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A Characterization of Factorable Surfaces in Euclidean 4-Space IE^4

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Abstract

In this paper, we consider a factorable surface in Euclidean space IE^4 with its curvature ellipse. We classify the origin of the normal space of such a surface according to whether it is hyperbolic, parabolic, or elliptic. Further, we give the necessary and sufficient condition of the factorable surface to become Wintgen ideal surface.

1. Introduction

Let S be a smooth surface given with the patch $X(u,v):(u,v)\in D\subset IE^2$ in IE^4. The tangent space to S at an arbitrary point $p=X(u,v)$ of S is spanned $\{X_u, X_v\}$. In the chart (u,v) the coefficients of first fundamental form of S are given by

$$E=\langle X_u, X_u\rangle,\ F=\langle X_u, X_v\rangle,\ G=\langle X_v, X_v\rangle,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. We assume that $W^2=EG-F^2\neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p\in S$, it is considered the decomposition $T_pIE^4=T_pS\oplus T_p^{\perp}S$ where $T_p^{\perp}S$ is the orthogonal component of T_pS in IE^4.

Let $\chi(S)$ and $\chi^{\perp}(S)$ be the spaces of smooth vector fields tangent to S and normal to S, respectively. Given any local vector fields X_1, X_2 tangent to S, it is considered the second fundamental map $h:\chi(S)\times\chi(S)\to\chi^{\perp}(S)$;

$$h(X_i,X_j)=\tilde{\nabla}_{X_i}X_j-\nabla_{X_i}X_j \quad 1\le i,j\le 2 \tag{1}$$

where ∇ and $\tilde{\nabla}$ are the induced connection of S and the Riemannian connection of IE^4, respectively. This map is well-defined, symmetric and bilinear [1].

For any arbitrary orthonormal frame field $\{N_1,N_2\}$ of S, recall the shape operator $A:\chi^{\perp}(S)\times\chi(S)\to\chi(S)$;

$$A_{N_k}X_j=-\left(\tilde{\nabla}_{X_j}N_k\right)^T,\ X_j\in\chi(S) \quad k=1,2.$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\begin{aligned}\langle A_{N_k}X_j, X_i\rangle&=\langle h(X_i,X_j), N_k\rangle\\&=c_{ij}^k,\quad 1\le i,j,k\le 2,\end{aligned} \tag{2}$$

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where c_{ij}^k are the coefficients of the second fundamental form [2].

The coefficients of the second fundamental form for a surface $S: X(u,v)$ in IE^n can be calculated by

$$\begin{aligned} c_{11}^k &= \langle X_{uu}, N_k \rangle, \\ c_{12}^k &= \langle X_{uv}, N_k \rangle, \quad 1 \le k \le n-2 \\ c_{22}^k &= \langle X_{vv}, N_k \rangle \end{aligned} \tag{3}$$

where $X_{uu} = \tilde{\nabla}_{X_u} X_u$, $X_{uv} = \tilde{\nabla}_{X_u} X_v$, $X_{vv} = \tilde{\nabla}_{X_v} X_v$.

Eq. (1) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \ 1 \le i, j \le 2. \tag{4}$$

Then, the Gaussian curvature and Gaussian torsion of a regular patch $X(u,v)$ are given by

$$K = \frac{1}{W^2} \sum_{k=1}^{2} \left(c_{11}^k c_{22}^k - \left(c_{12}^k\right)^2 \right) \tag{5}$$

and

$$K_N = \frac{1}{W^2} \begin{pmatrix} E\left(c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1\right) \\ -F\left(c_{11}^1 c_{22}^2 - c_{11}^2 c_{22}^1\right) \\ +G\left(c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1\right) \end{pmatrix} \tag{6}$$

,respectively.

Further, the mean curvature vector of a regular patch $X(u,v)$ is given by

$$H = \frac{1}{2W^2} \sum_{k=1}^{2} \left(c_{11}^k G + c_{22}^k E - 2c_{12}^k F \right) N_k. \tag{7}$$

The norm of the mean curvature vector $\|H\|$ is called the mean curvature of S.

A surface S is called a Wintgen ideal surface if it satisfies the equation $K + |K_N| = \|H\|^2$ [3]. This condition is related by the notion of curvature ellipse of a surface. Curvature ellipses of some surfaces and especially Wintgen ideal surfaces are investigated in studies [3, 4, 5, 6, 7, 8, 9].

