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Lie Ideals of Semiprime Rings with Generalized Derivations

Emine KOÇ SÖGÜTCÜ[1,*], Öznur GÖLBAŞI[1]

[1]*Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140 Sivas, Türkiye,*
eminekoc@cumhuriyet.edu.tr , ogolbasi@cumhuriyet.edu.tr

Abstract

Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R. A map F:R→R is called a generalized derivations if there exists a derivation d:R→R such that F(xy) = F(x)y + xd(y) for all x,y∈R. In the present paper, we shall prove that h is commuting map on U if any one of the following holds: i) F(u)u = ±uG(u), ii) [F(u),v] = ±[u,G(v)], iii) F(u)∘v = ±u∘G(v), iv) [F(u),v] = ±u∘G(v), v) F([u,v]) = [F(u),v] + [d(v),u] for all u,v∈U, where G:R→R is a generalized derivation associated with the derivation h:R→R.

Keywords: Semiprime ring, Lie ideal, Derivation, Generalized derivation.

Genelleştirilmiş Türevli Yarıasal Halkaların Lie İdealleri

Özet

R, 2-torsion free bir yarıasal halka ve U, R halkasının bir merkez tarafından kapsanılmayan kare-kapalı Lie ideali olsun. Eğer her x,y∈R için F(xy) = F(x)y + xd(y), koşulunu sağlayan bir d:R→R türevi varsa F dönüşümüne R halkasının d ile belirlenmiş bir genelleştirilmiş türevi denir. Bu çalışmada, aşağıdaki koşullardan biri sağlanırsa d dönüşümünün U üzerinde komüting dönüşüm olduğu gösterilecektir: i) F(u)u = ±uG(u),

* Corresponding Author

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ii) $[F(u),v] = \pm[u,G(v)]$, iii) $F(u)\circ v = \pm u\circ G(v)$, iv) $[F(u),v] = \pm u\circ G(v)$, v) $F([u,v]) = [F(u),v] + [d(v),u]$. Burada $G:R\to R$ dönüşümü $h:R\to R$ türevi ile belirlenmiş bir genelleştirilmiş türevdir.

Anahtar Kelimeler: Yarıasal halka, Lie ideal, Türev, Genelleştirilmiş türev.

Introduction

Throughout R will present an associative ring with center Z. For any $x,y\in R$, the symbol $[x,y]$ stands for the commutator $xy-yx$ and the symbol xoy denotes the anti-commutator $xy+yx$. Recall that a ring R is prime if $xRy=0$ implies $x=0$ or $y=0$, and R is semiprime if for $x\in R$, $xRx=0$ implies $x=0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r]\in U$, for all $u\in U$, $r\in R$. U is called a square closed Lie ideal of R if U is a Lie ideal and $u^2\in U$ for all $u\in U$. Let S be a nonempty subset of R. A mapping f from R to R is called centralizing on S if $[f(x),x]\in Z$, for all $x\in S$ and is called commuting on S if $[f(x),x]=0$, for all $x\in S$. A mapping $f:R\to R$ is called skew-centralizing on R if $f(x)x+xf(x)\in Z(R)$ holds for all $x\in R$; in particular, if $f(x)x+xf(x)=0$ holds for all $x\in R$, then it is called skew-commuting on R.

An additive mapping $d:R\to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. In [3], Bresar defined the following notation. An additive mapping $F:R\to R$ is called a generalized derivation if there exists a derivation $d:R\to R$ such that $F(xy)=F(x)y+xd(y)$ for all $x,y\in R$.

The history of commuting and centralizing mappings goes back to 1955 when Divinsky [5] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. The commutativity of prime rings with derivation was initiated by Posner [9]. He showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [1], Awtar considered centralizing derivations on Lie and Jordan ideals. For prime rings Awtar showed that a nontrivial derivation which is centralizing on Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two or three. In [8], Lee and Lee obtained the same result while removing the restriction of characteristic not three. The same result is showed for generalized derivations in [7].

