteration Method) [18], LDM, (Laplace Decomposition Method) [19], ADM; (Adomian Decomposition Method) [20].

2. Basic definitions

2.1. One-variable differential transformation

Using relevant references [1, 2, 4, 6, 9-12, 21-29] the fundamental definitions of the One-variable Differential Transformation are explained as follows:

Definition 1 The one-variable differential transformation of $\omega(t)$ function is defined as follows:

$$\Omega(n) = \frac{1}{n!} \left[\frac{d^n}{dt^n} \omega(t) \right]_{t=0} \tag{2.1}$$

$\Omega(n)$ is the transformed function, which is called the T- function of $\omega(t)$ function.

Definition 2 The inverse differential transform of $\Omega(n)$ is defined as follows:

$$\omega(t) = \sum_{n=0}^{\infty} \Omega(n) t^n \tag{2.2}$$

Definition 3 The one-variable DTM can be concluded from equations (2.1) and (2.2) as follows:

$$\omega(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{d^n}{dt^n} \omega(t) \right]_{t=0} t^n \tag{2.3}$$

By using the equations (2.1), (2.2) and (2.3), the basic operations of the one-variable differential transform functions can be summarized in the following table,

Table 1. One-variable differential transform functions.

Function	T- function
$\omega(t) = y(t) + z(t)$	$\Omega(n) = Y(n) \pm Z(n)$
$\omega(t) = u(t)v(t)s(t)$	$\Omega(n) = \sum_{r=0}^n \sum_{s=0}^{n-r} U(r)V(s)S(n-r-s)$
$\omega(t) = \lambda y(t)$	$\Omega(n) = \lambda Y(n)$
$\omega(t) = y(t)z(t)$	$\Omega(n) = \sum_{r=0}^n Y(r)Z(n-r)$
$\omega(t) = e^{\lambda t}$	$\Omega(n) = \frac{\lambda^n}{n!}$
$\omega(t) = \frac{d^m y(t)}{dt^m}$	$\Omega(n) = (n+1)(n+2)\dots(n+m)U(n+m)$
$\omega(t) = \frac{dy(t)}{dt}$	$\Omega(n) = (n+1)Y(n+1)$
$\omega(t) = u(t) \frac{d^2}{dt^2} v(t)$	$\Omega(n) = \sum_{r=0}^n (n-r+2)(n-r+1)U(r)V(n-r+2)$
$\omega(t) = \cos(ft + \alpha)$	$\Omega(n) = \frac{f^n}{n!} \cos\left(\frac{\pi n}{2} + \alpha\right)$
$\omega(t) = \sin(ft + \alpha)$	$\Omega(n) = \frac{f^n}{n!} \sin\left(\frac{\pi n}{2} + \alpha\right)$
$\omega(t) = \frac{d}{dt} u(t) \frac{d}{dt} v(t)$	$\Omega(n) = \sum_{r=0}^n (r+1)(n-r+1)U(r+1)V(n-r+1)$

Table 1. (Continued).

Function	T- function
$\omega(t) = t^m$	$\Omega(n) = \delta(n - m) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
$\omega(t) = \int_{t_0}^t u(t) dt$	$\Omega(n) = \frac{U(n-1)}{n}, n \geq 1$
$\omega(t) = u(t)v(t) \frac{d^2}{dt^2} s(t)$	$\Omega(n) = \sum_{r=0}^n \sum_{s=0}^{n-r} (n-r-s+2)^2 U(r)V(s)S(n-r-s+2)$
$\omega(t) = \theta^{\lambda t}$	$\Omega(n) = \frac{\lambda^n (\ln \theta)^n}{n!}$
$\omega(t) = sh(\lambda t)$	$\Omega(n) = \begin{cases} \frac{\lambda^n}{n!}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$
$\omega(t) = ch(\lambda t)$	$\Omega(n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{\lambda^n}{n!}, & \text{if } n \text{ is even} \end{cases}$
$\omega(t) = e^{\lambda t + b}$	$\Omega(n) = \frac{\lambda^n}{n!} e^b$

2.2. Two-variables differential transformation

Using relevant references [3, 5, 8, 24, 26, 30, 31] the fundamental definitions of the Two-variables Differential Transformation are explained as follows:

Definition 4 The two-variables differential transformation of $\omega(t, z)$ function is defined as follows:

$$\Omega(n, m) = \frac{1}{n!m!} \left[\frac{\partial^{n+m}}{\partial t^n \partial z^m} \omega(t, z) \right]_{t=0, z=0} \tag{2.4}$$

$\Omega(n, m)$ is the transformed function, which is called the T- function of $\omega(t, z)$ function.

Definition 5 The inverse differential transform of $\Omega(n, m)$ is defined as follows:

$$\omega(t, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Omega(n, m) t^n z^m \tag{2.5}$$

Definition 6 The two-variable DTM can be concluded from equations (2.4) and (2.5) as follows:

$$\omega(t, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \left[\frac{\partial^{n+m}}{\partial t^n \partial z^m} \omega(t, z) \right]_{\substack{t=0 \\ z=0}} t^n z^m \tag{2.6}$$

By using the equations (2.4), (2.5) and (2.6), the basic operations of the two-variable differential transform functions can be summarized in the following table,

Table 2. Two-variable differential transform functions.

Function	T- function
$\omega(t, z) = u(t, z) \pm v(t, z)$	$\Omega(n, m) = U(n, m) \pm V(n, m)$
$\omega(t, z) = \alpha u(t, z)$	$\Omega(n, m) = \alpha U(n, m)$
$\omega(t, z) = \frac{\partial u(t, z)}{\partial t}$	$\Omega(n, m) = (n + 1)U(n + 1, m)$
$\omega(t, z) = \frac{\partial u(t, z)}{\partial z}$	$\Omega(n, m) = (m + 1)U(n, m + 1)$
$\omega(t, z) = u(t, z)v(t, z)$	$\Omega(n, m) = \sum_{r=0}^n \sum_{s=1}^m U(r, m - s)V(n - r, s)$
$\omega(t, z) = t^p z^q$	$\Omega(n, m) = \delta(n - p, m - q) = \delta(n - p)\delta(m - q) = \begin{cases} 1, & n = p, m = q \\ 0, & \text{otherwise} \end{cases}$
$\omega(t, z) = \frac{\partial u(t, z)}{\partial t} \frac{\partial v(t, z)}{\partial t}$	$\Omega(n, m) = \sum_{r=0}^n \sum_{s=0}^m (r + 1)(n - r + 1)U(r + 1, m - s)V(n - r + 1, s)$
$\omega(t, z) = \frac{\partial u(t, z)}{\partial z} \frac{\partial v(t, z)}{\partial z}$	$\Omega(n, m) = \sum_{r=0}^n \sum_{s=0}^m (s + 1)(m - s + 1)U(r, m - s + 1)V(n - r, s + 1)$
$\omega(t, z) = \frac{\partial u(t, z)}{\partial t} \frac{\partial v(t, z)}{\partial z}$	$\Omega(n, m) = \sum_{r=0}^n \sum_{s=0}^m (n - r + 1)(m - s + 1)U(n - r + 1, s)V(r, m - s + 1)$
$\omega(t, z) = u(t, z)v(t, z)z(t, z)$	$\Omega(n, m) = \sum_{r=0}^n \sum_{q=0}^{n-r} \sum_{s=0}^m \sum_{p=0}^{m-s} U(r, m - s - p)V(q, s)Z(n - r - q, p)$
$\omega(t, z) = u(t, z) \frac{\partial^2 v(t, z)}{\partial t^2}$	$\Omega(n, m) = \sum_{r=0}^n \sum_{s=0}^m (n - r + 2)(n - r + 1)U(r, m - s)V(n - r + 2, s)$
$\omega(t, z) = \frac{\partial^{r+s} u(t, z)}{\partial t^r \partial z^s}$	$\Omega(n, m) = (n + 1) \dots (n + r)(m + 1) \dots (m + s)U(n + r, m + s)$
$\omega(t, z) = t^p e^{at}$	$\Omega(n, m) = \frac{a^m}{m!} \delta(n - p)$
$\omega(t, z) = e^{z-t}$	$\Omega(n, m) = \frac{1^m (-1)^n}{m! n!}$

