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INEQUALITIES INVOLVING k-CHEN INVARIANTS FOR SUBMANIFOLDS OF RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. An optimal inequality involving the scalar curvatures, the mean curvature and the k-Chen invariant is established for Riemannian submanifolds. Particular cases of this inequality is reported. Furthermore, this inequality is investigated on submanifolds, namely slant, F-invariant and F-anti invariant submanifolds of an almost constant curvature manifold.

1. Introduction

Riemannian invariants have an essential role in Riemannian geometry since they affect the intrinsic features of Riemannian manifolds. In this manner, these invariants are considered as DNA of a Riemannian manifold (cf. [6]). The most fundamental notions in Riemannian invariants are curvature invariants. Curvature invariants play key roles in physics as in geometry. According to Newton's laws, the magnitude of a force, required to move an object at constant speed, is a constant multiple of the curvature of the trajectory. According to Einstein, the motion of a body in a gravitational field is determined by the curvatures of space time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures (cf. [15]).

The main extrinsic curvature invariant is the squared mean curvature and the main intrinsic curvature invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. In [4], B.-Y. Chen introduced a new curvature invariant, now known as (first) Chen invariant. In [8], he introduced and investigated two strings of new types of curvature invariants. These new curvature invariants seem to play significant roles in several areas of mathematics including submanifold theory and Riemannian, spectral and symplectic geometries. For more details, we refer to [7] and [10].

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Beside these facts, the theory of almost product manifolds and their submanifolds have been developed in a similar manner with theories of almost complex manifolds and almost contact manifolds. In [16], S. Tachibana firstly introduced locally product manifolds and then submanifolds of locally product manifolds have been intensely studied by various geometers. Invariant and anti-invariant submanifold of a locally product manifold were studied by T. Adati in [1], semi-slant submanifolds of a locally product manifold were investigated by A. Bejancu in [3], slant submanifolds of Riemannian product manifolds were presented by B. Sahin in [14] and M. Atçeken in [2], almost semi-invariant submanifolds of a locally product manifold were studied by the second author in [17], and skew semi-invariant submanifolds (which are a special class of almost-semi-invariant submanifolds) of a locally product manifold were studied by X. Liu and F.-M. Shao in [13]. Recently, proper slant surfaces of locally product Riemannian manifolds were investigated by the first and third authors and S. Saraçoğlu Çelik [11]. Finally, Chen-Ricci inequalities for slant submanifolds of a Riemannian product manifold were established by the authors in [12].

Based on the above presented facts, we are going to give some relations involving the Chen invariants, the intrinsic and extrinsic curvature invariants of a Riemannian submanifold. Also, we are going to investigate these relations on submanifolds of a Riemannian product manifold and an almost constant curvature manifold.

2. Riemannian submanifolds

In this section, we are going to focus on some basic facts about Riemannian submanifolds by following the notations and formulas used in [7] and [10].

Let $(\widetilde{M},\widetilde{g})$ be an m-dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and (M,g) be submanifold of $(\widetilde{M},\widetilde{g})$ such that g is just the restriction of $\widetilde{g}$. For all vector fields X and Y in the tangent bundle TM and N in the normal bundle $T^{\perp}M$, the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma\left(X,Y\right) \tag{2.1}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, and σ is the second fundamental form related to the shape operator A_N by

$$\langle\sigma\left(X,Y\right),N\rangle = \langle A_N X, Y\rangle\,. \tag{2.3}$$

Here, $\langle\,,\rangle$ denotes the inner product notation for both the metric $\widetilde{g}$ and the induced metric g.

Let $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M respectively. For all $X, Y, Z, W \in TM$, the following relation between these tensors holds:

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) + \langle \sigma(X,W), \sigma(Y,Z) \rangle \\ &- \langle \sigma(X,Z), \sigma(Y,W) \rangle \end{aligned} \tag{2.4}$$

We note that the equation (2.4) is known as *the Gauss equation*.

Now, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, $p \in M$. The mean curvature vector, denoted by $H(p)$, is defined by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i). \tag{2.5}$$

The submanifold M is called *totally geodesic* in $\widetilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma(X,Y) = g(X,Y) H$ for all $X, Y \in TM$, then M is called *totally umbilical.*

Suppose that e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. Then we can write

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle. \tag{2.6}$$

In view of (2.4) and (2.6), we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right), \tag{2.7}$$

where K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$ respectively. Thus, we can say that K_{ij} and $\widetilde{K}_{ij}$ are the "intrinsic" and "extrinsic" sectional curvatures of the Span$\{e_i, e_j\}$ at p. From (2.7), it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2, \tag{2.8}$$

where $\widetilde{\tau}(T_pM)$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$ defined by

$$\widetilde{\tau}(T_pM) = \sum_{1 \le i < j \le n} \widetilde{K}_{ij}.$$

Thus, we can say that $\tau(p)$ and $\widetilde{\tau}(T_pM)$ are the "intrinsic" and "extrinsic" scalar curvature of the submanifold at p respectively.

The *relative null space* of a Riemannian submanifold M at p is defined by [5]

$$\mathcal{N}_p = \{X \in T_pM \mid \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\},$$

which is also known as the *kernel of the second fundamental form* [8].

