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Independent sets of axioms for boolean algebras

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Abstract

In this work, we review axiomatic systems and prove some of the equivalent axiomatizations of Boolean algebras. Also we prove the independence of three axioms, proposed by Huntington and then by Robbins, which form a minimal set of axioms for Boolean algebras.

***Keywords**: Classical logic, Boolean algebras, axiomatizability, consistence, independence.*

Boole cebirleri için bağımsız aksiyom kümeleri

Özet

Bu çalışmada, aksiyomatik sistemler araştırıldı ve Boole cebirlerinin denk aksiyomlaştırmalarının bazıları ispatlandı. Ayrıca, Huntington ve sonrasında Robbins tarafından ileri sürülen, Boole cebirleri için aksiyomların bir minimal kümesini oluşturan üç aksiyomun bağımsızlığını ispatlandı.

***Anahtar Kelimeler:** Klasik lojik, Boole cebirleri, aksiyomlaştırılabilirlik, tutarlılık, bağımsızlık.*

1. Introduction

An axiomatic system consists of certain undefined or primitive term(s) together with a set of statements, called axioms, that are presupposed to be true. A theorem is any statement that can be deduced from the axioms using inference rules. This raises the issue of what is an axiom and how a statement is recognized as an axiom. Over the centuries, nearly since Euclid, an axiom has been thought of as a statement that neither

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need nor admit a proof; more precisely, an axiom has been considered as a general presupposition or fundamental statement that was evident, infallible. From the nineteenth century, for logicians an axiom is utilized as a statement explicitly mentioned as a primary premise accepted without proof. In a way, the axioms act as definitions for primitive terms. For example, point and line are the undefined terms for the Euclidean

geometry, but in axiom we say that what is about points and lines that will be used in the development of geometry.

In order to create an axiomatic system, its axioms must be chosen wisely so that the system is

Consistent: Consistency can be explained in different equivalent ways:

- The axioms of the system do not lead to contradictions, i.e. the system is free of contradictions.
- The system does not allow to derive both a statement and its negation.
- The system has a model.

Meanwhile, recall that a model for an axiomatic system is a way to define the primitive terms so that axioms are true.

Independent: Any axiom deducible from the remaining ones is redundant for the system, that is, no axiom can be obtained from the other axioms of the system.

Complete: Given any statement about the objects of the system, if we are able to prove or disprove the statement from the axioms alone, then the system will be complete. This is a very delicate issue, and we do not consider it here.

In this work we deal with independent axiomatization of Boolean algebras. Independent axiomatization of formal systems initially caused many problems. To give an example, mathematicians thought that the fifth axiom of the Euclidean geometry needed proof, that is, it could be deduced from the other four axioms. But with the recognition of non-Euclidean geometries in 1818 by Schweitkart and around 1830 especially by Bolyai and Lobachevsky [1], it came out that the fifth postulate was indeed independent of Euclid's other axioms. Huntington strived at founding a smallest set of axioms for Boolean algebras which we prove its independency [2, 3]. Tarski and then Kreisel pointed out that independent axiomatizability in classical propositional logic can be proved for any set of formulas of cardinality 2 [4-6]. However, this claim was refuted through an example by Reznikoff [7]. More on the independence of countable sets of formulas can be found in [8].

In this work, we focus on different sets of axioms for Boolean algebras already studied by Hungtington and others, and we give some detailed proofs related to the independency of axiom sets. We would like to declare that concerning this matter, we are inspired a lot from [9].

2. Axiom sets for boolean algebras

A Boolean algebra is a mathematical structure $\mathfrak{B} = < B, \vee, \wedge, ', 0, 1 >$ of the type $< 2, 2, 1, 0, 0 >$, where B is a nonempty set with distinguished elements 0 and 1, satisfying the following axioms.

$(B_1)\ 0' = 1$	$(B_1)'\ 1' = 0$
$(B_2)\ x \wedge 0 = 0$	$(B_2)'\ x \vee 1 = 1$
$(B_3)\ x \wedge 1 = x$	$(B_3)'\ x \vee 0 = x$
$(B_4)\ x \wedge x' = 0$	$(B_4)'\ x \vee x' = 1$
$(B_5)\ x'' = x$	
$(B_6)\ x \wedge x = x$	$(B_6)'\ x \vee x = x$
$(B_7)\ (x \wedge y)' = x' \vee y'$	$(B_7)'\ (x \vee y)' = x' \wedge y'$
$(B_8)\ x \wedge y = y \wedge x$	$(B_8)'\ x \vee y = y \vee x$
$(B_9)\ x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$(B_9)'\ x \vee (y \vee z) = (x \vee y) \vee z$
$(B_{10})\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	$(B_{10})'\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in B$.