2.3. Three-variables differential transformation

Using relevant references [7, 24, 26, 31] the fundamental definitions of the Three-variables Differential Transformation are explained as follows:

Definition 7 The three-variables differential transformation of $\omega(t, z, y)$ function is defined as follows:

$$\Omega(n, m, p) = \frac{1}{n!m!p!} \left[\frac{\partial^{n+m+p}}{\partial t^n \partial z^m \partial y^p} \omega(t, z, y) \right]_{t=0, z=0, y=0} \tag{2.7}$$

$\Omega(n, m, p)$ is the transformed function, which is called the T- function of $\omega(t, z, y)$ function.

Definition 8 The inverse differential transform of $\Omega(n, m, p)$ is defined as follows:

$$\omega(t, z, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \Omega(n, m, p) t^n z^m y^p \quad (2.8)$$

Definition 9 The three-variable DTM can be concluded from equations (2.7) and (2.8) as follows:

$$\omega(t, z, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{n!m!p!} \left[\frac{\partial^{n+m+p}}{\partial t^n \partial z^m \partial y^p} \omega(t, z, y) \right]_{t=0, z=0, y=0} t^n z^m y^p \quad (2.9)$$

By using the equations (2.7), (2.8) and (2.9), the basic operations of the three-variable differential transform functions can be summarized in the following table,

Table 3. Three-variable differential transform functions.

Function	T- function
$\omega(t, z, y) = u(t, z, y) \pm v(t, z, y)$	$\Omega(n, m, p) = U(n, m, p) \pm V(n, m, p)$
$\omega(t, z, y) = \alpha u(t, z, y)$	$\Omega(n, m, p) = \alpha U(n, m, p)$
$\omega(t, z, y) = \frac{\partial u(t, z, y)}{\partial t}$	$\Omega(n, m, p) = (n+1)U(n+1, m, p)$
$\omega(t, z, y) = \frac{\partial u(t, z, y)}{\partial z}$	$\Omega(n, m, p) = (m+1)U(n, m+1, p)$
$\omega(t, z, y) = \frac{\partial u(t, z, y)}{\partial y}$	$\Omega(n, m, p) = (p+1)U(n, m, p+1)$
$\omega(t, z, y) = \frac{\partial^{(r+s+q)} u(t, z, y)}{\partial t^r \partial z^s \partial y^q}$	$\Omega(n, m, p) = \frac{(n+r)! (m+s)! (p+q)!}{n! m! p!} U(n+r, m+s, p+q)$
$\omega(t, z, y) = u(t, z, y)v(t, z, y)$	$\Omega(n, m, p) = \sum_{r=0}^n \sum_{s=0}^m \sum_{q=0}^p U(r, m-s, p-q)V(n-r, s, q)$
$\omega(t, z, y) = \frac{\partial u(t, z, y)}{\partial t} \frac{\partial v(t, z, y)}{\partial z}$	$\Omega(n, m, p) = \sum_{r=0}^n \sum_{s=0}^m \sum_{q=0}^p (n-r+1)(m-s+1)U(n-r+1, s, q)V(r, m-s+1, p-q)$

3. Applications of DTM to system of differential equations

In this section, we have applied DTM to the system of ordinary and partial differential equations. The solutions of examples that was discussed in this section have been checked by the MAPLE program. MAPLE programming codes describing the solution produced are given for all examples.

To our knowledge, the DTM solutions of the discussed examples have been newly submitted the literature.

Example 1 Consider the Biomass Transfer Equation (ODE) see [32, 33]; $x(t)$; biomass decayed into humus, $y(t)$; biomass of dead trees, $z(t)$; biomass of living trees, t ; time in decades (decade = 10 years)

Biological system of differential equations;

$$\begin{aligned} x'(t) &= -x(t) + 3y(t), \\ y'(t) &= -3y(t) + 5z(t), \\ z'(t) &= -5z(t). \end{aligned} \tag{3.1}$$

Suppose there are no dead tree and no humus at $t = 0$, with initially z_0 units of living tree biomass. These assumptions require that the initial conditions be in the form,

$$x(0) = y(0) = 0, z(0) = z_0. \tag{3.2}$$

Taking one-variable transformation of Eq. (3.1) and by using the related operations in Table 1, we get

$$\begin{aligned} (k+1)X(k+1) &= -X(k) + 3Y(k), \\ (k+1)Y(k+1) &= -3Y(k) + 5Z(k), \\ (k+1)Z(k+1) &= -5Z(k). \end{aligned} \tag{3.3}$$

the following with initial conditions

$$X(0) = 0, Y(0) = 0, Z(0) = z_0. \tag{3.4}$$

Using the Eqs. (3.1), (3.2), (3.3) and (3.4), we get the following results with MAPLE.

Ex.1-MAPLE code

```
[> X[0]:=0; Y[0]:=0; Z[0]:= #msub(mi("z"),mn("0"))` :
[> F1:=proc (k) options operator, arrow;(k+1)*X[k+1]=-X[k]+3*Y[k] end proc:
[> F2:=proc (k) options operator, arrow;(k+1)*Y[k+1]=-3*Y[k]+5*Z[k] end proc:
[> F3:=proc (k) options operator, arrow;(k+1)*Z[k+1]=-5*Z[k] end proc:
[> for i from 0 to 10 do eq1[i]:=F1(i); eq2[i]:=F2(i); eq3[i]:=F3(i);
    solve({eq1[i]}); assign(solve({eq1[i]}));
    solve({eq2[i]}); assign(solve({eq2[i]}));
    solve({eq3[i]}); assign(solve({eq3[i]})); end do:
[> x:=sum(X[k1]*t^k1,k1=0..10);y:=sum(Y[k1]*t^k1,k1=0..10);z:=sum(Z[k1]*t^k1,k1=0..10);
```

Some calculation steps of produce can be given as follows,

$$\begin{aligned} X(1) = 0, X(2) = \frac{15}{2}z_0, X(3) = \frac{-45}{2}z_0, X(4) = \frac{145}{4}z_0, X(5) = \frac{-165}{4}z_0, X(6) = \frac{1771}{48}z_0, \dots \\ Y(1) = 5z_0, Y(2) = -20z_0, Y(3) = \frac{245}{6}z_0, Y(4) = \frac{-170}{3}z_0, Y(5) = \frac{1441}{24}z_0, Y(6) = \frac{-931}{18}z_0, \dots \\ Z(1) = -5z_0, Z(2) = \frac{25}{2}z_0, Z(3) = \frac{-125}{6}z_0, Z(4) = \frac{625}{24}z_0, Z(5) = \frac{-625}{24}z_0, Z(6) = \frac{3125}{144}z_0, \dots \end{aligned} \tag{3.5}$$