3. Chen invariants

Let (M, g) be an n-dimensional Riemannian (sub)manifold and Π_k be a k-plane section of T_pM. Suppose that $\{e_1, \ldots, e_k\}$ to be an orthonormal basis of Π_k. For each $2 \leq i < k$, *k-Ricci curvature* of Π_k at e_i, denoted $\mathrm{Ric}_{\Pi_k}(X)$, is defined by [5]

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^{k} K_{ij}. \tag{3.1}$$

We note that

a. if $k = n$, then $\Pi_n = T_pM$ and an *n-Ricci curvature* $\mathrm{Ric}_{T_pM}(e_i)$ is the usual *Ricci curvature* of e_i, denoted $\mathrm{Ric}\,(e_i)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and for a fixed $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Ric}_{T_pM}(e_i) \equiv \mathrm{Ric}\,(e_i) = \sum_{j \neq i}^{n} K_{ij}.$$

b. if $k = 2$, then Π is a plane section of T_pM and the 2-Ricci curvature becomes the sectional curvature.

The scalar curvature $\tau\,(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau\,(\Pi_k) = \sum_{1 \leq i < j \leq k} K_{ij} \tag{3.2}$$

In view of (3.2), we get

$$\tau\,(\Pi_k) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j \neq i}^{k} K_{ij} = \frac{1}{2} \sum_{i=1}^{k} \mathrm{Ric}_{\Pi_k}(e_i). \tag{3.3}$$

The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is,

$$\tau\,(p) = \tau\,(T_pM)\,.$$

If Π_k is a 2-plane section, $\tau(\Pi_k)$ is nothing but the sectional curvature $K\,(\Pi_k)$ of Π_k. Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p.

Now, we shall recall the following definition of B.-Y. Chen in [9]:

Definition 1. *Let (M, g) be an n-dimensional Riemannian (sub)manifold. For $2 \leq k \leq n-1$, the k-Chen invariant δ_M^k is defined to be*

$$\delta_M^k(p) = \tau(p) - (\inf \tau\,(\Pi_k))\,(p)\,, \tag{3.4}$$

where

$$(\inf \tau\,(\Pi_k))\,(p) = \inf\{\tau(\Pi_k)\,|\,\Pi_k \text{ is a } k\text{-plane section} \subset T_pM\}.$$

We note that

a. if $k=2$, δ_M^k reduces to the well known Chen invariant [4] of M given by

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

b. if $k=n-1$, δ_M^k reduces to the maximum Ricci curvature of M given by

$$\widehat{\mathrm{Ric}}(p) = \max\left\{\mathrm{Ric}(X) \mid X \in T_p^1 M\right\} = \tau(p) - (\inf\tau(\Pi_{n-1}))(p).$$

Now, we are going to give the following algebraic lemma:

Lemma 1. *If $2 \leq k < 2$ and $a_1, \ldots, a_n, a$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + a\right), \tag{3.5}$$

then

$$2\sum_{1\leq i<j\leq k} a_i a_j \geq a,$$

with equality holding if and only if

$$a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n.$$

Proof. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq (n-k+1)((a_1+a_2+\cdots+a_k)^2 + a_{k+1}^2 + \cdots + a_n^2). \tag{3.6}$$

From (3.5) and (3.6), we get

$$\sum_{i=1}^{n} a_i^2 + a \leq (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}^2 + \cdots + a_n^2.$$

The above equation is equivalent to

$$2\sum_{1\leq i<j\leq k} a_i a_j \geq a.$$

The equality holds if and only if $a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n$. □

Theorem 1. *Let M be an n-dimensional $(n \geq 3)$ submanifold in an m-dimensional Riemannian manifold $\widetilde{M}$. Then, for each point $p \in M$ and each k-plane section $\Pi_k \subset T_pM$ $(n > k \geq 2)$, we have*

$$\delta_M^k(p) \leq \frac{n^2(n-k)}{2(n-k+1)}\|H\|^2 + \widetilde{\tau}(T_pM) - \widetilde{\tau}(\Pi_k). \tag{3.7}$$

The equality in (3.7) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such *that **(a)** $\Pi_k = \operatorname{Span}\{e_1, \dots, e_k\}$ and **(b)** the forms of shape operators A_{e_r}, $r = n+1, \dots, m$, become*

$$A_{e_{n+1}} = \begin{pmatrix} \begin{matrix} \sigma_{11}^{n+1} & 0 & \cdots & 0 \\ 0 & \sigma_{22}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{kk}^{n+1} \end{matrix} & 0 \\ 0 & \left(\sum_{i=1}^{k} \sigma_{ii}^{n+1}\right) I_{n-k} \end{pmatrix}, \tag{3.8}$$

$$A_{e_r} = \begin{pmatrix} \begin{matrix} \sigma_{11}^{r} & \sigma_{12}^{r} & \cdots & \sigma_{1k}^{r} \\ \sigma_{12}^{r} & \sigma_{22}^{r} & \cdots & \sigma_{2k}^{r} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k}^{r} & \sigma_{2k}^{r} & \cdots & -\sum_{i=1}^{k-1} \sigma_{ii}^{r} \end{matrix} & 0 \\ 0 & 0_{n-k} \end{pmatrix}, \qquad r \in \{n+2, \dots, m\}. \tag{3.9}$$