Factorable surfaces (also known homotethical surfaces) can be parametrized, locally, as $X(u,v) = (u, v, f(u)g(v))$, where f and g smooth functions [10, 11]. Some authors have considered factorable surfaces in Euclidean and semi-Euclidean spaces [11, 12, 13, 14]. In [10], Van de Woestyne proved that the only minimal factorable non-degenerate surfaces in IL^3 are planes and helicoids. Recently, the authors have studied spacelike factorable surfaces in four dimensional Minkowski space [15].

In the present study, we consider a factorable surface which locally can be written as a monge patch

$$X(u,v) = (u, v, f_1(u)g_1(v), f_2(u)g_2(v)),$$

for some differentiable functions, $f_i(u), g_i(v), i = 1,2$ [16, 17]. We characterize the factorable surfaces in Euclidean $4-$space with regards to their curvature ellipses. We classify the origin of normal space of a surface according to whether it is hyperbolic, parabolic, or elliptic. Further, we calculate Gaussian curvature, the normal curvature and mean curvature of the surface and give the necessary and sufficient condition for the factorable surfaces to become Wintgen ideal surface.

2. The Notion of Curvature Ellipse

Let $S \subset IE^4$ be a surface given with the regular patch $X(u,v)$ and consider a circle with the parameter $\theta \in [0, 2\pi]$ in the tangent space T_pS on the point p. Let the curve $\gamma(\theta)$ indicate the intersection of the surface S and the hyperplane which is the direct sum of the normal plane on the point p and the tangent vector

$$X = \cos\theta X_1 + \sin\theta X_2$$

where X_1, X_2 are orthonormal base of T_pS. This curve is called normal section curve of S on p in the direction X. Additionally, the normal curvature vector η_θ is a vector that lies on the normal plane. Varying θ from 0 to 2π, this vector defines an ellipse on normal plane. This ellipse is called as curvature ellipse of S on p. Thus; the curvature ellipse of S on p is defined by

$$E(p) = \{ h(X,X) : \quad X \in T_pS, \quad \|X\| = 1 \} \tag{8}$$

where $\gamma_\theta' = X = \cos\theta X_1 + \sin\theta X_2$ is the unit tangent vector of normal section curve and h is the second fundamental form of the patch $X(u,v)$. The reason why the Eq. (8) indicates an ellipse is that the second fundamental form satisfies the relation

$$h(X,X) = H + \cos\theta B + \sin\theta C,$$

where

$$B = \frac{1}{2}\left(h(X_1, X_1) + h(X_2, X_2)\right),\ C = h(X_1, X_2)$$

are normal vectors and H is the mean curvature vector of S. While the unit vector X makes one turn around the unit circle, the vector $h(X,X)$ takes two turns around the ellipse centered at H. It is possible that the ellipse $E(p)$ can degenerate into a point or a straight line.

Definition 1. Let $S \subset IE^4$ be a surface given with the regular patch $X(u,v)$. If the condition

$$\langle B, C\rangle = 0 \text{ and } \|B\| = \|C\|$$

is satisfied, then the curvature ellipse of S is congruent to a circle.

Remark 1. The curvature ellipse is congruent to a circle if and only if

$$\|H\|^2 - K - |K_N| = 0$$

is hold. The surface is called superconformal if its curvature ellipse is congruent to a circle. Thus, Wintgen ideal surfaces are superconformal at the same time.

Definition 2. Let $S \subset IE^4$ be a surface given with the regular patch $X(u,v)$. The determinant $\Delta(p)$ and the matrice $\alpha(p)$ are defined by

$$\Delta(p) = \frac{1}{W^2}\det\begin{pmatrix} c_{11}^1 & 2c_{12}^1 & c_{22}^1 & 0 \\ c_{11}^2 & 2c_{12}^2 & c_{22}^2 & 0 \\ 0 & c_{11}^1 & 2c_{12}^1 & c_{22}^1 \\ 0 & c_{11}^2 & 2c_{12}^2 & c_{22}^2 \end{pmatrix}(p) \tag{9}$$

and

$$\alpha(p) = \begin{pmatrix} c_{11}^1 & 2c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 & c_{22}^2 \end{pmatrix}(p) \tag{10}$$

where c_{ij}^k, $(i,j,k = 1,2)$ are the coefficients of the second fundamental form of S [18].

The following classification can be given for the origin p of the normal space $T_p^{\perp}S$:

(i) If $\Delta(p) < 0$, then the point p is outside the curvature ellipse $E(p)$ and is called hyperbolic point.

(ii) If $\Delta(p) = 0$, then the point p is on the curvature ellipse $E(p)$ and is called parabolic point. According to this

(a) If $\Delta(p) = 0$ and $K(p) > 0$, then the point p is an inflection point of imaginary type.