In [4], Bresar has proved that if R is a 2-torsion-free semiprime ring and $f:R\rightarrow R$ is an additive skew-commuting mapping on R, then f=0. This result extended for semiprime rings in [11].

In the present paper, we shall extend the above results for a noncentral square closed Lie ideal of semiprime rings with generalized derivation.

Preliminaries

Make some extensive use of the basic commutator identities:

$[x,yz]=y[x,z]+[x,y]z,$

$[xy,z]=[x,z]y+x[y,z],$

$xo(yz)=(xoy)z-y[x,z]=y(xoz)+[x,y]z,$

$(xy)oz=x(yoz)-[x,z]y=(xoz)y+x[y,z].$

Moreover, we shall require the following lemmas.

Lemma 1 [2, Lemma 4] *Let R be a prime ring with characteristic not two, $a,b\in R$. If U a noncentral Lie ideal of R and aUb=0, then a=0 or b=0.*

Lemma 2 [2, Lemma 5] *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that d(U)=(0), then $U\subseteq Z$.*

Lemma 3 [2, Lemma 2] *Let R be a prime ring with characteristic not two. If U a noncentral Lie ideal of R, then $C_R(U)=Z$.*

Lemma 4 [10, Lemma 2] *Let R be a 2-torsion free semiprime ring, U is a Lie ideal of R such that $U\subseteq Z(R)$ and $a\in U$. If aUa=0, then $a^2=0$ and there exists a nonzero ideal K=R[U,U]R of R generated by [U,U] such that $[K,R]\subseteq U$ and Ka=aK=0.*

Corollary 1 [6, Corollary] *Let R be a 2-torsion free semiprime ring, U a noncentral Lie ideal of R and $a,b\in U$.*

(i) *If aUa=0, then a=0.*

(ii) *If aU=0(or Ua=0), then a=0.*

(iii) *If U is square-closed and aUb=0, then ab=0 and ba=0.*

Main Results

Now we can prove the main results of this paper.

Theorem 1 *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F,G generalized derivations associated to the derivations d, h of R respectively such that h(U)⊆U. If F(u)u=±uG(u) for all u∈U, then h is commuting map on U.*

Proof. Let F(u)u=uG(u) for all u∈U. The linearization of the above relation gives

$$F(u)v+F(v)u=uG(v)+vG(u) \text{ for all } u,v\in U. \quad (3.1)$$

Replacing u by uv in above relation, we get

$$F(uv)v+F(v)uv=uvG(v)+vG(uv) \text{ for all } u,v\in U,$$

that is,

$$F(u)v^2+ud(v)v+F(v)uv=uvG(v)+vG(u)v+vuh(v) \text{ for all } u,v\in U. \quad (3.2)$$

Right multiplication of (3.1) by v gives

$$F(u)v^2+F(v)uv=uG(v)v+vG(u)v \text{ for all } u,v\in U. \quad (3.3)$$

Subtracting (3.2) from (3.3), we obtain

$$ud(v)v=uvG(v)-uG(v)v+vuh(v) \text{ for all } u,v\in U. \quad (3.4)$$

Replacing u by uw, w∈U in above relation, we get

$$uwd(v)v=vuwh(v)+uw[v,G(v)] \text{ for all } u,v,w\in U,$$

that is,

$$uwd(v)v-uw[v,G(v)]=vuwh(v) \text{ for all } u,v,w\in U.$$

Using (3.4),

$$uvwh(v)=vuwh(v) \text{ for all } u,v,w\in U,$$

which reduces to

$$[u,v]wh(v)=0 \text{ for all } u,v,w\in U.$$

Replacing u by h(v), in above relation, we get

$$[h(v),v]wh(v)=0 \text{ for all } u,v,w\in U. \quad (3.5)$$

Right multiplication of (3.5) by v gives

$$[h(v),v]wh(v)v=0 \text{ for all } u,v,w\in U. \quad (3.6)$$

Replacing w by wv in (3.5), we have

$$[h(v),v]wvh(v)=0 \text{ for all } u,v,w\in U. \quad (3.7)$$

Subtracting (3.6) from (3.7), we arrive that

$$[h(v),v]w[h(v),v]=0 \text{ for all } u,v,w\in U.$$

By Corollary 1, we conclude that

$$[h(v),v]=0 \text{ for all } v\in U$$

and so, h is commuting map on U.