Proof. Let $\Pi_k \subset T_pM$ be a k-plane section. We choose an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for T_pM and $\{e_{n+1}, \dots, e_m\}$ for the normal space $T_p^{\perp}M$ at p such that $\Pi_k = \operatorname{Span}\{e_1, \dots, e_k\}$, the mean curvature vector H is in the direction of the normal vector to e_{n+1}, and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \operatorname{diag}\left(\sigma_{11}^{n+1}, \sigma_{22}^{n+1}, \dots, \sigma_{nn}^{n+1}\right), \tag{3.10}$$

$$A_{e_r} = \left(\sigma_{ij}^{r}\right), \quad \operatorname{trace} A_{e_r} = \sum_{i=1}^{n} \sigma_{ii}^{r} = 0 \tag{3.11}$$

for all $i, j = 1, \dots, n$ and $r = n+2, \dots, m$. Thus, we rewrite (2.8) as

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} \left(\sigma_{ii}^{n+1}\right)^2 + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{r}\right)^2 + \omega\right), \tag{3.12}$$

where

$$\omega = 2\tau(p) - 2\widetilde{\tau}(T_pM) - \frac{n^2(n-2)}{n-1}\|H\|^2. \tag{3.13}$$

Applying Lemma 1 to equation (3.12), we get

$$2 \sum_{1 \le i < j \le k} \sigma_{ii}^{n+1} \sigma_{jj}^{n+1} \ge \omega + \sum_{r=n+2}^{m} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{r}\right)^2. \tag{3.14}$$

From equation (2.7) it also follows that

$$\tau(\Pi_k) = \widetilde{\tau}(\Pi_k) + \sum_{1 \le i < j \le k} \sigma_{ii}^{n+1} \sigma_{jj}^{n+1} + \sum_{r=n+2}^{m} \sum_{1 \le i < j \le k} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - \left(\sigma_{ij}^{r}\right)^2\right). \tag{3.15}$$

From (3.14) and (3.15) we get

$$\begin{aligned}\tau(\Pi_k) \quad &\geq \quad \widetilde{\tau}(\Pi_k) + \frac{1}{2}\omega + \sum_{r=n+2}^{m}\sum_{j>k}\{(\sigma_{1j}^r)^2 + (\sigma_{2j}^r)^2 + \cdots + (\sigma_{kj}^r)^2\} \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{m}(\sigma_{11}^r + \sigma_{22}^r + \cdots + \sigma_{kk}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{m}\sum_{i,j>k}(\sigma_{ij}^r)^2,\end{aligned}$$

or

$$\tau(\Pi_k) \geq \widetilde{\tau}(\Pi_k) + \frac{1}{2}\omega. \tag{3.16}$$

In view of (3.13) and (3.16), we get (3.7).

If the equality in (3.7) holds, then the inequalities given by (3.14) and (3.16) become equalities. In this case, for $r = n+2, \ldots, m$ we have

$$\begin{cases} \sigma_{1j}^{n+1} = \sigma_{2j}^{n+1} = \sigma_{kj}^{n+1} = 0, \qquad j = k+1, \ldots, n, \\ \sigma_{ij}^r = 0, \qquad i, j = k+1, \ldots, n, \\ \sigma_{11}^r + \sigma_{22}^r + \cdots + \sigma_{kk}^r = 0. \end{cases} \tag{3.17}$$

Applying Lemma 1 we also have

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} + \cdots + \sigma_{kk}^{n+1} = \sigma_{ll}^{n+1}, \qquad l = k+1, \ldots, n. \tag{3.18}$$

Thus, after choosing a suitable orthonormal basis $\{e_1, \ldots, e_m\}$, the shape operator of M becomes of the form given by (3.8) and (3.9). The converse is easy to follow. □

In particular case of $k = 2$, we have the following:

Theorem 2. *Let M be an n-dimensional $(n \geq 3)$ submanifold in an m-dimensional Riemannian manifold $\widetilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi_2 \subset T_pM$, we have*

$$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \widetilde{\tau}(T_pM) - \widetilde{K}(\Pi_2). \tag{3.19}$$

*The equality in (3.19) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that **(a)** $\Pi_2 = \mathrm{Span}\{e_1, e_2\}$ and **(b)** the forms of shape operators A_{e_r}, $r = n+1, \ldots, m$, become*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{pmatrix}, \tag{3.20}$$

$$A_{e_r} = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \ldots, m\}. \tag{3.21}$$

In particular case of $k = n-1$, we have the following:

Theorem 3. *Let M be an n-dimensional submanifold in a Riemannian manifold. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. Then,*

1. *For each unit vector $U \in T_pM$, we have*

$$\mathrm{Ric}\,(U) \leq \frac{1}{4}n^2\|H\|^2 + \widetilde{\tau}_{(T_pM)}\,(U)\,. \tag{3.22}$$

2. *If the mean curvature $H(p) = 0$, then a unit vector $U \in T_pM$ satisfies the equality case of (3.22) if and only if U lies in the relative null space $\mathcal{N}_p$ at p.*

3. *The equality case of (3.22) holds for all unit vectors $U \in T_pM$, if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let M be an n-dimensional submanifold in an m-dimensional Riemannian manifold $\widetilde{M}$. Now, we use Theorem 1. Thus, for each point $p \in M$ and each $(n-1)$-plane section $\Pi_{n-1} \subset T_pM$, we have

$$\tau\,(p) - K\,(\Pi_{n-1}) \leq \frac{1}{4}\,\|H\|^2 + \widetilde{\tau}\,(T_pM) - \widetilde{K}\,(\Pi_{n-1})\,. \tag{3.23}$$