(b) If $\Delta(p) = 0$ and $K(p) < 0$, then

$\operatorname{rank}\alpha(p) = 2 \Rightarrow$ the point p is non-degenerate.

$\operatorname{rank}\alpha(p) = 1 \Rightarrow$ the point p is an inflection point of real type.

(c) If $\Delta(p) = 0$ and $K(p) = 0$, then the point p is an inflection point of real type.

(iii) If $\Delta(p) > 0$, then the point p is inside the curvature ellipse $E(p)$ and is called elliptic point.

3. Factorable Surfaces in IE^4

Definition 3. Let S be a surface in $4-$dimensional Euclidean space IE^4. If the surface is given by an explicit form $z(u,v) = f_1(u)g_1(v)$ and $w(u,v) = f_2(u)g_2(v)$ where u, v, z, w are Cartesian coordinates in IE^4 and $f_i, g_i,\quad i \in \{1,2\}$ are smooth functions, then the surface is called a factorable surface in IE^4. Thus, the factorable surface can be written as a monge patch

$$X(u,v) = \left(u, v, f_1(u)g_1(v), f_2(u)g_2(v)\right). \tag{11}$$

Let S be a factorable surface with the parametrization Eq. (11). Then, we have the following:

The tangent space of S is spanned by the vector fields

$$X_u = \left(1, 0, f_1'(u)g_1(v), f_2'(u)g_2(v)\right),$$
$$X_v = \left(0, 1, f_1(u)g_1'(v), f_2(u)g_2'(v)\right).$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} E &= \langle X_u, X_u\rangle = 1 + (f_1'g_1)^2 + (f_2'g_2)^2 \\ F &= \langle X_u, X_v\rangle = f_1'f_1g_1'g_1 + f_2'f_2g_2'g_2 \\ G &= \langle X_v, X_v\rangle = 1 + (f_1g_1')^2 + (f_2g_2')^2 \end{aligned} \tag{12}$$

where $\langle\ ,\ \rangle$ is standard scalar product in IE^4. Since the surface S is non-degenerate, then $\|X_u \times X_v\| = \sqrt{EG - F^2} \neq 0$. For the later use we define a smooth function W as $W = \|X_u \times X_v\|$.

The second partial derivatives of $X(u,v)$ are expressed as follows;

$$\begin{aligned} X_{uu} &= (0,0,f_1''(u)g_1(v),f_2''(u)g_2(v)), \\ X_{uv} &= (0,0,f_1'(u)g_1'(v),f_2'(u)g_2'(v)), \\ X_{vv} &= (0,0,f_1(u)g_1''(v),f_2(u)g_2''(v)). \end{aligned} \tag{13}$$

Further, the normal space of S is spanned by the orthonormal vector fields

$$N_1 = \frac{1}{\sqrt{\tilde{E}}}(-f_1'(u)g_1(v),-f_1(u)g_1'(v),1,0),$$

$$N_2 = \frac{1}{\sqrt{\tilde{E}}W}\begin{pmatrix} \tilde{F}f_1'(u)g_1(v)-\tilde{E}f_2'(u)g_2(v), \\ \tilde{F}f_1(u)g_1'(v)-\tilde{E}f_2(u)g_2'(v),-\tilde{F},\tilde{E} \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} \tilde{E} &= 1+(f_1'g_1)^2+(f_1g_1')^2, \\ \tilde{F} &= f_1'f_2'g_1g_2+f_1f_2g_1'g_2', \\ \tilde{G} &= 1+(f_2'g_2)^2+(f_2g_2')^2. \end{aligned} \tag{15}$$

Also, $\tilde{E}\tilde{G}-\tilde{F}^2 = W^2$. Using Eq. (13) and Eq. (14), we can calculate the coefficients of the second fundamental form as follows;

$$\begin{aligned} c_{11}^1 &= \frac{f_1''g_1}{\sqrt{\tilde{E}}}, \quad c_{22}^1 = \frac{f_1g_1''}{\sqrt{\tilde{E}}} \\ c_{12}^1 &= \frac{f_1'g_1'}{\sqrt{\tilde{E}}}, \quad c_{12}^2 = \frac{\tilde{E}f_2'g_2'-\tilde{F}f_1'g_1'}{\sqrt{\tilde{E}}W}, \\ c_{11}^2 &= \frac{\tilde{E}f_2''g_2-\tilde{F}f_1''g_1}{\sqrt{\tilde{E}}W}, \\ c_{22}^2 &= \frac{\tilde{E}f_2g_2''-\tilde{F}f_1g_1''}{\sqrt{\tilde{E}}W}. \end{aligned} \tag{16}$$