In similar manner, we can prove that the same conclusion holds for $F(u)u+uG(u)=0$ for all $u\in U$.

The following example shows that the semiprimeness condition in Theorem 1 is not superfluous.

Example 1 Let $\mathbb{Z}$ be the set of integers and R= $\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a,b,c \in \mathbb{Z}\right\}$. For any 0≠b∈$\mathbb{Z}$, $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0)$, then R is not a semiprime ring. Take U= $\left\{\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} | a,b \in \mathbb{Z}\right\}$. It can be easily cheked that U is a Lie ideal of R and U is not square-closed Lie ideal of R. Define maps F, d, G, h:R→R as follows:

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & b+c \\ 0 & c \end{pmatrix},\ d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix},\ h\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}.$$

Then is easy to check that d, h are two derivations, F,G are two generalized derivations associated to the derivations d, h of R and F(u)u=uG(u), for all u∈U. However, h is not commuting map on U.

Corollary 2 *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F generalized derivation associated to the derivation d of R such that d(U)⊆U. If F is commuting map (or skew-commuting) on U, then d is commuting map on U.*

Corollary 3 *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively. If F(u)u=±uG(u) for all u∈U, then U⊆Z.*

Proof. Using the same methods in the proof of Theorem 1, we have

[u,v]wh(v)=0 for all u,v,w∈U.

By Lemma 1, we get either [u,v]=0 or h(v)=0 for each v∈U. We set

K={v∈U|[u,v]=0 for all u∈U} and L={v∈U|h(v)=0}. Clearly each of K and L is additive subgroup of U. Morever, U is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence K=U or L=U. In the first case, we have $U \subseteq Z$ by Lemma 3. In the latter case, we have $U \subseteq Z$ by Lemma 2. This completes the proof.

Theorem 2 *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively such that $h(U) \subseteq U$. If $[F(u),v]=\pm[u,G(v)]$ for all $u,v \in U$, then h is commuting map on U.*

Proof. Let $[F(u),v]=[u,G(v)]$ for all $u,v \in U$. Replacing v by vu in above relation, we get

$$[F(u),vu]=[u,G(vu)] \text{ for all } u,v \in U.$$

Using the hypothesis, we obtain

$$v[F(u),u]=[u,v]h(u)+v[u,h(u)] \text{ for all } u,v \in U.$$

Replacing v by vw, $w \in U$ in above relation, we get

$$vw[F(u),u]=[u,vw]h(u)+vw[u,h(u)] \text{ for all } u,v,w \in U,$$

that is,

$$vw[F(u),u]=[u,v]wh(u)+v[u,w]h(u)+vw[u,h(u)] \text{ for all } u,v,w \in U,$$

which reduces to

$$[u,v]wh(u)=0 \text{ for all } u,v,w \in U.$$

Replacing v by h(u) in above relation, we get

$$[u,h(u)]wh(u)=0 \text{ for all } u,w \in U. \quad (3.8)$$

Right multiplication of (3.8) by u gives

$$[u,h(u)]wh(u)u=0 \text{ for all } u,w \in U. \quad (3.9)$$

Replacing w by wu in (3.8), we get

$$[u,h(u)]wuh(u)=0 \text{ for all } u,w\in U. \quad (3.10)$$

Subtracting (3.9) from (3.10), we arrive that

$$[u,h(u)]w[u,h(u)]=0 \text{ for all } u,w\in U,$$

that is,

$$[u,h(u)]U[u,h(u)]=0 \text{ for all } u\in U.$$

By Corollary 1, we conclude that

$$[u,h(u)]=0 \text{ for all } u\in U,$$

and so, h is commuting on U. This completes the proof.