The equality in (3.23) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that **(a)** $\Pi_k = \mathrm{Span}\,\{e_1, \ldots, e_{n-1}\}$ and **(b)** the forms of shape operators A_{e_r}, $r = n+1, \ldots, m$, become

$$A_{e_{n+1}} = \begin{pmatrix} \sigma_{11}^{n+1} & 0 & \cdots & 0 & 0 \\ 0 & \sigma_{22}^{n+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{(n-1)(n-1)}^{n+1} & 0 \\ 0 & 0 & \cdots & 0 & \left(\sum\limits_{i=1}^{n-1} \sigma_{ii}^{n+1}\right) \end{pmatrix}, \tag{3.24}$$

$$A_{e_r} = \begin{pmatrix} \sigma_{11}^{r} & \sigma_{12}^{r} & \cdots & \sigma_{1(n-1)}^{r} & 0 \\ \sigma_{12}^{r} & \sigma_{22}^{r} & \cdots & \sigma_{2(n-1)}^{r} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{1(n-1)}^{r} & \sigma_{2(n-1)}^{r} & \cdots & -\sum\limits_{i=1}^{n-2} \sigma_{ii}^{r} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad r \in \{n+2, \ldots, m\}\,. \tag{3.25}$$

Now, we assume that the unit vector U is e_n. Then, from (3.23), then we get (3.22). Assuming $U = e_n$ from (3.24) and (3.25), we see that the equality in (3.22) is valid if and only if

$$\begin{cases} \sigma_{nn}^{r} = \sigma_{11}^{r} + \sigma_{22}^{r} + \cdots + \sigma_{(n-1)(n-1)}^{r} \\ \sigma_{1n}^{r} = \sigma_{2n}^{r} = \cdots = \sigma_{(n-1)n}^{r} = 0. \end{cases} \tag{3.26}$$

for $r \in \{n+1, \ldots, m\}$. If $H(p) = 0$, then (3.26) implies that $U = e_n$ lies in the relative null space $\mathcal{N}_p$. Conversely, if $U = e_n$ lies in the relative null space, then (3.26) is true because $H(p) = 0$ is assumed. Thus **(2)** is proved.

Now we prove **(3)**. Assuming the equality case of (3.22) for all unit tangent vectors to M at p, in view of (3.26), for each $r \in \{n+1, \ldots, m\}$, we have

$$\begin{cases} 2\sigma_{ii}^r = \sigma_{11}^r + \sigma_{22}^r + \cdots + \sigma_{nn}^r, \\ \sigma_{ij}^r = 0, \quad i \neq j \end{cases} \tag{3.27}$$

for all $i \in \{1, ..., n\}$ and $r \in \{n+1, \ldots, m\}$. Thus, we have two cases, namely either $n = 2$ or $n \neq 2$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point.

The proof of converse part is straightforward. $\square$

4. Almost product manifolds

Let $\widetilde{M}$ be an m-dimensional smooth manifold. A system of coordinate neighborhood is called a *separating coordinate system* if, in the intersection of any two coordinate neighborhoods $\left(x^i\right)$ and $\left(x^{i'}\right)$, there exist the following relations:

$$x^{a'} = x^{a'}\left(x^a\right), \qquad x^{\alpha'} = x^{\alpha'}\left(x^\alpha\right),$$

with

$$\det\left(\frac{\partial x^{a'}}{\partial x^a}\right) \neq 0, \qquad \det\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right) \neq 0,$$

where the indices a, b, c, d run over the range $1, \ldots, m_1$, the indices $\alpha, \beta, \gamma, \nu$ run over $m_1 + 1, \ldots, m_1 + m_2 = m$, and the indices i, j, k, h run over $1, \ldots, m$.

Now, let $\widetilde{M}$ be a manifold covered by a separating coordinate system. Suppose that $\widetilde{M}_1$ is a subspace defined by

$$x^\alpha = \text{constant}, \qquad \alpha \in \{m_1 + 1, \ldots, m_1 + m_2 = m\},$$

and by $\widetilde{M}_2$ is a subspace defined by

$$x^a = \text{constant}, \qquad a \in \{1, \ldots, m_1\}.$$

Then it follows that $\widetilde{M}$ is locally the product $\widetilde{M}_1 \times \widetilde{M}_2$ of two manifolds. Such a manifold is called a *locally product manifold*. If we define (F_j^i) as the following matrix form

$$F_j^i = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_\beta^\alpha \end{pmatrix}, \tag{4.1}$$

then it is obvious that there exists always a natural tensor field F of type $(1,1)$ on $\widetilde{M}$ satisfied

$$F^2 = I, \tag{4.2}$$

where I denotes the identity transformation.

