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Positive Solutions of Second-order Neutral Differential Equations with Distributed Deviating Arguments

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Abstract: In this study we shall obtain some sufficient conditions for the existence of positive solutions of variable coefficient nonlinear second order neutral differential equation with distributed deviating arguments. The main tool for proving our results is the Banach contraction principle. For this reason, we define a conversion and we show that it's a contraction transformation. The example at the end of the article is given to illustrate the effectiveness of our results. Our results improve and extend some existing results.

Dağıtılmış Sapma Argümentlerine Sahip İkinci Mertebeden Nötral Diferensiyel Denklemlerin Pozitif Çözümleri

Anahtar Kelimeler
Sabit Nokta,
Nötral Diferensiyel Denklem,
Pozitif Çözüm,
Dağıtılmış Sapma Argümenti

Öz: Bu çalışmada, dağıtılmış sapma argümentlere sahip değişken katsayılı lineer olmayan ikinci mertebeden nötral diferensiyel denklemlerin pozitif çözümlerinin varlığı için yeterli koşulları elde edeceğiz. Sonuçlarımızı kanıtlamanın ana aracı Banach daralma ilkesidir. Bunun için bir dönüşüm tanımlayıp daralma dönüşümü olduğunu göstereceğiz. Makalenin sonundaki örnek, sonuçlarımızın etkililiğini göstermek için verilmiştir. Sonuçlarımız bazı mevcut sonuçları iyileştirmekte ve genişletmektedir.

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1. Introduction

In this work we consider the second-order neutral nonlinear differential equation with distributed deviating arguments of form

$$\left(x(t) - \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi\right)'' + \int_{a_2}^{b_2} f\left(t, x\big(\sigma(t,\xi)\big)\right)d\xi = 0\,, \tag{1}$$

where $P(t,\xi) \in C([t_0,\infty) \times [\mathrm{a}_1,\mathrm{b}_1]\,,\mathbb{R})$ for $0 < a_1 < b_1$ and $\sigma(t,\xi) \in C([t_0,\infty) \times [\mathrm{a}_2,\mathrm{b}_2]\,,\mathbb{R})$ with $\lim_{t\to\infty} \sigma(t,\xi) = \infty$ and $0 \leq a_2 < b_2$.

In this paper, we assume that $f(t,x) \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$ is nondecreasing in $x, xf(t,x) > 0$ for $x \neq 0$ and satisfies

$$|f(t,x) - f(t,y)| \leq q(t)|x-y| \quad \text{for } t \in [t_0,\infty) \quad \text{and} \quad x,y \in [e,f], \tag{2}$$

where $q \in C([t_0,\infty)\,,\mathbb{R}^+)$ and $[e,f]$ $(0 < e < f$ or $e < f < 0)$ is any closed interval.

Furthermore, suppose that

$$\int_{t_0}^{\infty} sq(s)ds < \infty\,, \tag{3}$$

$$\int_{t_0}^{\infty} s|f(s,d|ds < \infty, \qquad \text{for some } d \neq 0. \tag{4}$$

The nonoscillatory behavior of solutions of neutral differential equations has been considered by different authors in the past. Yang, Zang and Ge in [1] concerned with the existence of nonoscillatory solutions of second-order differential equation of the form

$$(x(t) - p(t)x(t-\tau))'' + f_1\left(t, x(\sigma_1(t))\right) - f_2\left(t, x(\sigma_2(t))\right) = 0. \tag{5}$$

T. Candan and R. S. Dahiya in [2] studied with the existence of first and second order neutral differential equations of the form

$$\frac{d^k}{dt^k}[x(t) + P(t)x(t-\tau)] + \int_a^b q_1(t,\xi)x(t-\xi)d\xi - \int_c^d q_2(t,\mu)x(t-\mu)d\mu = 0. \tag{6}$$

This work was motivated by the (5) and (6) equations. For the other works and related books concerning oscillatory and nonoscillatory of neutral differential equations, we refer to [3-11].