The same argument can be adopted in case $[F(u),v]+[u,G(v)]=0$ for all $u,v\in U$.

Corollary 4 *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively. If* $[F(u),v]=\pm[u,G(v)]$ *for all* $u,v\in U$*, then* $U\subseteq Z$.

Theorem 3 *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively such that* $h(U)\subseteq U$*. If* $F(u)\circ v=\pm\, u\circ G(v)$ *for all* $u,v\in U$*, then h is commuting map on U.*

Proof. Let $F(u)\circ v= u\circ G(v)$ for all $u,v\in U$. Replacing v by vu and using this equation, we obtain

$$v[u,h(u)]-v[F(u),u]-(u\circ v)h(u)=0 \text{ for all } u,v\in U. \quad (3.11)$$

Replacing v by h(u)v in the (3.11), we get

$$h(u)v[u,h(u)]-h(u)v[F(u),u]-h(u)(u\circ v)h(u)-[u,h(u)]vh(u)=0 \text{ for all } u,v\in U. \quad (3.12)$$

Left multiplication of (3.11) by h(u), we have

$$h(u)v[u,h(u)]-h(u)v[F(u),u]-h(u)(u\circ v)h(u)=0 \text{ for all } u,v\in U.$$

Subtract it from (3.12), we obtain

$$[u,h(u)]vh(u)=0 \text{ for all } u,v\in U.$$

The proof is completed following equation (3.8) in Theorem 2.

The same argument can be adopted in case $F(u)\circ v+u\circ G(v)=0$ for all $u,v\in U$.

Corollary 5 *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively. If $F(u)\circ v=\pm\, u\circ G(v)$ for all $u,v\in U$, then $U\subseteq Z$.*

Theorem 4. *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively such that $h(U)\subseteq U$. If $[F(u),v]=\pm u\circ G(v)$ for all $u,v\in U$, then h is commuting map on U.*

Proof. Let $[F(u),v]-u\circ G(v)=0$ for all $u,v\in U$. Replace v by vu in this equation, we get

$$v[F(u),u]+([F(u),v]-u\circ G(v))u-u\circ vh(u)=0 \text{ for all } u,v\in U.$$

Using the hypothesis in above equation, we obtain

$$v[F(u),u]-u\circ vh(u)=0 \text{ for all } u,v\in U,$$

that is,

$$v[F(u),u]-(u\circ v)h(u)+v[u,h(u)]=0 \text{ for all } u,v\in U. \quad (3.13)$$

Replacing v by h(u)v in (3.13), we get

$$h(u)v[F(u),u]-(u\circ h(u)v)h(u)+h(u)v[u,h(u)]=0 \text{ for all } u,v\in U,$$

that is,

$$h(u)v[F(u),u]-h(u)(u\circ v)h(u)-[u,h(u)]vh(u)+h(u)v[u,h(u)]=0 \text{ for all } u,v\in U. \quad (3.14)$$

Left multiplication of (3.13) by h(u), we have

$$h(u)v[F(u),u]-h(u)(u\circ v)h(u)+h(u)v[u,h(u)]=0 \text{ for all } u,v\in U.$$

Subtract from (3.14), we get

$$[u,h(u)]vh(u)=0 \text{ for all } u,v\in U.$$

Further, the proof follows from Theorem 2, after equation (3.8). The same technique can be followed in case $[F(u),v]+u\circ G(v)=0$ for all $u,v\in U$. This completes the proof.

Corollary 6 *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F, G generalized derivations associated to the derivations d, h of R respectively. If* $[F(u),v]=\pm u\circ G(v)$ *for all* $u,v\in U$*, then* $U\subseteq Z$.