As a result, we can give the solution of $x(t)$, $y(t)$ and $z(t)$ in series form as follows:

$$\begin{aligned}
 x(t) &= \frac{15}{2}z_0t^2 - \frac{45}{2}z_0t^3 + \frac{145}{4}z_0t^4 - \frac{165}{4}z_0t^5 + \frac{1771}{48}z_0t^6 + \dots \\
 y(t) &= 5z_0t - 20z_0t^2 + \frac{245}{6}z_0t^3 - \frac{170}{3}z_0t^4 + \frac{1441}{24}z_0t^5 - \frac{931}{18}z_0t^6 + \dots \\
 z(t) &= z_0 - 5z_0t + \frac{25}{2}z_0t^2 - \frac{125}{6}z_0t^3 + \frac{625}{24}z_0t^4 - \frac{625}{24}z_0t^5 + \frac{3125}{144}z_0t^6 + \dots
 \end{aligned} \tag{3.6}$$

which are exactly the same as the solutions obtained by [32, 33]. Finally, the closed forms of solutions are obtained by means of Taylor expansion:

$$\begin{aligned}
 x(t) &= \frac{15}{8}z_0(e^{-5t} - 2e^{-3t} + e^{-t}), \\
 y(t) &= \frac{5}{2}z_0(-e^{-5t} + e^{-3t}), \\
 z(t) &= z_0e^{-5t}.
 \end{aligned} \tag{3.7}$$

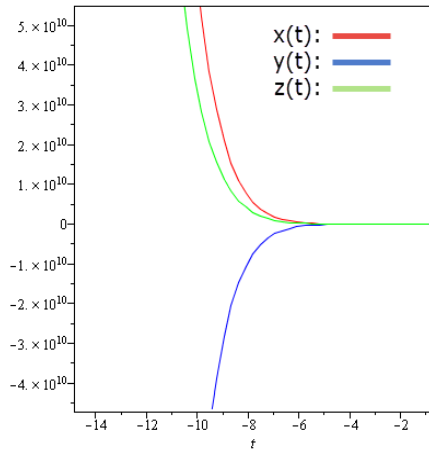


Figure 1: Solution with $(k = 0, \dots, 10)$.

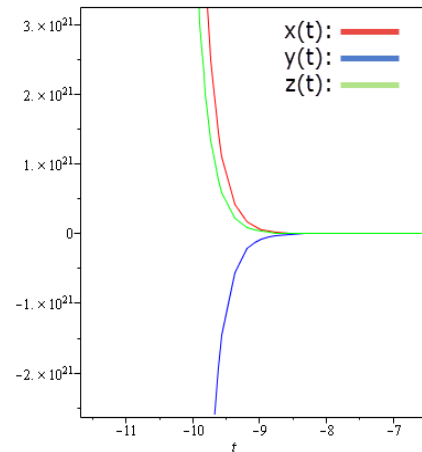


Figure 2: $x(t)$, $y(t)$, and $z(t)$; $z_0 = 1$.

Example 2 Consider the following nonlinear system of PDE see [20]:

$$\begin{aligned}
 u_t + u_x v_x &= 2, \\
 v_t + u_x v_x &= 0.
 \end{aligned} \tag{3.8}$$

with the initial conditions

$$u(x, 0) = x, \quad v(x, 0) = x. \tag{3.9}$$

Taking two-variables transformation of Eq. (3.8) and by using the related operations in Table 2, we obtain the initial conditions Eq. (2.5).

$$\begin{aligned}
 (h+1)U(k, h+1) + \sum_{r=0}^k \sum_{s=0}^h (r+1)(k-r+1)U(r+1, h-s)V(k-r+1, s) &= 2\delta(k, h), \\
 (h+1)V(k, h+1) + \sum_{r=0}^k \sum_{s=0}^h (r+1)(k-r+1)U(r+1, h-s)V(k-r+1, s) &= 0.
 \end{aligned} \tag{3.10}$$

And, transformed initial steps are as follows;

$$\sum_{k=0}^{\infty} U(k,0)x^k = x, \tag{3.11}$$

$$\sum_{k=0}^{\infty} V(k,0)x^k = x.$$

Hence from Eq. (3.11), we get

$$\begin{aligned} U(0,0) &= 0, & V(0,0) &= 0 \\ U(1,0) &= 1, & V(1,0) &= 1 \\ U(2,0) &= 0, & V(2,0) &= 0 \\ U(3,0) &= 0, & V(3,0) &= 0 \\ &\vdots & &\vdots \end{aligned} \tag{3.12}$$

equations, which can be given by the following general form,

$$U(k,0) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

$$V(k,0) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the Eqs. (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) we get the following results with MAPLE.

```
Ex.2-MAPLE code
[> UU:=proc(m,n) local UU; UU[m,n]:=piecewise(m=1 and n=0,1,m<>1 and
n=0,0,U[m,n]); printf("%a:=%a",u[m,n], UU[m,n]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do
SOL:=parse(UU(k,h)); U[k,h]:=SOL; 'U[k,h]' end do end do;
[> VV:=proc (m,n) local VV; VV[m,n]:=piecewise(m=1 and n=0,1,m<>1 and
n=0,0,V[m,n]); printf("%a:=%a",v[m,n], VV[m,n]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do
SOL:=parse(VV(k,h)); V[k,h]:=SOL; 'V[k,h]' end do end do;
[> Mu:=proc(m,n) local Mu; Mu[m,n]; Mu[m,n]:=piecewise(m=0 and n=0,1,0);
printf("%a:=%a",delta[m,n], Mu[m,n]) end proc;
[> for k from 0 to 50 do for h from 0 to 50 do
SOL:=parse(Mu(k,h)); delta[k,h]:=SOL; 'delta[k,h]' end do end do;
[> F1:=proc(k,h) options operator, arrow; (h+1)*U[k,h+1]+sum(sum((r+1)*(k-r+1)*
U[r+1,h-s]*V[k-r+1,s],s=0..h),r=0..k)=2*delta[k,h] end proc;
[> F2:=proc(k,h) options operator, arrow; (h+1)*V[k,h+1]+sum(sum((r+1)*(k-r+1)*
U[r+1,h-s]*V[k-r+1,s],s=0..h),r=0..k)=0 end proc;
[> for k from 0 to 10 do for h from 0 to 10 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> u:=sum(sum(U[kkk,hhh]*x^kkk*t^hhh,hhh=0..4),kkk=0..4);
v:=sum(sum(V[kkk,hhh]*x^kkk*t^hhh,hhh=0..4),kkk=0..4);
```