The purpose of this article is to give some sufficient conditions for the nonoscillatory solutions of (1) according to different cases of the range of $p(t)$ by using Banach Contraction Principle.

Let $T_0 = \min\left\{t_1 - b_1, \inf_{t \geq t_1} \min_{\xi \in [a_2, b_2]} \sigma(t,\xi)\right\}$ for $t_1 \geq t_0$. By a solution of equation (1), we mean a function $x \in C([T_1,\infty), \mathbb{R})$ in the sense that $x(t) - \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi$ is two times continuously differentiable on $[t_1,\infty)$ and such that equation (1) is satisfied for $t \geq t_1$.

As is customary, a solution of (1) is said to be oscillatory if it has arbitrary large zeros. Otherwise the solution is called nonoscillatory.

2. Main Results

Theorem 2.1. *Assume that (3)-(4) hold,* $P(t,\xi) \geq 0$ *and* $\int_{a_1}^{b_1} P(t,\xi)d\xi \leq p < 1$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_2 = d$.

Set

$$A = \{x \in X : N_1 \leq x(t) \leq N_2, \ t \geq t_0\},$$

where N_1 and N_2 are positive constants such that

$$N_1 < (1-p)N_2.$$

It is obvious that A is a closed, bounded and convex subset of X. Because of (3)-(4), we can take a $t_1 > t_0$ sufficiently large such that $t - b_1 \geq t_0$, $\sigma(t,\xi) \geq t_0$, $\xi \in [a_i, b_i]$ for $t \geq t_1$, $i = 1,2$ and

$$p + \int_{t_1}^{\infty} s(b_2 - a_2)q(s)ds \leq \theta_1 < 1, \tag{7}$$

$$\int_{t_1}^{\infty} s(b_2 - a_2)f(s,d)ds \leq \alpha - N_1, \tag{8}$$

where $\alpha \in (N_1, (1-p)N_2]$. Define a mapping $S : A \to X$ as follows:

$$(Sx)(t) = \begin{cases} \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds, & t \geq t_1 \\ (Sx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that Sx is continuous. For every $x \in A$ and $t \geq t_1$ we can get

$$\begin{aligned}(Sx)(t) &= \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds \\ &\leq \alpha + pN_2 \\ &\leq N_2\end{aligned}$$

and taking (8) in to account, we can get

$$\begin{aligned}(Sx)(t) &= \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds \\ &\geq \alpha - \int_{t_1}^{\infty} s(b_2-a_2)f(s,d)ds \\ &\geq N_1.\end{aligned}$$

Thus we proved that $SA \subset A$. Now we shall show that S is a contraction mapping on A.

In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (7) we have

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| &\leq \int_{a_1}^{b_1} P(t,\xi)|x(t-\xi) - y(t-\xi)|d\xi \\ &\quad + \int_t^{\infty}(s-t)\int_{a_2}^{b_2}\left|f\left(s, x(\sigma(s,\xi))\right) - f\left(s, y(\sigma(s,\xi))\right)\right| d\xi\, ds \\ &\leq \int_{a_1}^{b_1} P(t,\xi)|x(t-\xi) - y(t-\xi)|d\xi \\ &\quad + \int_{t_1}^{\infty} s\int_{a_2}^{b_2} q(s)\left|x(\sigma(s,\xi)) - y(\sigma(s,\xi))\right| d\xi\, ds \\ &\leq \|x-y\|\left[p + \int_{t_1}^{\infty} s(b_2-a_2)q(s)ds\right] \\ &\leq \theta_1\|x-y\|,\end{aligned}$$

which implies the sup norm that

$$\|Sx - Sy\| \leq \theta_1\|x-y\|.$$

Since $\theta_1 < 1$, S is a contraction mapping on A. By Banach Contraction Mapping Principle, there exist a unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof.