Theorem 5 *Let R be a 2-torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and F generalized derivation associated to the derivation d of R such that* $d(U)\subseteq U$*. If* $F([u,v])=[F(u),v]+[d(v),u]$ *for all* $u,v\in U$*, then d is commuting map on U.*

Proof. Let $F([u,v])=[F(u),v]+[d(v),u]$ for all $u,v\in U$. Replacing v by vu, in above relation, we get

$$F([u,vu])=[F(u),vu]+[d(vu),u] \text{ for all } u,v\in U.$$

Using the hypothesis, we obtain

$$[u,v]d(u)=v[F(u),u]+[v,u]d(u)+v[d(u),u] \text{ for all } u,v\in U,$$

that is,

$$2[u,v]d(u)=v[F(u),u]+v[d(u),u] \text{ for all } u,v\in U.$$

Replacing v by vw, $w \in U$ in above relation, we get

$$2[u,vw]d(u)=vw[F(u),u]+vw[d(u),u] \text{ for all } u,v,w \in U,$$

that is,

$$2[u,v]wd(u)+2v[u,w]d(u)=vw[F(u),u]+vw[d(u),u] \text{ for all } u,v,w \in U,$$

and so,

$$2[u,v]wd(u)=0 \text{ for all } u,v,w \in U.$$

Since R be a 2-torsion free semiprime ring, we get

$$[u,v]wd(u)=0 \text{ for all } u,v,w \in U.$$

Replacing v by d(u), we have

$$[u,d(u)]wd(u)=0 \text{ for all } u,w \in U. \tag{3.15}$$

Multiplying (3.15) on the right by u, we get

$$[u,d(u)]wd(u)u=0 \text{ for all } u,w \in U. \tag{3.16}$$

Taking w by wu in equation (3.15), we have

$$[u,d(u)]wud(u)=0 \text{ for all } u,w \in U. \tag{3.17}$$

Subtracting (3.16) from (3.17), we have

$$[u,d(u)]w[u,d(u)]=0 \text{ for all } u,w \in U.$$

By Corollary 1, we conclude that $[u,d(u)]=0$ for all $u \in U$. Hence, d is commuting on U.

Corollary 7 *Let R be a 2-torsion free prime ring, U a square-closed Lie ideal of R and F generalized derivation associated to the derivation d of R. If $F([u,v])=[F(u),v]+[d(v),u]$ for all $u,v \in U$, then $U \subseteq Z$.*

References

[1] Awtar, R., *Lie structure in prime rings with derivations,* Publ. Math. Debrecen, 31, 209-215, 1984.

[2] Bergen, J., Herstein, I. N., Kerr, W., *Lie ideals and derivation of prime rings,* J. Algebra, 71, 259-267, 1981.

[3] Bresar, M., *On the distance of the composition of two derivations to the generalized derivations,* Glasgow Math. J., 33, 89-93, 1991.

[4] Bresar, M., *On skew-commuting mappings of rings,* Bull. Austral. Math. Soc., 47, 291-296, 1993.

[5] Divinsky, N., *On commuting automorphisms of rings,* Trans. Roy. Soc. Canada Sect. III., 49, 19-52, 1955.

[6] Hongan, M., Rehman, N., Al-Omary, R. M., *Lie ideals and Jordan triple derivations in rings*, Rend. Semin. Mat. Univ. Padova, 125, 147-156, 2011.

[7] Gölbaşı, Ö., Koç, E., *Generalized derivations on Lie ideals in prime rings,* Turk. J. Math., 35, 23-28, 2011.

[8] Lee, P. H., Lee, T. K., *Lie ideals of prime rings with derivations,* Bull. Institute Math. Academia Sinica, 11, 75-79, 1983.

[9] Posner, E. C., *Derivations in prime rings,* Proc. Amer. Math. Soc., 8, 1093-1100, 1957.

[10] Rehman, N., Hongan, M., *Generalized Jordan derivations on Lie ideals associate with Hochschild 2-cocycles of rings,* Rend. Circ. Mat. Palermo, 60 (3), 437-444, 2011.

[11] Vukman, J., *Identities with derivations and autommorphisms on semiprime rings*, Int. J. Math. Math. Sci., 2005 (7), 1031-1038, 2005.