Some steps are

$$\begin{aligned}
 U(0,0) &= 0, & U(0,1) &= 1, & U(0,2) &= 0, & U(0,3) &= 0, & U(0,4) &= 0, \\
 U(1,0) &= 1, & U(1,1) &= 0, & U(1,2) &= 0, & U(1,3) &= 0, & U(1,4) &= 0, \\
 U(2,0) &= 0, & U(2,1) &= 0, & U(2,2) &= 0, & U(2,3) &= 0, & U(2,4) &= 0, \\
 U(3,0) &= 0, & U(3,1) &= 0, & U(3,2) &= 0, & U(3,3) &= 0, & U(3,4) &= 0.
 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
 V(0,0) &= 0, & V(0,1) &= -1, & V(0,2) &= 0, & V(0,3) &= 0, & V(0,4) &= 0, \\
 V(1,0) &= 1, & V(1,1) &= 0, & V(1,2) &= 0, & V(1,3) &= 0, & V(1,4) &= 0, \\
 V(2,0) &= 0, & V(2,1) &= 0, & V(2,2) &= 0, & V(2,3) &= 0, & V(2,4) &= 0, \\
 V(3,0) &= 0, & V(3,1) &= 0, & V(3,2) &= 0, & V(3,3) &= 0, & V(3,4) &= 0.
 \end{aligned} \tag{3.15}$$

and these values are continuing. If we generalize these values, we get following demonstrations

$$U(k,h) = \begin{cases} 1 & \text{if } k=1 \text{ and } h=0 \\ 1 & \text{if } k=0 \text{ and } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

$$V(k,h) = \begin{cases} 1 & \text{if } k=1 \text{ and } h=0 \\ -1 & \text{if } k=0 \text{ and } h=1 \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

As a result, we can give the series solutions of $u(x,t)$ and $v(x,t)$ as follows:

$$u(x,t) = \underbrace{\sum_{k=1}^1 \sum_{h=0}^0 \frac{U(k,h)}{k!h!} x^k t^h}_{U(1,0)x} + \underbrace{\sum_{k=0}^0 \sum_{h=1}^1 \frac{U(k,h)}{k!h!} x^k t^h}_{U(0,1)t} + \underbrace{0}_{\text{otherwise}} \tag{3.18}$$

$$u(x,t) = x + t \tag{3.19}$$

$$v(x,t) = \underbrace{\sum_{k=1}^1 \sum_{h=0}^0 \frac{V(k,h)}{k!h!} x^k t^h}_{V(1,0)x} + \underbrace{\sum_{k=0}^0 \sum_{h=1}^1 \frac{V(k,h)}{k!h!} x^k t^h}_{V(0,1)t} + \underbrace{0}_{\text{otherwise}} \tag{3.20}$$

$$v(x,t) = x - t \tag{3.21}$$

which are exactly the same as the solutions obtained by VIM [18,34] and ADM [20].

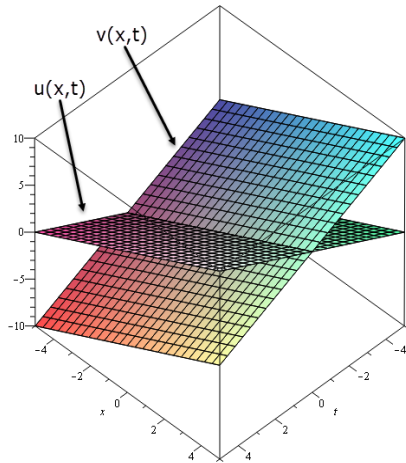


Figure 3: Solution with $(k, h = 0, \dots, 10)$.

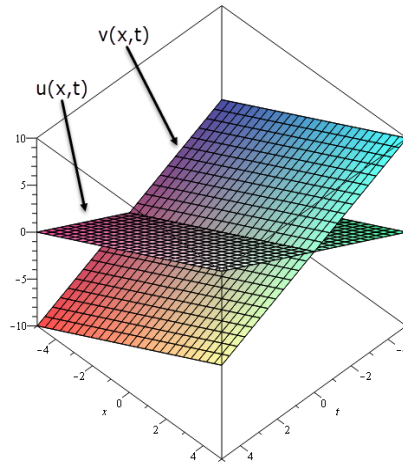


Figure 4: $u(x,t)$ and $v(x,t)$.

Example 3 Consider the following nonlinear system of PDE see [20]:

$$\begin{aligned} u_t + 2vu_x - u &= 2, \\ v_t - 3uv_x + v &= 3. \end{aligned} \tag{3.22}$$

with the initial conditions

$$u(x,0) = e^x, \quad v(x,0) = e^{-x}. \tag{3.23}$$

Taking two-variables transformation of Eq. (3.22) and by using the related operations in Table 2, we obtain the initial conditions Eq. (2.5).

$$\begin{aligned} (h+1)U(k, h+1) + 2 \sum_{r=0}^k \sum_{s=0}^h (k-r+1)V(r, h-s)U(k-r+1, s) - U(k, h) &= 2\delta(k, h) \\ (h+1)V(k, h+1) - 3 \sum_{r=0}^k \sum_{s=0}^h (k-r+1)U(r, h-s)V(k-r+1, s) + V(k, h) &= 3\delta(k, h) \end{aligned} \tag{3.24}$$

and, transformed initial steps are as follows;

$$\begin{aligned} \sum_{k=0}^{\infty} U(k,0)x^k &= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots \\ \sum_{k=0}^{\infty} V(k,0)x^k &= \frac{1}{0!}x^0 - \frac{1}{1!}x^1 + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \dots \end{aligned} \tag{3.25}$$

Hence from Eq. (3.25), we get

$$\begin{aligned}
U(0,0) &= \frac{1}{0!}, & V(0,0) &= \frac{1}{0!} \\
U(1,0) &= \frac{1}{1!}, & V(1,0) &= -\frac{1}{1!} \\
U(2,0) &= \frac{1}{2!}, & V(2,0) &= \frac{1}{2!} \\
U(3,0) &= \frac{1}{3!}, & V(3,0) &= -\frac{1}{3!} \\
&\vdots & & \vdots
\end{aligned} \tag{3.26}$$

equations, which can be given by the following general form,

$$\begin{aligned}
U(k,0) &= \frac{1}{k!} \text{ if } k = 0,1,2,\dots \\
V(k,0) &= \begin{cases} \frac{1}{k!} & \text{if } k = 0,2,4,\dots \\ -\frac{1}{k!} & \text{if } k = 1,3,5,\dots \end{cases}
\end{aligned} \tag{3.27}$$

Using the Eqs. (3.22), (3.23), (3.24), (3.25), (3.26) and (3.27) we get the following results with MAPLE.

