Konuralp Journal of Mathematics, 6 (2) (2018) 240-245

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Characteristic Directions of Closed Planar Homothetic Inverse Motions

Ayhan Tutar[1*], Esra İnan[2] and Önder Şener[3]

[1] *Ondokuz Mayıs University, Arts and Science Faculty Department of Mathematics Samsun, Turkey*
[2] *Ondokuz Mayıs University, Arts and Science Faculty Department of Mathematics Samsun, Turkey*
[3] *Ondokuz Mayıs University, Arts and Science Faculty Department of Mathematics Samsun, Turkey*
[*] *Corresponding author E-mail: atutar@omu.edu.tr*

Abstract

In this paper, during one-parameter closed planar homothetic inverse motions, the Steiner area formula and the polar moment of inertia were expressed. The Steiner point or Steiner normal concepts were described according to whether rotation number is different zero or equal to zero, respectively. The fixed pole point was given with its components and its relation between Steiner point or Steiner normal was specified. The sagittal motion of a crane was considered as an example. This motion was described by a double hinge consisting of the fixed control panel of the crane and the moving arm of the crane. The results obtained in the first sections of this study were applied to this motion.

Keywords: *Steiner formula, polar moment of inertia, planar kinematics, homothetic inverse motions*
2010 Mathematics Subject Classification: *53A17, 70B10*

1. Introduction

For a geometrical object rolling on a line and making a complete turn, some properties of the area of a path of a point were given by [6]. The Steiner area formula and the Holditch theorem during one-parameter closed planar homothetic motions were expressed by [8]. We calculated the expression of the Steiner formula relative to the fixed coordinate system under one parameter closed planar motions. If the points of the fixed plane which enclose the same area lie on a circle, then the centre of this circle is called the Steiner point $(h = 1)$ by [4,5]. If these points lie on a line, we use Steiner normal instead of Steiner point. Then we obtained the fixed pole point for the closed planar homothetic inverse motions. We dealt with the polar moment of inertia of a path generated by a closed planar homothetic inverse motion. Furthermore, we expressed the relationship between the area enclosed by a path and the polar moment of inertia. As an example, the sagittal motion of a crane which is described by a double hinge being fixed and moving was considered. The Steiner area formula, the moving pole point and the polar moment of inertia were calculated for this motion. Moreover, the relationship between the Steiner formula and the polar moment of inertia was expressed.

2. Closed paths in planar homothetic inverse motion

We consider one-parameter closed planar homothetic motion between two reference systems: the fixed E' and the moving E, with their origins (O', O) and orientations. Then, we take into account motion relative to the moving coordinate system (inverse motion). We know the motion defined by the transformation $X'(t) = R(t)(h(t)X - U(t))$ is called one-parameter closed planar homothetic direct motion.
By taking displacement vector $OO' = U$ and $O'O = U'$, the total angle of rotation $\alpha(t)$, the motion defined by the transformation

$$X(t) = \frac{1}{h(t)}(R(t))^T (X' - U'(t)) \tag{2.1}$$

is called one-parameter closed planar homothetic inverse motion and denoted by E'/E, where h is a homothetic scale of the motion E'/E, X and X' are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X \in E$, respectively. The homothetic scale h and the vectors X and U, U' are continuously differentiable functions of a real parameter t.
In Eq. (2.1), $X(t)$ is the trajectory with respect to the moving system of a point X' belonging to the fixed system. While Eq. (2.1) is written, we use $U = -R^T U'$. The coordinates of the equation above,

Email addresses**:** **atutar@omu.edu.tr (Ayhan Tutar), esra-unsal@hotmail.com (Esra İnan), ondersener_55@hotmail.com (Önder Şener)**

$$X' = \begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix}, U^{'}(t) = \begin{pmatrix} u_1^{'}(t) \\ u_2^{'}(t) \end{pmatrix}, X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

and the rotation matrix

$$R(t) = \begin{pmatrix} \cos(\alpha(t)) & -\sin(\alpha(t)) \\ \sin(\alpha(t)) & \cos(\alpha(t)) \end{pmatrix} \tag{2.2}$$

can be written. The components of Eq. (2.1) may be given as

$$\left.\begin{array}{l} x_1(t) = \frac{1}{h(t)}\left[(\cos\alpha(t))\left(x_1' - u_1'(t)\right) + (\sin\alpha(t))\left(x_2' - u_2'(t)\right)\right], \\ x_2(t) = \frac{1}{h(t)}\left[(-\sin\alpha(t))\left(x_1' - u_1'(t)\right) + (\cos\alpha(t))\left(x_2' - u_2'(t)\right)\right]. \end{array}\right\} \tag{2.3}$$