3.1. Curvature Ellipse of the Factorable Surface

Theorem 1 Let S be a factorable surface given with the parametrization Eq. (11) in Euclidean $4-$space $\mathbb{E}^4$. Then the origin p of $T_p^{\perp}S$ can be characterized by the followings:

(i) If

$$\begin{aligned} &4(f_1''g_1f_2'g_2'-f_2''g_2f_1'g_1')(f_2g_2''f_1'g_1'-f_1g_1''f_2'g_2') \\ &= (f_1''g_1f_2g_2''-f_1g_1''f_2''g_2)^2 \end{aligned} \tag{17}$$

is hold, then the point p is on the curvature ellipse. This point is the parabolic point of S.

(ii) If

$$\begin{aligned} &4(f_1''g_1f_2'g_2'-f_2''g_2f_1'g_1')(f_2g_2''f_1'g_1'-f_1g_1''f_2'g_2') \\ &< (f_1''g_1f_2g_2''-f_1g_1''f_2''g_2)^2 \end{aligned} \tag{18}$$

is hold, then the point p is outside the curvature ellipse. This point is the hyperbolic point of S.

(iii) If

$$\begin{aligned} &4(f_1''g_1f_2'g_2'-f_2''g_2f_1'g_1')(f_2g_2''f_1'g_1'-f_1g_1''f_2'g_2') \\ &> (f_1''g_1f_2g_2''-f_1g_1''f_2''g_2)^2 \end{aligned} \tag{19}$$

is hold, then the point p is inside the curvature ellipse. This point is the elliptic point of S.

Proof. Let S be a factorable surface given with the parametrization Eq. (11) in Euclidean $4-$space $\mathbb{E}^4$. By the use of the Equation (9) and the second fundamental form coefficients, we get

$$\Delta(p) = \frac{\begin{bmatrix} 4(f_1''g_1f_2'g_2'-f_2''g_2f_1'g_1')(f_2g_2''f_1'g_1'-f_1g_1''f_2'g_2') \\ -(f_1''g_1f_2g_2''-f_1g_1''f_2''g_2)^2 \end{bmatrix}}{4W^2}$$

.

With the help of the Definition 2, if $\Delta(p)<0$, then the point p is outside the curvature ellipse. If $\Delta(p)>0$, then the point p is outside the curvature ellipse. If $\Delta(p)=0$, then the point p is on the curvature ellipse. Thus, desired result is obtained.

Example 1. Let S be a factorable surface given with the parametrization Eq. (11) in Euclidean $4-$space $\mathbb{E}^4$. If the functions are choosen as:

$$\begin{aligned} f_1(u) &= 3u+1, \quad g_1(v) = \cos v, \\ f_2(u) &= 2u+2, \quad g_2(v) = \sin v, \end{aligned} \tag{20}$$

then the origin p of $T_p^{\perp}S$ is on the curvature ellipse. p is parabolic point. Also, projection of the surface onto $\mathbb{E}^3$ is given in Figure 1.

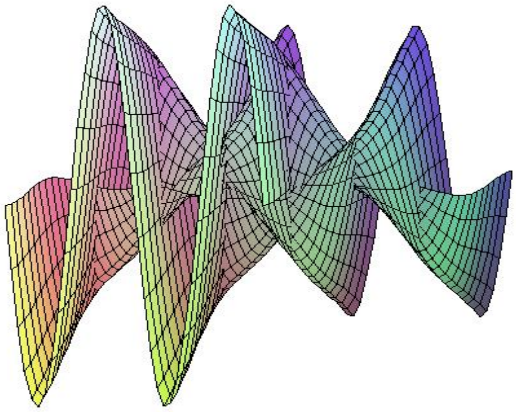

Figure 1. A Factorable Surface Given by the Functions in Eq. (20)

3.2. Wintgen Ideal Surface

In 1979, P. Wintgen [3] proved a basic relationship between the Gaussian curvature K, normal curvature K_N, and the mean curvature H of a surface S in Euclidean $4-$space $\mathbb{E}^4$ that is

$$K+|K_N|\leq\|H\|^2. \tag{21}$$

The equality is hold if and only if the curvature ellipse is a circle.