A locally product manifold $\widetilde{M}$ equipped with a Riemannian metric defined by

$$ds^2 = \widetilde{g}_{ij}\left(x\right)dx^i dx^j \tag{4.3}$$

is called a *locally product Riemannian manifold*. If we define $F_{ji} = (F_j^t)g_{ti}$, $t \in \{1, \ldots m\}$ such that in fact

$$F_{ji} = \begin{pmatrix} g_{ba} & 0 \\ 0 & -g_{\beta\alpha} \end{pmatrix}. \tag{4.4}$$

Thus, we have $F_{ij} = F_{ji}$. In view of (4.3) and (4.4), there exists always a natural tensor field F of type $(1,1)$ on any locally product Riemannian manifold satisfied

$$\widetilde{g}\left(FX, FY\right) = \widetilde{g}\left(X, Y\right) \tag{4.5}$$

for any $X, Y \in T\widetilde{M}$. If the metric $\widetilde{g}$ of a locally product Riemannian manifold $\widetilde{M}$ has the form

$$ds^2 = \widetilde{g}_{ab}\left(x^c\right)dx^a dx^b + \widetilde{g}_{\alpha\beta}\left(x^\gamma\right)dx^\alpha dx^\beta,$$

then $\widetilde{M}$ is called a *locally decomposable Riemannian manifold*. We note that a locally product Riemannian manifold is a locally decomposable manifold if and only if $\widetilde{\nabla}F = 0$, where $\widetilde{\nabla}$ is the Riemannian connection of $(\widetilde{M}, \widetilde{g})$.

Theorem 4. [20, Theorem 2.4, p. 421] *Let $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ be a locally decomposable Riemannian manifold with $\dim(\widetilde{M}_\ell) = m_\ell > 2$, $\ell = 1, 2$. Then, both the manifolds $\widetilde{M}_1$ and $\widetilde{M}_2$ are Einstein if and only if the Ricci tensor $\widetilde{S}$ of $\widetilde{M}$ has the form*

$$\widetilde{S}_{ij} = k_1\widetilde{g}_{ij} + k_2 F_{ij}$$

for certain constants k_1 and k_2.

Theorem 5. [20, Theorem 2.5, p. 422] *Let $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ be a locally decomposable Riemannian manifold with $\dim(\widetilde{M}_\ell) = m_\ell > 2$, $\ell = 1, 2$. both the manifolds $\widetilde{M}_1$ and $\widetilde{M}_2$ are of constant sectional curvatures λ_1 and λ_2, respectively, that is, the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ has the form*

$$\widetilde{R}_{abcd} = \lambda_1(\widetilde{g}_{ad}\widetilde{g}_{bc} - \widetilde{g}_{ac}\widetilde{g}_{bd}), \quad \widetilde{R}_{\alpha\beta\gamma\nu} = \lambda_2(\widetilde{g}_{\alpha\nu}\widetilde{g}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\nu})$$

if and only if

$$\begin{aligned} \widetilde{R}_{hijk} \quad = \quad & a\{(\widetilde{g}_{hk}\widetilde{g}_{ij} - \widetilde{g}_{hj}\widetilde{g}_{ik}) + (F_{hk}F_{ij} - F_{hj}F_{ik})\} \\ & + b\{(F_{hk}\widetilde{g}_{ij} - F_{hj}\widetilde{g}_{ik}) + (\widetilde{g}_{hk}F_{ij} - \widetilde{g}_{hj}F_{ik})\}, \end{aligned}$$

where

$$a = \frac{1}{4}\left(\lambda_1 + \lambda_2\right), \qquad b = \frac{1}{4}\left(\lambda_1 - \lambda_2\right).$$

A locally decomposable Riemannian manifold is called a *manifold of almost constant curvature*, denoted $\widetilde{M}(a,b)$, if its curvature tensor $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= a\{(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) \\ &\quad + (\langle X,FW\rangle\langle Y,FZ\rangle - \langle X,FZ\rangle\langle Y,FW\rangle)\} \\ &\quad + b\{(\langle X,FW\rangle\langle Y,Z\rangle - \langle X,FZ\rangle\langle Y,W\rangle) \\ &\quad + (\langle X,W\rangle\langle Y,FZ\rangle - \langle X,Z\rangle\langle Y,FW\rangle)\}\end{aligned} \tag{4.6}$$

for all vector fields X,Y,Z,W in $\widetilde{M}$ (See [16], [18], [19] and [20]).

Let $\widetilde{M}$ be a smooth manifold equipped with a tensor of type $(1,1)$ which is satisfies (4.2). Then $\widetilde{M}$ is called an *almost product manifold* and F is called an *almost product structure* on $\widetilde{M}$. If an almost product manifold $\widetilde{M}$ admits a Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX,FY) = \widetilde{g}(X,Y) \tag{4.7}$$

for all vector fields X and Y on $\widetilde{M}$, then $\widetilde{M}$ is called an *almost product Riemannian manifold* [20].

Now, let (M,g) be an n-dimensional Riemannian submanifold of a Riemannian product manifold $(\widetilde{M},\widetilde{g})$. For any vector field X tangent to M, we can write

$$FX = fX + \omega X, \tag{4.8}$$

where fX is the tangential part of FX and ωX is the normal part of FX. From (4.7) and (4.8), we see that

$$g(fX,Y) = g(X,fY) \tag{4.9}$$

for all vector fields in M.

Furthermore, we note that the squared norm of f at $p \in M$ is given by

$$\|f\|^2 = \sum_{i,j=1}^{n} g(fe_i,e_j)^2,$$

where $\{e_1,\ldots,e_n\}$ is any orthonormal basis of the tangent space T_pM.