By differentiating Eq. (2.4) with respect to the time parameter t, we have

$$\begin{array}{rcl} dx_1 & = & -\frac{dh}{h^2}\left[\cos\alpha\left(x_1^{'} - u_1^{'}\right) + \sin\alpha\left(x_2^{'} - u_2^{'}\right)\right] \\ & + & \frac{1}{h}\left[-du_1^{'}.\cos\alpha - \left(x_1^{'} - u_1^{'}\right)\sin\alpha d\alpha - du_2^{'}.\sin\alpha + \left(x_2^{'} - u_2^{'}\right)\cos\alpha d\alpha\right] \\ dx_2 & = & -\frac{dh}{h^2}\left[-\sin\alpha\left(x_1^{'} - u_1^{'}\right) + \cos\alpha\left(x_2^{'} - u_2^{'}\right)\right] \\ & + & \frac{1}{h}\left[du_1^{'}.\sin\alpha - \left(x_1^{'} - u_1^{'}\right)\cos\alpha d\alpha - du_2^{'}.\cos\alpha - \left(x_2^{'} - u_2^{'}\right)\sin\alpha d\alpha\right] \end{array} \tag{2.4}$$

2.1. The Steiner formula in planar homothetic inverse motion

The formula for the area F' of a closed planar curve of the point X is given by

$$F' = \frac{1}{2}\oint (x_1 dx_2 - x_2 dx_1). \tag{2.5}$$

If Eqs. (2.3) and (2.4) are placed in Eq. (2.5), we have

$$\begin{array}{rcl} 2F' & = & -\left(x_1'^2 + x_2'^2\right)\oint \frac{1}{h^2} d\alpha + x_1' \oint \left(2\frac{1}{h^2} u_1' d\alpha - \frac{1}{h^2} du_2'\right) \\ & + & x_2' \oint \left(2\frac{1}{h^2} u_2' d\alpha + \frac{1}{h^2} du_1'\right) \\ & - & \oint \left\{\frac{1}{h^2}\left(u_1'^2 + u_2'^2\right) d\alpha - \frac{1}{h^2}\left(u_1' du_2' - u_2' du_1'\right)\right\}. \end{array} \tag{2.6}$$

The following expressions are used in Eq. (2.6) with the aim of shortness:

$$\left.\begin{array}{rll} & \oint & \left(2\frac{1}{h^2} u_1' d\alpha - \frac{1}{h^2} du_2'\right) = a'^* \\ & \oint & \left(2\frac{1}{h^2} u_2' d\alpha + \frac{1}{h^2} du_1'\right) = b'^* \\ - & \oint & \left\{\frac{1}{h^2}\left(u_1'^2 + u_2'^2\right) d\alpha - \frac{1}{h^2}\left(u_1' du_2' - u_2' du_1'\right)\right\} = c' \end{array}\right\} \tag{2.7}$$

The scalar term c' which is related to the trajectory of the origin of the fixed system may be given as by taking $F'_{O'} := F'\left(x_1' = 0,\ x_2' = 0\right)$, we have

$$2F'_{O'} = c'. \tag{2.8}$$

The coefficient m' is defined by

$$m' = \oint \frac{1}{h^2} d\alpha = \frac{1}{h^2(t_0)} \oint d\alpha = \frac{1}{h^2(t_0)} 2\pi\nu \tag{2.9}$$

with the rotation number ν establishes whether the lines with $F' = const.$ describing circles or straight lines. If $\nu \neq 0$, then we have circles. If $\nu = 0$, the circles reduce to straight lines. If Eqs. (2.7), (2.8) and (2.9) are substituted in Eq. (2.6), then

$$2(F' - F'_{O'}) = -(x_1'^2 + x_2'^2)m' + a'^* x'_1 + b'^* x_2' \tag{2.10}$$

can be obtained.

2.1.1. A different parametrization for the integral coefficients

Eq. (2.1) by differentiation with respect to t yields

$$dX = \frac{1}{h} d(R^T)(X' - U') - \frac{1}{h} R^T dU' - \frac{dh}{h^2} R^T (X' - U')$$

and if $X' = P' = \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}$ (the pole point) is taken,

$$0 = dX = \frac{1}{h} d(R^T)(P' - U') - \frac{1}{h} R^T dU' - \frac{dh}{h^2} R^T (P' - U') \tag{2.11}$$

can be written. Then if $U' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix}$ is solved from Eq. (2.11),

$$\left.\begin{aligned} u'_1 &= p'_1 + \tfrac{h}{h^2(d\alpha)^2+(dh)^2}(dhdu'_1 + hd\alpha du'_2) \\ u'_2 &= p'_2 + \tfrac{h}{h^2(d\alpha)^2+(dh)^2}(dhdu'_2 - hd\alpha du'_1) \end{aligned}\right\} \tag{2.12}$$

are found.
If Eq. (2.12) is replaced in Eq. (2.7),

$$\left.\begin{aligned} a'^* &= 2\oint \tfrac{1}{h^2} p'_1 d\alpha + 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2+h(dh)^2}(dhdu'_1 + hd\alpha du'_2) - \oint \tfrac{1}{h^2} du'_2, \\ b'^* &= 2\oint \tfrac{1}{h^2} p'_2 d\alpha + 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2+h(dh)^2}(dhdu'_2 - hd\alpha du'_1) + \oint \tfrac{1}{h^2} du'_1 \end{aligned}\right\} \tag{2.13}$$