Definition 4. A surface S in $\mathbb{E}^4$ is called a Wintgen ideal surface if it satisfies the equality case in the inequality (21), namely

$$K+|K_N|=\|H\|^2. \tag{22}$$

Theorem 2. Let S be a factorable surface given with the parameterization (11) in Euclidean $4-$space $\mathbb{E}^4$. Then the Gaussian curvature of S is

$$K=\frac{\begin{bmatrix}\left(f_1''f_1g_1''g_1-f_1'^2g_1'^2\right)\tilde{G}\\ -\left(f_1''f_2g_1g_2''+f_1f_2''g_1''g_2-2f_1'f_2'g_1'g_2'\right)\tilde{F}\\ +\left(f_2''f_2g_2''g_2-f_2'^2g_2'^2\right)\tilde{E}\end{bmatrix}}{W^4} \tag{23}$$

where $\tilde{E}$, $\tilde{F}$, and $\tilde{G}$ are given by the Equation (15).

Proof. Let S be a factorable surface in $\mathbb{E}^4$. Substituting the second fundamental form coefficients of S into the Eq. (5), we obtain the desired result.

Theorem 3. Let S be a factorable surface given with the parametrization Eq. (11) in Euclidean $4-$space $\mathbb{E}^4$. Then the normal curvature function of S is

$$K_N=\frac{\begin{bmatrix}E\left(f_1'f_2g_1'g_2''-f_1f_2'g_1''g_2'\right)\\ -F\left(f_1''f_2g_1g_2''-f_1f_2''g_1''g_2\right)\\ +G\left(f_1''f_2'g_1g_2'-f_1'f_2''g_1'g_2\right)\end{bmatrix}}{W^4} \tag{24}$$

where E, F, and G are the first fundamental form coefficients which are given by Eq. (12).

Proof. Let S be a factorable surface in $\mathbb{E}^4$. Substituting the second fundamental form coefficients of S into the Eq. (6), we obtain the desired result.

Theorem 4. Let S be a factorable surface given with the parameterization Eq. (11) in Euclidean $4-$space $\mathbb{E}^4$. Then the mean curvature of S is

$$H=\frac{f_1''g_1G+f_1g_1''E-2f_1'g_1'F}{2\sqrt{\tilde{E}}W^2}N_1+\frac{\begin{bmatrix}\tilde{E}\left(f_2''g_2G+f_2g_2''E-2f_2'g_2'F\right)\\ -\tilde{F}\left(f_1''g_1G+f_1g_1''E-2f_1'g_1'F\right)\end{bmatrix}}{2\sqrt{\tilde{E}}W^3}N_2. \tag{25}$$

Proof. Let S be a factorable surface in $\mathbb{E}^4$. Substituting the second fundamental form coefficients of S into the Eq. (7), we obtain the desired result.

Theorem 5. Let S be a factorable surface given with the parametrization Eq. (11) in $\mathbb{E}^4$. Then the surface is Wintgen ideal (superconformal) surface if and only if

$$\begin{aligned}&\tilde{G}\left[\left(f_1''g_1G+f_1g_1''E-2f_1'g_1'F\right)-4W^2\left(f_1''f_1g_1''g_1-f_1'^2g_1'^2\right)\right]\\ &+\tilde{E}\begin{bmatrix}\left(f_2''g_2G+f_2g_2''E-2f_2'g_2'F\right)\\ -4W^2\left(f_2''f_2g_2''g_2-f_2'^2g_2'^2\right)\end{bmatrix}\\ &-2\tilde{F}\begin{bmatrix}\left(f_1''g_1G+f_1g_1''E-2f_1'g_1'F\right)\\ \left(f_2''g_2G+f_2g_2''E-2f_2'g_2'F\right)\\ -2W^2\left(f_1''f_2g_1g_2''+f_1f_2''g_1''g_2-2f_1'f_2'g_1'g_2'\right)\end{bmatrix}\\ &=\pm4W^2\begin{bmatrix}E\left(f_1'f_2g_1'g_2''-f_1f_2'g_1''g_2'\right)\\ -F\left(f_1''f_2g_1g_2''-f_1f_2''g_1''g_2\right)\\ +G\left(f_1''f_2'g_1g_2'-f_1'f_2''g_1'g_2\right)\end{bmatrix}\end{aligned}$$

is hold.

Proof. Let S be a factorable surface given with the parameterization Eq. (11) in $\mathbb{E}^4$. Substituting Eq. (23), (24), and (25) into the relation (22), we obtain the desired result.

Example 2. The plane given with the parameterization

$$X(u,v) = (u, v, c_1 v + c_2, c_3 u + c_4) \tag{26}$$

is obviously a Wintgen ideal and factorable surface in $\mathbb{E}^4$ where $f_1(u) = 1$, $g_2(v) = 1$, and c_i, $i = 1,..,4$ are real constants.

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