Ex.3-MAPLE code

```

[> for kk1 from 0 to 30 do U[kk1,0]:=1/factorial(kk1) end do;
[> G1:=-1/factorial(kk2); G2:=1/factorial(kk2); for kk2 from 0 to 30 do
  V[kk2,0]:=piecewise(type(kk2,odd),G1,type(kk2,even),G2) end do;
[> Mu:=proc(m,n) local Mu; Mu[m,n]; Mu[m,n]:=piecewise(m=0 and n=0,1,0);
  sprintf("%a:=%a",delta[m,n],Mu[m,n]) end proc;
[> for k from 0 to 100 do for h from 0 to 100 do
  SOL:=parse(Mu(k,h)); delta[k,h]:=SOL; 'delta[k,h]' end do end do;
[> F1:=proc(k,h)options operator,arrow;(h+1)*U[k,h+1]+2*(sum(sum((k-r+1)*
  V[r,h-s]*U[k-r+1,s],s=0..h),r=0..k))-U[k,h]=2*delta[k,h] end proc;
[> F2:=proc(k,h)options operator,arrow;(h+1)*V[k,h+1]-3*(sum(sum((k-r+1)*
  U[r,h-s]*V[k-r+1,s],s=0..h),r=0..k))+V[k,h]=3*delta[k,h] end proc;
[> for k from 0 to 10 do for h from 0 to 0 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 9 do for h from 1 to 1 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 8 do for h from 2 to 2 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 7 do for h from 3 to 3 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 6 do for h from 4 to 4 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> u:=sum(sum(U[kkk,hhh]*x^kkk*t^hhh, hhh=0..4), kkk=0..4);
  v:=sum(sum(V[kkk,hhh]*x^kkk*t^hhh, hhh=0..4), kkk=0..4);

```

Some steps are

$$\begin{aligned}
 U(0,0) &= \frac{1}{0!0!}, & U(0,1) &= \frac{1}{0!1!}, & U(0,2) &= \frac{1}{0!2!}, & U(0,3) &= \frac{1}{0!3!}, & U(0,4) &= \frac{1}{0!4!}, \\
 U(1,0) &= \frac{1}{1!0!}, & U(1,1) &= \frac{1}{1!1!}, & U(1,2) &= \frac{1}{1!2!}, & U(1,3) &= \frac{1}{1!3!}, & U(1,4) &= \frac{1}{1!4!}, \\
 U(2,0) &= \frac{1}{2!0!}, & U(2,1) &= \frac{1}{2!1!}, & U(2,2) &= \frac{1}{2!2!}, & U(2,3) &= \frac{1}{2!3!}, & U(2,4) &= \frac{1}{2!4!}, \\
 U(3,0) &= \frac{1}{3!0!}, & U(3,1) &= \frac{1}{3!1!}, & U(3,2) &= \frac{1}{3!2!}, & U(3,3) &= \frac{1}{3!3!}, & U(3,4) &= \frac{1}{3!4!}.
 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
 V(0,0) &= \frac{1}{0!0!}, & V(0,1) &= -\frac{1}{0!1!}, & V(0,2) &= \frac{1}{0!2!}, & V(0,3) &= -\frac{1}{0!3!}, & V(0,4) &= \frac{1}{0!4!}, \\
 V(1,0) &= -\frac{1}{1!0!}, & V(1,1) &= \frac{1}{1!1!}, & V(1,2) &= -\frac{1}{1!2!}, & V(1,3) &= \frac{1}{1!3!}, & V(1,4) &= -\frac{1}{1!4!}, \\
 V(2,0) &= \frac{1}{2!0!}, & V(2,1) &= -\frac{1}{2!1!}, & V(2,2) &= \frac{1}{2!2!}, & V(2,3) &= -\frac{1}{2!3!}, & V(2,4) &= \frac{1}{2!4!}, \\
 V(3,0) &= -\frac{1}{3!0!}, & V(3,1) &= \frac{1}{3!1!}, & V(3,2) &= -\frac{1}{3!2!}, & V(3,3) &= \frac{1}{3!3!}, & V(3,4) &= -\frac{1}{3!4!}.
 \end{aligned} \tag{3.29}$$

and these values are continuing. If we generalize these values, we get following demonstrations

$$U(k, h) = \frac{1}{k!h!} \quad k = 0, 1, 2, \dots \tag{3.30}$$

$$V(k, h) = \frac{(-1)^{k+h}}{k!h!} \quad k = 0, 1, 2, \dots \tag{3.31}$$

As a result, we can give the series solutions of $u(x, t)$ and $v(x, t)$ as follows:

$$u(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} x^k t^h \tag{3.32}$$

$$u(x, t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + xt + \frac{1}{2}xt^2 + \frac{1}{6}xt^3 + \frac{1}{2}x^2t + \frac{1}{4}x^2t^2 + \frac{1}{12}x^2t^3 + \dots \tag{3.33}$$

$$u(x, t) = e^{x+t} \tag{3.34}$$

$$v(x, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{(-1)^{k+h}}{k!h!} x^k t^h \tag{3.35}$$

$$v(x, t) = 1 - x - t + \frac{1}{2}x^2 + xt + \frac{1}{2}t^2 - \frac{1}{6}x^3 - \frac{1}{2}tx^2 - \frac{1}{2}t^2x - \frac{1}{6}t^3 + \frac{1}{24}x^4 + \frac{1}{6}tx^3 + \frac{1}{4}t^2x^2 + \dots \tag{3.36}$$

$$v(x,t) = e^{-x-t} \tag{3.37}$$

which are exactly the same as the solutions obtained by VIM [18, 34] and ADM [20].

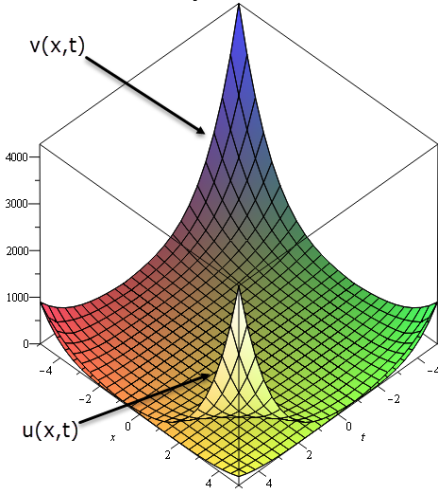


Figure 5: Solution with $(k, h = 0, \dots, 10)$.

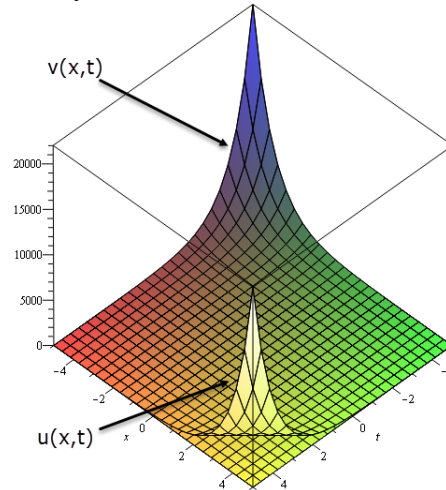


Figure 6: $u(x,t)$ and $v(x,t)$.