Let us denote by $\mathbb{B}$ the set of these axioms. Clearly, $\mathbb{B}$ is too large for our purpose. The problem of selecting enough small subsets of $\mathbb{B}$ that imply them all is sometimes a difficult task. One solution is given by the subset $\mathbb{A} = \{(B_i) - (B_i)' : i = 3, 4, 8, 10\}$ [2]. Note that for this axiomatization to prove that $\mathbb{A}$ implies (B_7), $(B_7)'$ and (B_9), $(B_9)'$ are particularly not so easy.

In this section, we shall try to find a smallest set of axioms for Boolean algebras. Recall that if S is a set of formulas and φ is a formula written in the same language, then $S \vdash \varphi$ signifies that φ is deducible from S.

2.1. Definition Let S be a set of formulas. S is said to be ***independent*** if for every $\varphi \in S$, φ is not a logical consequence of $S - \{\varphi\}$, in symbols, $S - \{\varphi\} \nvDash \varphi$.

2.2. Proposition [2] Let $\mathfrak{B}$ be a Boolean algebra. Then the set $\mathbb{A}$ constitutes a set of axioms for $\mathfrak{B}$.

2.3. Proposition The set $\mathbb{A}$ is not an independent set of axioms.
Proof In fact, $\mathbb{A} - (B_3) \vdash (B_3)$:

$$
\begin{aligned}
x \wedge 1 &= x \wedge (x \vee x') && \text{(by } (B_4)'\text{)}\\
&= (x \vee 0) \wedge (x \vee x') && \text{(by } (B_3)'\text{)}\\
&= x \vee (0 \wedge x') && \text{(by } (B_{10})'\text{)}\\
&= x \vee 0 && \text{(by } (B_2)'\text{)}\\
&= x. \square && \text{(by } (B_3)'\text{)}
\end{aligned}
$$

2.4. Remark As each of the sets $\{\wedge, '\}$, $\{\vee, '\}$ and $\{\rightarrow, '\}$ is "complete" for propositional calculus, a Boolean algebra may also be thought of as a structure $\mathfrak{B} = < B, \vee, ', 0, 1 >$ of the type $< 2, 1, 0, 0 >$. In this case the operation of meet and the distinguished elements can be defined by the equations:

$x \wedge y = (x' \vee y')', \; 0 = (x \vee x')', 1 = x \vee x'.$ $(*)$

The following result is due to Hungtington [3]; we give a proof of it in modern notation.

2.5. Theorem The axioms

$(B_8)'$ $x \vee y = y \vee x$
$(B_9)'$ $x \vee (y \vee z) = (x \vee y) \vee z$
(H) $(x' \vee y')' \vee (x' \vee y)' = x$

form a minimal set of axioms for Boolean algebras. (H) is called Huntington's axiom. Note that if one chooses the set $\{\wedge, '\}$, then (H) becomes

$(x' \wedge y')' \vee (x' \wedge y)' = x.$

Let us denote by $\mathbb{H}$ the set of these three axioms.

Proof According to Proposition 2.2, we have $\mathbb{A} \vdash \mathbb{B}$, and in particular, $(B_8)'$ and $(B_9)'$ are so derivable.

The definition of meet given in $(*)$ follows from:

$(x \wedge y)' = x' \vee y'$ (by (B_7))
$(x' \wedge y')' = (x' \vee y')'$
$x \wedge y = (x' \vee y')'.$ (by (B_5))

The definition of 1 follows from $(B_4)'$, and the definition of 0 follows from $(B_1)'$ and $(B_4)'$. Now let us give a deduction of (H).

$$
\begin{aligned}
(x' \vee y')' \vee (x' \vee y)' &= ((x')' \wedge (y')') \vee ((x')' \wedge y') && \text{(by } (B_7)'\text{)}\\
&= (x \wedge y) \vee (x \wedge y') && \text{(by } (B_5)\text{)}\\
&= x \wedge (y \vee y') && \text{(by } (B_{10})\text{)}\\
&= x \wedge 1 && \text{(by } (B_4)'\text{)}\\
&= x && \text{(by } (B_3)\text{)}
\end{aligned}
$$

Thus we have shown that $\mathbb{B} \vdash \mathbb{H}$.