Let $(\widetilde{M},\widetilde{g})$ be an almost product Riemannian manifold and (M,g) be a submanifold of $(\widetilde{M},\widetilde{g})$. For each non-zero vector X to M at p, if the angle $\theta(p)$ between FX and X given by

$$\cos\theta = \frac{\langle FX, fX\rangle}{\|X\|\|fX\|} \tag{4.10}$$

is independent of the choice of $p \in M$ and $X \in T_pM$, then M is called a *slant submanifold*. From this definition, it can be shown that M is a slant manifold there exists a constant $\lambda \in [0,1]$ such that

$$f^2 = \lambda. \tag{4.11}$$

A slant submanifold is called

a. an *F-invariant submanifold* if $\theta = 0$,
b. an *F-anti-invariant submanifold* or *totally real submanifold* if $\theta = \frac{\pi}{2}$,
c. a *proper slant submanifold* if it is neither non-invariant nor anti-invariant,
d. a *product slant submanifold* if the endomorphism f is parallel [14].

We shall need the following results:

Theorem 6. [12, Theorem 4.4, p. 45] *Let M be an n-dimensional proper slant submanifold of almost product Riemannian manifold $\widetilde{M}$. Then an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, $p \in M$ satisfies the following condition:*

For any e_a vector belongs to the basis $\{e_1, \ldots, e_n\}$, there exists an e_b vector belongs to the basis $\{e_1, \ldots, e_n\}$ such that

$$\langle fe_a, e_b \rangle = \langle e_a, fe_b \rangle = \cos\theta$$

and

$$\langle fe_a, e_c \rangle = \langle fe_b, e_c \rangle = 0$$

for $c \neq a$ and $c \neq b$.

Theorem 7. [12, Theorem 4.7, p. 48] *Let M be a proper θ-slant submanifold of an almost product Riemannian manifold $\widetilde{M}$. Then $\nabla_X f = 0$ for all $X \in TM$ if and only if either $\langle fe_i, e_i \rangle = \cos\theta$ or e_i is parallel for each $i \in \{1, \ldots, n\}$.*

5. Some Optimal inequalities for submanifolds of almost constant curvature manifolds

We shall begin this section with the following lemma for later uses:

Lemma 2. *Let M be an n-dimensional submanifold of an almost constant curvature manifold and $\Pi_k = Span\{e_1, \ldots, e_k\}$ be a k-plane section of T_pM. Denote f_k by the projection morphism of T_pM onto Π_k. For any orthonormal vector pair $\{e_i, e_j\}$ in Π_k, we have*

$$\begin{aligned} \widetilde{K}_{ij} &= a\left\{1 + \langle e_i, f_k e_i \rangle \langle e_j, f_k e_j \rangle - \langle e_i, f_k e_j \rangle^2\right\} \\ &\quad + b\left\{\langle e_i, f_k e_i \rangle + \langle e_j, f_k e_j \rangle\right\}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} \widetilde{\mathrm{Ric}}_{\Pi_k}(e_i) &= a\left\{(k-1) + \langle e_i, f_k e_i \rangle \,\mathrm{trace}\,(f_k) - \|f_k e_i\|^2\right\} \\ &\quad + b\left\{(k-2)\langle e_i, f_k e_i \rangle + \mathrm{trace}\,(f_k)\right\}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \widetilde{\tau}(\Pi_k) &= \frac{a}{2}\left\{(k-1)k + (\mathrm{trace}\,(f_k))^2 - \|f_k\|^2\right\} \\ &\quad + b(k-1)\mathrm{trace}\,(f_k), \end{aligned} \tag{5.3}$$

where trace (f_k) *denotes the trace restricted to Π_k with respect to the metric g.*

Proof. We get (5.1) from (4.6). Considering (3.1) and (5.1), we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{\Pi_k}(e_i) &= a\left\{(k-1)+\langle e_i, f_k e_i\rangle\sum_{j=2}^{k}\langle e_j, f_k e_j\rangle-\sum_{j=2}^{k}\langle e_i, f_k e_j\rangle^2\right\}\\ &\quad + b\left\{(k-1)\langle e_i, f_k e_i\rangle+\sum_{j=2}^{k}\langle e_j, f_k e_j\rangle\right\},\end{aligned}$$

which implies (5.2). Next, using (3.2) and (5.2), we obtain (5.3). □

Theorem 8. *Let M be an n-dimensional submanifold of an almost constant curvature manifold $\widetilde{M}(a,b)$. For each point $p\in M$ and each k-plane section $\Pi_k\subset T_pM$, $(n>k\geq 2)$, we have*

$$\begin{aligned}\delta_M^k(p) &\leq \frac{n^2(n-k)}{2(n-k+1)}\|H\|^2+\frac{a}{2}\Big\{(n-k)(n+k-1)+(\mathrm{trace}\,(f))^2-(\mathrm{trace}\,(f_k))^2\\ &\quad +\|f_k\|^2-\|f\|^2\Big\}+\ b\{(n-1)\mathrm{trace}\,(f)-(k-1)\mathrm{trace}\,(f_k)\}. \end{aligned}\tag{5.4}$$

The equality in (5.4) *holds at $p\in M$ if and only if the shape operators take forms as* (3.8) *and* (3.9)*.*