Theorem 2.2. *Assume that (3)-(4) hold,* $P(t,\xi) \leq 0$ *and* $-1 < -p \leq \int_{a_1}^{b_1} P(t,\xi)d\xi \leq p$. *Then (1) has a bounded nonoscillatory solution.*

Proof. Suppose (4) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_4 = d$.

Set

$$A = \{x \in X : N_3 \leq x(t) \leq N_4,\ t \geq t_0\},$$

where N_3 and N_4 are positive constants such that

$$pN_4 + N_3 < N_4.$$

It is clear that A is a closed, bounded and convex subset of X. Because of (3)-(4), we can take a $t_1 > t_0$ sufficiently large such that $t - b_1 \geq t_0$, $\sigma(t,\xi) \geq t_0$, $\xi \in [a_i, b_i]$ for $t \geq t_1$, $i = 1,2$ and

$$p + \int_{t_1}^{\infty} s(b_2 - a_2)q(s)ds \leq \theta_2 < 1\,, \tag{9}$$

$$\int_{t_1}^{\infty} s(b_2 - a_2)f(s,d)ds \leq \alpha - pN_4 - N_3\,, \tag{10}$$

where $\alpha \in (pN_4 + N_3,\ N_4]$. Define a mapping $S : A \to X$ as follows:

$$(Sx)(t) = \begin{cases} \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds, & t \geq t_1 \\ (Sx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

It is easy to see that Sx is continuous. For every $x \in A$ and $t \geq t_1$ we can get

$$\begin{aligned} (Sx)(t) &= \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds \\ &\leq \alpha \\ &\leq N_4 \end{aligned}$$

and taking (10) in to account, we can get

$$\begin{aligned} (Sx)(t) &= \alpha + \int_{a_1}^{b_1} P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\int_{a_2}^{b_2} f\left(s, x(\sigma(s,\xi))\right)d\xi\, ds \\ &\geq \alpha - pN_4 - \int_{t_1}^{\infty} s(b_2 - a_2)f(s,d)ds \\ &\geq N_3. \end{aligned}$$

Thus we proved that $SA \subset A$. Now we shall show that S is a contraction mapping on A.

In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (9) we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| &\leq \int_{a_1}^{b_1}(-P(t,\xi))|x(t-\xi) - y(t-\xi)|d\xi \\ &\quad + \int_t^{\infty}(s-t)\int_{a_2}^{b_2}\left|f\left(s, x(\sigma(s,\xi))\right) - f\left(s, y(\sigma(s,\xi))\right)\right| d\xi\, ds \\ &\leq \int_{a_1}^{b_1}(-P(t,\xi))|x(t-\xi) - y(t-\xi)|d\xi \\ &\quad + \int_{t_1}^{\infty} s \int_{a_2}^{b_2} q(s)\left|x(\sigma(s,\xi)) - y(\sigma(s,\xi))\right| d\xi\, ds \\ &\leq \|x - y\|\left[\, p + \int_{t_1}^{\infty} s(b_2 - a_2)q(s)ds \,\right] \\ &\leq \theta_2\|x - y\|, \end{aligned}$$

which implies the sup norm that

$$\|Sx - Sy\| \leq \theta_2 \|x - y\|.$$

Since $\theta_2 < 1$, S is a contraction mapping on A. By Banach Contraction Mapping Principle, there exist a unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof.

Example 2.3. For $t > 4$, consider the equation

$$\left(x(t) - \int_3^4 \exp(t-\xi)\, x(t-\xi) d\xi\right)'' + \int_1^2 \exp(2\xi) x(t-2\xi)\, d\xi = 0. \quad (11)$$

Note that $P(t,\xi) = \exp(t-\xi)$, $\sigma(t,\xi) = t - 2\xi$, $f(t,u) = \exp(2\xi)u$. We can check that the conditions of Theorem 2.1 are all satisfied. We note that $x(t) = \exp(-t)$ is a nonoscillatory solution of (11).

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