Conversely, we have to prove that $\mathbb{H}$ with $(*)$ imply $\mathbb{A}$, i.e. $\mathbb{H} \vdash \mathbb{A}$. But the derivation of them seems to necessitate the derivation of almost every axiom in $\boldsymbol{B}$. In fact, we do not need to derive all of the axioms in $\mathbb{B}$; the main axiom is (B_5), since many of them follow from it. This is why we shall be contented only with a proof of (B_5). To do this, it will be helpful to have some instances of (H) obtained by replacing x and y by appropriate x', x''. Here are some useful instances:

(1) $(x' \vee x''')' \vee (x' \vee x'')' = x$
(2) $(x'' \vee x''')' \vee (x'' \vee x'')' = x'$
(3) $(x'' \vee x'')' \vee (x'' \vee x')' = x'$
(4) $(x''' \vee x')' \vee (x''' \vee x')' = x$
(5) $(x' \vee x')' \vee (x' \vee x)' = x$
(6) $(x'' \vee x')' \vee (x'' \vee x)' = x'.$

2.6. Lemma We have $x \vee x' = x' \vee x''$.

Proof Using the given instances of (H), we can write the following identities:

$$x \vee x' = [(x' \vee x''')' \vee (x' \vee x'')'] \vee x' \qquad \text{(by (1))}$$
$$= [(x' \vee x''')' \vee (x' \vee x'')'] \vee [(x'' \vee x''')' \vee (x'' \vee x'')'] \qquad \text{(by (2))}$$

and

$$x' \vee x'' = [(x'' \vee x'')' \vee (x'' \vee x')'] \vee x'' \qquad \text{(by (3))}$$
$$= [(x'' \vee x'')' \vee (x'' \vee x')'] \vee [(x''' \vee x'')' \vee (x'' \vee x')']. \qquad \text{(by (4))}$$

It is clear that by $(B_8)'$, the right hand sides of both identities are equal, hence $x \vee x' = x' \vee x''$. □

Derivation of (B_5): The proof of this is similar to that given in Lemma 2.6. If we look at (1) and (4) closely, then we find by Lemma 2.6 and commutativity that

$$x' \vee x'' = x'' \vee x''' = x''' \vee x''.$$

On the other hand, we have $x' \vee x'' = x''' \vee x'$ by commutativity. Hence the left sides of (1) and (4) are equal by commutativity, so the right sides must also be equal, and this gives (B_5). Thus $\mathbb{H} \vdash \mathbb{B}$, and the proof is complete. □

Theorem 2.7 $\mathbb{H}$ is an independent set of axioms.

Proof We have to find a model such that $\mathbb{H} - \{\varphi_i\}$ holds in that model but not $\{\varphi_i\}$, for i = 1, 2, 3.

The independence of $(B_8)'$: Consider the model $\mathfrak{M} = \langle M, \vee, ' \rangle$, where M = {0, 1, 2, 3, 4, 5} and the tables for ∨ and ′:

∨	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	1	1	2
3	3	1	1	3	3	1
4	4	1	1	4	4	1
5	5	1	5	1	1	5

′	
0	1
1	0
2	3
3	2
4	5
5	4

One can verify that $\mathfrak{M} \vdash (B_9)'$ and $\mathfrak{M} \vdash (H)$. The verification of (H) is irksome; the table needed to verify it has 36 rows (corresponding to the 36 possible pairs of values for the variables x and y) that we do not give here. As for $(B_9)'$, the table needed to verify it has 216 rows! But $\mathfrak{M} \nvDash (B_8)'$. Indeed, take, for example, x = 5 and y = 2. Then

5 $x \vee y = 5 \vee 2 = 5 \neq 2 = 2 \vee 5 = y \vee x$.

Hence, $(B_8)'$ is independent of $(B_9)'$ and (H).

The independence of $(B_8)'$: Consider the model $\mathfrak{N} = \langle N, \vee, ' \rangle$, where $N = \{0, 1, 2, 3\}$ and the tables for $\vee$ and $'$:

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	0
2	2	2	2	1
3	3	0	1	1

$'$	
0	1
1	0
2	3
3	2

Clearly, $\mathfrak{N} \vdash (B_8)'$, since the table for $\vee$ is symmetric with respect to the main diagonal. Also $\mathfrak{N} \vdash (H)$; the table for (H) has 16 rows. However, $\mathfrak{N} \nvDash (B_9)'$.