Proof. Putting $k=n$ in equation (5.3) we have

$$\begin{aligned}\widetilde{\tau}(\Pi) &= \frac{a}{2}\Big\{(n-1)n+(\mathrm{trace}\,(f))^2-\|f\|^2\Big\}\\ &\quad + b(n-1)\mathrm{trace}\,(f).\end{aligned}\tag{5.5}$$

From (5.4), (5.3) and (5.5), the proof of theorem is straightforward. □

Corollary 1. *Let M be an n-dimensional submanifold of an almost constant curvature manifold $\widetilde{M}(a,b)$ and $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM. Then we* *have the following table:*

	M	*Inequality*
(1)	*proper θ-slant*	$\delta^k_M(p) \leq \frac{n^2(n-k)}{2(n-k+1)} \lVert H\rVert^2 + \frac{a}{2}\Big\{(n-k)(n+k-1) + (\mathrm{trace}\,(f))^2 - (\mathrm{trace}\,(f_k))^2 + \lVert f_k\rVert^2 - n\cos^2\theta\Big\} + \; b\,\{(n-1)\mathrm{trace}\,(f) - (k-1)\mathrm{trace}\,(f_k)\}$
(2)	*proper product θ-slant*	$\delta^k_M(p) \leq \frac{n^2(n-k)}{2(n-k+1)} \lVert H\rVert^2 + \frac{a}{2}\Big\{(n-k)(n+k-1) + (n-k)(n+k-1)\cos^2\theta\Big\} + b\cos\theta\,\{n(n-1) - k(k-1)\}$
(3)	*F-invariant*	$\delta^k_M(p) \leq \frac{n^2(n-k)}{2(n-k+1)} \lVert H\rVert^2 + \frac{a}{2}\Big\{n(n-2) + k(1-k) + (\mathrm{trace}\,(f))^2 - (\mathrm{trace}\,(f_k))^2 + \lVert f_k\rVert^2\Big\} + \; b\,\{(n-1)\mathrm{trace}\,(f) - (k-1)\mathrm{trace}\,(f_k)\}$
(4)	*F-totally real*	$\delta^k_M(p) \leq \frac{n^2(n-k)}{2(n-k+1)} \lVert H\rVert^2 + \frac{a}{2}\,\{(n-k)(n+k-1)\}\;.$

The equality case of inequalities given by the table holds at $p \in M$ if and only if the shape operators of M take forms as (3.8) *and* (3.9)*.*

Proof. Suppose that M is a proper θ-slant submanifold. We have from Theorem 6 that

$$\lVert f\rVert^2 = \cos^2\theta.$$

Using (**??**) in (5.4) we find the inequality (1). Next, if M is a product θ-slant submanifold with all e_i are parallel we have from Theorem 7 that

$$g(fe_i, e_i) = \cos\theta.$$

Using (**??**) in (5.4) we find the inequality (2). Putting $\theta = \frac{\pi}{2}$ and 0 in the inequality (1), we get the inequalities (3) and (4), respectively. □

In particular case of $k = 2$, we have the followings:

Theorem 9. *Let M be an n-dimensional submanifold of an almost constant curvature manifold $\widetilde{M}(a,b)$. For any plane section $\Pi = Span\{e_1, e_2\} \subset T_pM$ at a point $p \in M$, we have*

$$\delta_M(p) \leq \frac{n^2\,(n-2)}{2\,(n-1)} \lVert H\rVert^2 + \frac{a}{2}\Big\{(n-2)(n+1) + (\mathrm{trace}\,(f))^2 - \lVert f\rVert^2 - 2\langle fe_1, e_2\rangle^2 + 2\langle fe_1, e_1\rangle\langle fe_2, e_2\rangle\Big\} + \; b\,\{(n-1)\mathrm{trace}\,(f) - \langle fe_1, e_1\rangle - \langle fe_2, e_2\rangle\}\,. \quad (5.6)$$

The equality in (5.6) *holds at $p \in M$ if and only if the shape operators take forms as* (3.20) *and* (3.21)*.*

Corollary 2. *Let M be an n-dimensional submanifold of an almost constant curvature manifold $\widetilde{M}(a,b)$ and $\{e_1,...,e_n\}$ be an orthonormal basis of T_pM. For any plane section $\Pi = Span\{e_i, e_j\}$, we have the following table:*

	M	*Inequality*
(1)	proper θ-slant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{a}{2}\left\{(n-2)(n+1) + (\text{trace}(f))^2 - n\cos^2\theta + 2\langle fe_i, e_j\rangle^2 - 2\langle fe_i, e_i\rangle\langle fe_j, e_j\rangle\right\} + b\left\{(n-1)\text{trace}(f) - \langle fe_i, e_i\rangle - \langle fe_j, e_j\rangle\right\}$
(2)	proper product θ-slant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{a}{2}\left\{(n-2)(n+1) + (n-2)(n+1)\cos^2\theta\right\} + b\cos\theta(n-2)(n+1)$
(3)	F-invariant	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{a}{2}\left\{n^2 - 2n - 2 + (\text{trace}(f))^2 + 2\langle fe_i, e_j\rangle^2 - 2\langle fe_i, e_i\rangle\langle fe_j, e_j\rangle\right\} + b\left\{(n-1)\text{trace}(f) - \langle fe_i, e_i\rangle - \langle fe_j, e_j\rangle\right\}$
(4)	F-totally real	$\delta_M(p) \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{a}{2}\{(n-2)(n+1)\}$.