Example 4 Consider the following nonlinear system of PDE see [20]:

$$\begin{aligned} u_t + vu_x - 3u &= 2, \\ v_t - uv_x + 3v &= 2, \end{aligned} \tag{3.38}$$

with the initial conditions

$$u(x,0) = e^{2x}, \quad v(x,0) = e^{-2x}. \tag{3.39}$$

Taking two-variables transformation of Eq. (3.38) and by using the related operations on Table 2, we obtain the initial conditions Eq. (2.5).

$$\begin{aligned} (h+1)U(k, h+1) + \sum_{r=0}^k \sum_{s=0}^h (k-r+1)V(r, h-s)U(k-r+1, s) - 3U(k, h) &= 2\delta(k, h) \\ (h+1)V(k, h+1) - \sum_{r=0}^k \sum_{s=0}^h (k-r+1)U(r, h-s)V(k-r+1, s) + 3V(k, h) &= 2\delta(k, h) \end{aligned} \tag{3.40}$$

and, transformed initial steps are as follows;

$$\begin{aligned} \sum_{k=0}^{\infty} U(k,0)x^k &= \frac{2^0}{0!}x^0 + \frac{2^1}{1!}x^1 + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + \dots \\ \sum_{k=0}^{\infty} V(k,0)x^k &= \frac{2^0}{0!}x^0 - \frac{2^1}{1!}x^1 + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 - \frac{2^5}{5!}x^5 + \dots \end{aligned} \tag{3.41}$$

Hence from Eq. (3.41), we get

$$\begin{aligned}
 U(0,0) &= \frac{2^0}{0!}, & V(0,0) &= \frac{2^0}{0!} \\
 U(1,0) &= \frac{2^1}{1!}, & V(1,0) &= -\frac{2^1}{1!} \\
 U(2,0) &= \frac{2^2}{2!}, & V(2,0) &= \frac{2^2}{2!} \\
 U(3,0) &= \frac{2^3}{3!}, & V(3,0) &= -\frac{2^3}{3!} \\
 &\vdots & &\vdots
 \end{aligned}
 \tag{3.42}$$

equations, which can be given by the following general form,

$$\begin{aligned}
 U(k,0) &= \frac{2^k}{k!} \quad \text{if } k = 0,1,2,\dots \\
 V(k,0) &= \begin{cases} \frac{2^k}{k!} & \text{if } k = 0,2,4,\dots \\ -\frac{2^k}{k!} & \text{if } k = 1,3,5,\dots \end{cases}
 \end{aligned}
 \tag{3.43}$$

Using the Eqs. (3.38), (3.39), (3.40), (3.41), (3.42) and (3.43) we get the following results with MAPLE.

Ex.4-MAPLE code

```

[> for kk1 from 0 to 30 do U[kk1,0]:=2^kk1/factorial(kk1) end do;
[> G1:=-2^kk2/factorial(kk2); G2:=2^kk2/factorial(kk2); for kk2 from 0 to 30 do
  V[kk2,0]:=piecewise(type(kk2,odd),G1,type(kk2,even),G2)end do;
[> Mu:=proc(m,n) local Mu; Mu[m,n]; Mu[m,n]:=piecewise(m=0 and n=0,1,0);
  sprintf("%a:=%a",delta[m,n],Mu[m,n]) end proc;
[> for k from 0 to 100 do for h from 0 to 100 do
  SOL:=parse(Mu(k,h)); delta[k,h]:=SOL;'delta[k,h]' end do end do;
[> F1:=proc(k,h) options operator, arrow; (h+1)*U[k,h+1]+sum(sum((k-r+1)*
  V[r,h-s]*U[k-r+1,s], s=0..h),r=0..k)-3*U[k,h]=2*delta[k,h] end proc;
[> F2:=proc(k,h) options operator, arrow; (h+1)*V[k,h+1]-(sum(sum((k-r+1)*
  U[r,h-s]*V[k-r+1,s], s=0..h),r=0..k))+3*V[k,h]=2*delta[k,h] end proc;
[> for k from 0 to 10 do for h from 0 to 0 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 9 do for h from 1 to 1 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 8 do for h from 2 to 2 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 7 do for h from 3 to 3 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> for k from 0 to 6 do for h from 4 to 4 do eq1[k,h]:=F1(k,h); eq2[k,h]:=F2(k,h);
  solve({eq1[k,h]}); assign(solve({eq1[k,h]}));
  solve({eq2[k,h]}); assign(solve({eq2[k,h]})); end do end do;
[> u:=sum(sum(U[kkk,hhh]*x^kkk*t^hhh,hhh=0..4),kkk=0..4);
  v:=sum(sum(V[kkk,hhh]*x^kkk*t^hhh,hhh=0..4),kkk=0..4);
  
```

Some steps are

$$\begin{aligned}
 U(0,0) &= \frac{2^0 3^0}{0!0!}, & U(0,1) &= \frac{2^0 3^1}{0!1!}, & U(0,2) &= \frac{2^0 3^2}{0!2!}, & U(0,3) &= \frac{2^0 3^3}{0!3!}, & U(0,4) &= \frac{2^0 3^4}{0!4!}, \\
 U(1,0) &= \frac{2^1 3^0}{1!0!}, & U(1,1) &= \frac{2^1 3^1}{1!1!}, & U(1,2) &= \frac{2^1 3^2}{1!2!}, & U(1,3) &= \frac{2^1 3^3}{1!3!}, & U(1,4) &= \frac{2^1 3^4}{1!4!}, \\
 U(2,0) &= \frac{2^2 3^0}{2!0!}, & U(2,1) &= \frac{2^2 3^1}{2!1!}, & U(2,2) &= \frac{2^2 3^2}{2!2!}, & U(2,3) &= \frac{2^2 3^3}{2!3!}, & U(2,4) &= \frac{2^2 3^4}{2!4!}, \\
 U(3,0) &= \frac{2^3 3^0}{3!0!}, & U(3,1) &= \frac{2^3 3^1}{3!1!}, & U(3,2) &= \frac{2^3 3^2}{3!2!}, & U(3,3) &= \frac{2^3 3^3}{3!3!}, & U(3,4) &= \frac{2^3 3^4}{3!4!}.
 \end{aligned} \tag{3.44}$$

and

$$\begin{aligned}
 V(0,0) &= \frac{2^0 3^0}{0!0!}, & V(0,1) &= -\frac{2^0 3^1}{0!1!}, & V(0,2) &= \frac{2^0 3^2}{0!2!}, & V(0,3) &= -\frac{2^0 3^3}{0!3!}, & V(0,4) &= \frac{2^0 3^4}{0!4!}, \\
 V(1,0) &= -\frac{2^1 3^0}{1!0!}, & V(1,1) &= \frac{2^1 3^1}{1!1!}, & V(1,2) &= -\frac{2^1 3^2}{1!2!}, & V(1,3) &= \frac{2^1 3^3}{1!3!}, & V(1,4) &= -\frac{2^1 3^4}{1!4!}, \\
 V(2,0) &= \frac{2^2 3^0}{2!0!}, & V(2,1) &= -\frac{2^2 3^1}{2!1!}, & V(2,2) &= \frac{2^2 3^2}{2!2!}, & V(2,3) &= -\frac{2^2 3^3}{2!3!}, & V(2,4) &= \frac{2^2 3^4}{2!4!}, \\
 V(3,0) &= -\frac{2^3 3^0}{3!0!}, & V(3,1) &= \frac{2^3 3^1}{3!1!}, & V(3,2) &= -\frac{2^3 3^2}{3!2!}, & V(3,3) &= \frac{2^3 3^3}{3!3!}, & V(3,4) &= -\frac{2^3 3^4}{3!4!}.
 \end{aligned} \tag{3.45}$$

and these values are continuing. If we generalize these values, we get following demonstrations

$$U(k,h) = \frac{2^k 3^h}{k!h!} \quad \text{if } k=0,1,2,\dots \text{ and } h=0,1,2,\dots \tag{3.46}$$

$$V(k,h) = (-1)^{k+h} \frac{2^k 3^h}{k!h!} \quad \text{if } k=0,1,2,\dots \text{ and } h=0,1,2,\dots \tag{3.47}$$

As a result, we can give the series solutions of $u(x,t), v(x,t)$ as follows:

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{2^k 3^h}{k!h!} x^k t^h \tag{3.48}$$

$$u(x,t) = 1 + 2x + 3t + 2x^2 + 6xt + \frac{9}{2}t^2 + \frac{4}{3}x^3 + 6x^2t + 9xt^2 + \frac{9}{2}t^3 + \frac{2}{3}x^4 + 4x^3t + 9x^2t^2 + 9xt^3 + \dots \tag{3.49}$$

$$u(x,t) = e^{2x+3t} \tag{3.50}$$

$$v(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} (-1)^{k+h} \frac{2^k 3^h}{k!h!} x^k t^h \tag{3.51}$$

$$v(x,t) = 1 - 2x - 3t + 2x^2 + 6xt + \frac{9}{2}t^2 - \frac{4}{3}x^3 - 6x^2t - 9xt^2 - \frac{9}{2}t^3 + \frac{2}{3}x^4 + 4x^3t + 9x^2t^2 + \dots \quad (3.52)$$

$$v(x,t) = e^{-2x-3t} \quad (3.53)$$

which are exactly the same as the solutions obtained by VIM [18] and ADM [20].

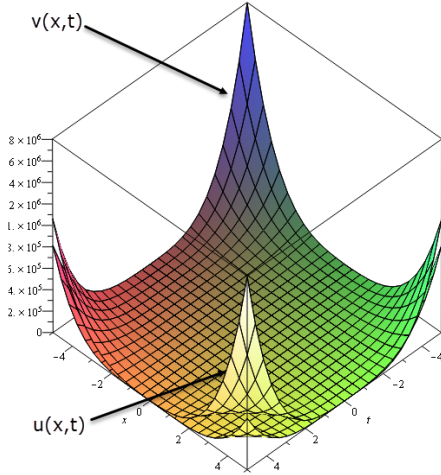


Figure 7: Solution with $(k, h = 0, \dots, 10)$.

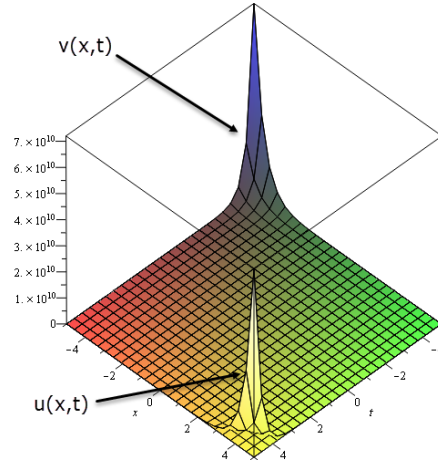


Figure 8: $u(x,t)$ and $v(x,t)$.

Example 5 Consider the following nonlinear system of PDE see [20]:

$$\begin{aligned} u_t - v_x w_y &= 1 \\ v_t - w_x u_y &= 5 \\ w_t - u_x v_y &= 5 \end{aligned} \quad (3.54)$$

with the initial conditions

$$u(x,y,0) = x + 2y, \quad v(x,y,0) = x - 2y, \quad w(x,y,0) = -x + 2y. \quad (3.55)$$

Taking two-variables transformation of Eq. (3.54) and by using the related operations in Table 3, we obtain the initial conditions Eq. (2.8).

$$\begin{aligned} (m+1)U(k,h,m+1) - \sum_{r=0}^k \sum_{s=0}^h \sum_{p=0}^m (k-r+1)(h-s+1)V(k-r+1,s,p)W(r,h-s+1,m-p) &= \delta(k,h,m) \\ (m+1)V(k,h,m+1) - \sum_{r=0}^k \sum_{s=0}^h \sum_{p=0}^m (k-r+1)(h-s+1)W(k-r+1,s,p)U(r,h-s+1,m-p) &= 5\delta(k,h,m) \\ (m+1)W(k,h,m+1) - \sum_{r=0}^k \sum_{s=0}^h \sum_{p=0}^m (k-r+1)(h-s+1)U(k-r+1,s,p)V(r,h-s+1,m-p) &= 5\delta(k,h,m) \end{aligned} \quad (3.56)$$

and, transformed initial steps are as follows;

$$\begin{aligned} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} U(k,h,0)x^r y^s &= x + 2y \\ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} V(k,h,0)x^r y^s &= x - 2y \\ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} W(k,h,0)x^r y^s &= -x + 2y \end{aligned} \quad (3.57)$$

Hence from Eq. (3.57), we get

$$\begin{aligned} U(1,0,0) &= 1, & V(1,0,0) &= 1, & W(1,0,0) &= -1, \\ U(0,1,0) &= 2, & V(0,1,0) &= -2, & W(0,1,0) &= 2, \\ U(1,1,0) &= 0, & V(1,1,0) &= 0, & W(1,1,0) &= 0, \end{aligned} \quad (3.58)$$

equations, which can be given by the following general form,

$$\begin{aligned} U(k,h,0) &= \begin{cases} 1 & \text{if } k=1 \text{ and } h=0 \\ 2 & \text{if } k=0 \text{ and } h=1 \\ 0 & \text{otherwise} \end{cases} \\ V(k,h,0) &= \begin{cases} 1 & \text{if } k=1 \text{ and } h=0 \\ -2 & \text{if } k=0 \text{ and } h=1 \\ 0 & \text{otherwise} \end{cases} \\ W(k,h,0) &= \begin{cases} -1 & \text{if } k=1 \text{ and } h=0 \\ 2 & \text{if } k=0 \text{ and } h=1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3.59)$$

Using the Eqs. (3.54), (3.55), (3.56), (3.57), (3.58) and (3.59) we get the following results with MAPLE.