Indeed, take $x = 2$, $y = 1$, and $z = 3$. Then

$$x \vee (y \vee z) = 2 \vee (1 \vee 3) = 2 \vee 0 = 2 \neq 1 = 2 \vee 3 = (2 \vee 1) \vee 3 = (x \vee y) \vee z.$$

Hence, $(B_9)'$ is independent of $(B_8)'$ and (H).

The independence of *(H):* Consider the model $\mathfrak{P} = \langle P, \vee, ' \rangle$, where $P = \{0, 1, 2, 3, 4, 5\}$ and the tables for $\vee$ and $'$:

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	1	1	1
3	3	1	1	3	1	1
4	4	1	1	1	4	1
5	5	1	1	1	1	5

$'$	
0	1
1	0
2	3
3	2
4	5
5	4

Then $\mathfrak{P} \vdash (B_8)'$, since the table for $\vee$ is symmetric across the main diagonal. Also it can be easily verified that $\mathfrak{P} \vdash (B_9)'$. But we have that $\mathfrak{P} \nvDash (H)$. Indeed, if we take $x = 3$ and $y = 5$, then

$$\begin{aligned}(x' \vee y')' \vee (x' \vee y)' &= (3' \vee 5')' \vee (3' \vee 5)' \\ &= (2 \vee 4)' \vee (2 \vee 5)' \\ &= 1' \vee 1' \\ &= 0 \vee 0 \\ &= 0 \neq 3 = x.\end{aligned}$$

Therefore, (H) is independent of $(B_8)'$ and $(B_9)'$, and this proves the theorem.□

3. Huntington algebra and Robbins algebra

Sometimes, Boolean algebra with the axiom set $\mathbb{H}$ is called Huntington algebra.

3.1. Robbins conjecture

After Huntington proved Theorem 2.5, in 1930s H. Robbins conjectured that the axiom (*H*) can be replaced by his own, somewhat similar looking axiom (which is simpler than (*H*) since it has one fewer occurrence of $'$):

$$\big((x \vee y)' \vee (x \vee y')'\big)' = x.$$

Thus a Robbins algebra has the following independent axioms:

(R_1) $x \vee y = y \vee x$,
(R_2) $x \vee (y \vee z) = (x \vee y) \vee z$,
(R_3) $\big((x \vee y)' \vee (x \vee y')'\big)' = x$.

Let us denote by $\mathbb{R}$ the set of these axioms.

3.1. Proposition Every Boolean algebra is a Robbins algebra: $\mathbb{A} \vdash \mathbb{R}$.

Proof All we need to prove is (R_3), since (R_1) and (R_2) are identical in both algebras. Note that (R_3) can be rewritten under the form

$$(x \vee y)' \wedge (x \vee y') = x.$$

Then

$$\begin{aligned}(x \vee y)' \wedge (x \vee y') &= x \vee (y \wedge y') && \text{(by } (B_{10})')\\ &= x \vee 0 && \text{(by } (B_4))\\ &= x && \text{(by } (B_3)).\square\end{aligned}$$

But is the converse true? The question “Are all Robbins algebras Boolean?” became known as the Robbins conjecture (or problem).

The problem remained unsolved for decades. In the 1980s, S. Winker proved several conditions sufficient to make a Robbins algebra Boolean [10]. The problem was finally solved in 1997 by EQP, a theorem prover created at Aragonne National Laboratory under the direction of W. McCune [11]. In the end, in 2003, the conjecture was completely proved by A. L. Mann (see [12]).

Conclusion In an axiomatic system, the lesser the number of axioms harder the proof of theorems becomes. In this work, we see such an example by means of Theorem 2.5. Of course, the number of axioms depend on the operations defined on the system. Sheffer (see [4]) proved that the theory of Boolean algebras is definitionally equivalent to the theory of the binary operation stroke ,|, axiomatized by three identities:

(S1) $(x \mid x) \mid (x \mid x) = x$,
(S2) $(x \mid \big(y \mid (y \mid y)\big)) = x \mid x$,
(S3) $(x \mid (y \mid z)) \mid (x \mid (y \mid z)) = ((y \mid y) \mid x) \mid ((z \mid z) \mid x)$.

But this restriction makes proofs very complicated and dull.

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