The equality case of inequalities given by the table holds at $p \in M$ if and only if the shape operators of M take forms as (3.20) *and* (3.21).

In particular case of $k = n - 1$, we have the following:

Theorem 10. *Let M be an n-dimensional submanifold of an almost constant curvature manifold $\widetilde{M}(a,b)$ and $\{e_1,...,e_n\}$ be an orthonormal basis of T_pM. Then,*

1. *For each unit vector $U \in T_pM$, we have*
$$\begin{aligned}\text{Ric}(U) \quad \leq \quad & \frac{1}{4}n^2\|H\|^2 + \frac{a}{2}\left\{(k-1)k + (\text{trace}(f_k))^2 - \|f_k\|^2\right\} \\ & + b(k-1)\text{trace}(f_k). \end{aligned} \tag{5.7}$$
2. *If the mean curvature $H(p) = 0$, then a unit vector $U \in T_pM$ satisfies the equality case of* (5.7) *if and only if U lies in the relative null space $\mathcal{N}_p$ at p.*
3. *The equality case of* (5.7) *holds for all unit vectors $U \in T_pM$, if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Now, we shall give some examples of submanifolds of almost curvature manifolds which are satisfying the inequalities obtained throughout the paper.

Example 1. *Consider a submanifold $\widetilde{M}$ in $\mathtt{E}^9$ given by*

$$\widetilde{M} \quad = \quad \{(t, -t, 0, t, -t, \cos u\cos v\cos w, \cos u\cos v\sin w, \cos u\sin v, \sin u)$$

for $t \in R$ and $u, v, w \in [0, \frac{\pi}{2})$. Let F be an almost product structure on $\mathtt{E}^9$ defined by

$$FX = (x^2, x^1, x^3, x^5, x^4, x^6, x^7, x^8, x^9),$$

where $X = (x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9)$. Then we have

$$PX = \frac{1}{2}(x^1 + x^2, x^1 + x^2, 2x^3, x^4 + x^5, x^4 + x^5, 2x^6, 2x^7, 2x^8, 2x^9)$$

and

$$QX = \frac{1}{2}(x^1 - x^2, x^1 - x^2, 0, x^4 - x^5, x^5 - x^4, 0, 0, 0, 0),$$

which show that $\widetilde{M}$ is a locally product of the unit 3-sphere S^3 given by the spherical coordinates in $\mathtt{E}^9$ as

$$(\cos u \cos v \cos w, \cos u \cos v \sin w, \cos u \sin v, \sin u, 0, 0, 0, 0, 0)$$

for u ranges over $[0, \frac{\pi}{2})$ and the other coordinates range over $[0, \frac{\pi}{2}]$, and a plane section M_1 in $\mathtt{E}^9$ given by

$$M_1 = \{(t, -t, 0, t, t, 0, 0, 0, 0) : t \in R\}.$$

Thus, it follows from Theorem 5 that $\widetilde{M}$ is an almost constant curvature manifold with $a = b = \frac{1}{4}$. By a straightforward computation, we have

$$\begin{aligned}
e_1 &= (0, 0, 0, 0, 0, -\sin u \cos v \cos w, -\sin u \cos v \sin w, -\sin u \sin v, \cos u),\\
e_2 &= (0, 0, 0, 0, 0, -\cos u \sin v \cos w, -\cos u \sin v \sin w, \cos u \cos v, 0),\\
e_3 &= (0, 0, 0, 0, 0, -\cos u \cos v \sin w, -\cos u \cos v \cos w, 0, 0),\\
e_4 &= (1, -1, 0, 1, -1, 0, 0, 0, 0).
\end{aligned}$$

For each unit vector U and each plane section Π on T_pS^3, we see that

$$\mathrm{Ric}\,(U) = 2, \quad H(p) = 0, \quad \mathrm{trace}\,(f_k) = 2, \quad \|f_k\|^2 = 2 \tag{5.8}$$

By a straightforward computation, it is clear that S^3 satisfies the conditions of Theorem 8, Corollary 1 and Theorem 10.

Example 2. *Consider*

$$\mathbb{R}^4 \times S^3 = \{(x_1, x_2, x_3, x_4, z_1, z_2) : x_i \in \mathbb{R}, 1 \le i \le 4 \text{ and } z_j \in \mathbb{C}, 1 \le j \le 2\},$$

where

$$|z_1|^2 + |z_2|^2 = 1.$$

Let F be an almost product structure on $\mathbb{R}^4 \times S^3$ defined by

$$F(x_1, x_2, x_3, x_4, z_1, z_2) = (x_3, x_4, x_1, x_2, z_1, z_2). \tag{5.9}$$

Then it is clear that $(\mathbb{R}^4 \times S^3, F)$ is of almost constant curvature manifold with $a = b = \frac{1}{4}$.

Consider a flat submanifold M of $\mathbb{R}^4 \times S^3$ given by

$$\{(u\cos\theta, u\cos\theta, v, w, 0, 0) : u, v, w \in \mathbb{R}\},$$

where θ is constant. Then, one can see that the submanifold M is a θ-slant submanifold $\mathbb{R}^4 \times S^3$ and satisfies the conditions of Theorem 8, Corollary 1 and Theorem 10.

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