Ex.5-MAPLE code

```
[> UU:=proc(m,n,g) local UU; UU[m,n,g]:=piecewise(m=1 and n=0 and g=0,1,m=0 and
n=1 and g=0,2,g<>0,U[m,n,g],0); printf("%a:=%a",u[m,n,g],UU[m,n,g]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do for m from 0 to 10 do
SOL:=parse(UU(k,h,m)); U[k,h,m]:=SOL; 'U[k,h,m]' end do end do end do;
[> VV:=proc(m,n,g) local VV; VV[m,n,g]:=piecewise(m=1 and n=0 and g=0,1,m=0 and
n=1 and g=0,-2,g<>0,V[m,n,g],0);printf("%a:=%a",v[m,n,g],VV[m,n,g]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do for m from 0 to 10 do
SOL:=parse(VV(k,h,m)); V[k,h,m]:=SOL; 'V[k,h,m]' end do end do end do;
[> WW:=proc(m,n,g) local WW; WW[m,n,g]:=piecewise(m=1 and n=0 and g=0,-1,m=0 and
n=1 and g=0,2,g<>0,W[m,n,g],0); printf("%a:=%a",w[m,n,g],WW[m,n,g]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do for m from 0 to 10 do
SOL:=parse(WW(k,h,m)); W[k,h,m]:=SOL; 'W[k,h,m]' end do end do end do;
[> Mu:=proc(m,n,g) local Mu; Mu[m,n,g]; Mu[m,n,g]:=piecewise(m=0 and n=0 and
g= 0,1,0); printf("%a:=%a", delta[m, n, g], Mu[m, n, g]) end proc;
[> for k from 0 to 10 do for h from 0 to 10 do for m from 0 to 10 do
SOL:=parse(Mu(k,h,m)); delta[k,h,m]:=SOL; 'delta[k,h,m]' end do end do end do;
[> F1:=proc(k,h,m) options operator, arrow; (m+1)*U[k,h,m+1]-
(sum(sum(sum((k-r+1)*(h-s+1)*V[k-r+1,s,p]*W[r,h-s+1,m-p],p=0..m),
s=0..h),r=0..k))=delta[k,h,m] end proc;
```

```
[> F2:=proc(k,h,m) options operator, arrow; (m+1)*V[k,h,m+1]-
  (sum(sum(sum((k-r+1)*(h-s+1)*W[k-r+1,s,p]*U[r,h-s+1,m-p],p=0..m),
  s=0..h),r=0..k))=5*delta[k,h,m] end proc;
[> F3:=proc(k,h,m) options operator, arrow; (m+1)*W[k,h,m+1]-
  (sum(sum(sum((k-r+1)*(h-s+1)*U[k-r+1,s,p]*V[r,h-s+1,m-p],p=0..m),
  s=0..h),r=0..k))=5*delta[k,h,m] end proc;
[> for k from 0 to 5 do for h from 0 to 5 do for m from 0 to 5 do
  eq1[k,h,m]:=F1(k,h,m); eq2[k,h,m]:=F2(k,h,m); eq3[k,h,m] :=F3(k,h,m);
  solve({eq1[k,h,m]}); assign(solve({eq1[k,h,m]}));
  solve({eq2[k,h,m]}); assign(solve({eq2[k,h,m]}));
  solve({eq3[k,h,m]}); assign(solve({eq3[k,h,m]})); end do end do end do;
[>
u:=sum(sum(sum(U[kkk,hhh,mmm]*x^kkk*y^hhh*t^mmm,mmm=0..3),hhh=0..3),kkk=0..3);

v:=sum(sum(sum(V[kkk,hhh,mmm]*x^kkk*y^hhh*t^mmm,mmm=0..3),hhh=0..3),kkk=0..3);

w:=sum(sum(sum(W[kkk,hhh,mmm]*x^kkk*y^hhh*t^mmm,mmm=0..3),hhh=0..3),kkk=0..3);
```

Some steps are

$$\begin{aligned}
 U(0,0,0) &= 0 & V(0,0,0) &= 0 & W(0,0,0) &= 0 \\
 U(0,0,1) &= 3 & V(0,0,1) &= 3 & W(0,0,1) &= 3 \\
 U(0,1,0) &= 2 & V(0,1,0) &= -2 & W(0,1,0) &= 2 \\
 U(0,1,1) &= 0 & V(0,1,1) &= 0 & W(0,1,1) &= 0 \\
 U(1,0,0) &= 1 & V(1,0,0) &= 1 & W(1,0,0) &= -1 \\
 U(1,0,1) &= 0 & V(1,0,1) &= 0 & W(1,0,1) &= 0 \\
 U(1,1,0) &= 0 & V(1,1,0) &= 0 & W(1,1,0) &= 0 \\
 U(1,1,1) &= 0 & V(1,1,1) &= 0 & W(1,1,1) &= 0
 \end{aligned}
 \tag{3.60}$$

and these values are continuing. If we generalize these values, we get following demonstrations

$$U(k,h,m) = \begin{cases} 3 & \text{if } k = 0, h = 0 \text{ and } m = 1 \\ 2 & \text{if } k = 0, h = 1 \text{ and } m = 0 \\ 1 & \text{if } k = 1, h = 0 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}
 \tag{3.61}$$

$$V(k,h,m) = \begin{cases} 3 & \text{if } k = 0, h = 0 \text{ and } m = 1 \\ -2 & \text{if } k = 0, h = 1 \text{ and } m = 0 \\ 1 & \text{if } k = 1, h = 0 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}
 \tag{3.62}$$

$$W(k,h,m) = \begin{cases} 3 & \text{if } k = 0, h = 0 \text{ and } m = 1 \\ 2 & \text{if } k = 0, h = 1 \text{ and } m = 0 \\ -1 & \text{if } k = 1, h = 0 \text{ and } m = 0 \\ 0 & \text{otherwise} \end{cases}
 \tag{3.63}$$

As a result, we can give the series solutions of $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ as follows:

$$u(x,y,t) = \underbrace{\sum_{k=0}^0 \sum_{h=0}^0 \sum_{m=1}^1 \frac{U(k,h,m)}{k!h!m!} x^k y^h t^m}_{U(0,0,1)t} + \underbrace{\sum_{k=0}^0 \sum_{h=1}^1 \sum_{m=0}^0 \frac{U(k,h,m)}{k!h!m!} x^k y^h t^m}_{U(0,1,0)y} + \underbrace{\sum_{k=1}^1 \sum_{h=0}^0 \sum_{m=0}^0 \frac{U(k,h,m)}{k!h!m!} x^k y^h t^m}_{U(1,0,0)x} + \underbrace{0}_{\text{otherwise}} \quad (3.64)$$

$$u(x,y,t) = x + 2y + 3t \quad (3.65)$$

$$v(x,y,t) = \underbrace{\sum_{k=0}^0 \sum_{h=0}^0 \sum_{m=1}^1 \frac{V(k,h,m)}{k!h!m!} x^k y^h t^m}_{V(0,0,1)t} + \underbrace{\sum_{k=0}^0 \sum_{h=1}^1 \sum_{m=0}^0 \frac{V(k,h,m)}{k!h!m!} x^k y^h t^m}_{V(0,1,0)y} + \underbrace{\sum_{k=1}^1 \sum_{h=0}^0 \sum_{m=0}^0 \frac{V(k,h,m)}{k!h!m!} x^k y^h t^m}_{V(1,0,0)x} + \underbrace{0}_{\text{otherwise}} \quad (3.66)$$

$$v(x,y,t) = x - 2y + 3t \quad (3.67)$$

$$w(x,y,t) = \underbrace{\sum_{k=0}^0 \sum_{h=0}^0 \sum_{m=1}^1 \frac{W(k,h,m)}{k!h!m!} x^k y^h t^m}_{W(0,0,1)t} + \underbrace{\sum_{k=0}^0 \sum_{h=1}^1 \sum_{m=0}^0 \frac{W(k,h,m)}{k!h!m!} x^k y^h t^m}_{W(0,1,0)y} + \underbrace{\sum_{k=1}^1 \sum_{h=0}^0 \sum_{m=0}^0 \frac{W(k,h,m)}{k!h!m!} x^k y^h t^m}_{W(1,0,0)x} + \underbrace{0}_{\text{otherwise}} \quad (3.68)$$

$$w(x,y,t) = x - 2y + 3t \quad (3.69)$$

which are exactly the same as the solutions obtained by LDM [19] and VIM [20].

4. Conclusion

This study shows that the Differential Transformation Method can be applied to solve some systems of ordinary and partial differential equations. The results in examples show that the solutions obtained with DTM coincide with the solutions obtained with other methods (VIM, LDM, ADM, etc.) in the literature. Besides this, the DTM method can be easily adapted to many computer programs. Because it uses only simple operations. For all these reasons, DTM can be used easily to solve some different type of ODEs and PDEs.

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