can be rewritten. Also Eqs. (2.13) can be expressed as separately

$$a' := 2\oint \frac{1}{h^2} p'_1 d\alpha, \; b' := \oint \frac{1}{h^2} p'_2 d\alpha \tag{2.14}$$

$$\left.\begin{aligned} \mu'_1 &:= 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2+h(dh)^2}(dhdu'_1 + hd\alpha du'_2) - \oint \tfrac{1}{h^2} du'_2, \\ \mu'_2 &:= 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2+h(dh)2}(dhdu'_2 - hd\alpha du'_1) + \oint \tfrac{1}{h^2} du'_1 \end{aligned}\right\} \tag{2.15}$$

where $\mu' = \begin{pmatrix} \mu'_1 \\ \mu'_2 \end{pmatrix}$.
Using Eqs. (2.14) and (2.15), we found the formula for the area

$$2(F' - F'_{O'}) = -\left({x'_1}^2 + {x'_2}^2\right) m' + a'x'_1 + b'x'_2 + \mu'_1 x'_1 + \mu'_2 x'_2. \tag{2.16}$$

2.2. Steiner Point or Steiner Normal in planar homothetic inverse motion

By taking $m' \neq 0$, the Steiner point $S' = (s'_1, s'_2)$ for the closed planar homothetic inverse motion can be written as

$$s'_j = \frac{\oint \frac{1}{h^2} p'_j d\alpha}{\oint \frac{1}{h^2} d\alpha}, \quad j = 1, 2. \tag{2.17}$$

Then

$$\oint \frac{1}{h^2} p'_1 d\alpha = s'_1 m', \; \oint \frac{1}{h^2} p'_2 d\alpha = s'_2 m' \tag{2.18}$$

are found. If Eq. (2.18) is replaced in Eq. (2.15) by considering Eq. (2.17), we obtain the Steiner area formula as

$$2(F' - F'_{O'}) = -m'\left({x'_1}^2 + {x'_2}^2 - 2s'_1 x'_1 - 2s'_2 x'_2\right) + \mu'_1 x'_1 + \mu'_2 x'_2. \tag{2.19}$$

Eq. (2.19) is called the Steiner area formula for the closed planar homothetic inverse motion.
By dividing this by m' and by completing the squares, one obtains the equation of a circle

$$\begin{aligned} \left(x'_1 - (s'_1 - \tfrac{\mu'_1}{2m'})\right)^2 &+ \left(x'_2 - (s'_2 - \tfrac{\mu'_2}{2m'})\right)^2 \\ &- (s'_1 - \tfrac{\mu'_1}{2m'})^2 - (s'_2 - \tfrac{\mu'_2}{2m'})^2 = -\tfrac{2(F'-F'_{o'})}{m'}. \end{aligned} \tag{2.20}$$

All the moving points of the fixed plane which pass around equal orbit areas under the motion E/E' lie on the same circle with the center

$$M' = \left(s'_1 - \frac{\mu'_1}{2m'}, \, s_2 - \frac{\mu'_2}{2m'}\right) \tag{2.21}$$

in the fixed plane.
In the case of $h(t) = 1$, since $\mu'_1 = \mu'_2 = 0$, the point M' and the Steiner point S' coincide [3]. Also by taking $m' = 0$, if it is replaced in Eq. (2.16), then we have

$$(a' + \mu'_1)x'_1 + (b' + \mu'_2)x'_2 - 2(F' - F'_{O'}) = 0. \tag{2.22}$$

Eq. (2.22) is of a straight line. If no complete loop occurs, then $\eta = 0$ and the circle are reduced to straight lines, namely, to a circle whose center lies at infinity. The normal to the lines of equal areas in Eq. (2.22) is given by

$$n' = \begin{pmatrix} a' + \mu'_1 \\ b' + \mu'_2 \end{pmatrix} \tag{2.23}$$

which is called the Steiner normal [2].

2.3. The fixed pole point in planar homothetic inverse motion

Using Eq. (2.12), if $P' = \begin{pmatrix} p'_1 \\ p'_2 \end{pmatrix}$ is solved, then at the pole point P' of the motion

$$\left.\begin{aligned} p'_1 &= \frac{h}{h^2(d\alpha)^2+(dh)^2}(-dhdu'_1 - hd\alpha du'_2) + u'_1 \\ p'_2 &= \frac{h}{h^2(d\alpha)^2+(dh)^2}(-dhdu'_2 + hd\alpha du'_1) + u'_2 \end{aligned}\right\} \tag{2.24}$$

are found.
For $m' \neq 0$, using Eqs. (2.9) and (2.17), we arrive at the relation in Eq. (2.18) between the Steiner point and the pole point.
For $m' = 0$, using Eqs. (2.14), (2.17) and (2.23), we arrive at the relation between the Steiner normal and the pole point below:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 2\oint \frac{1}{h^2} p'_1 d\alpha \\ 2\oint \frac{1}{h^2} p'_2 d\alpha \end{pmatrix} = n' - \mu' \tag{2.25}$$

2.4. The polar moments of inertia in planar homothetic inverse motion

Because of their mathematical structure, these averages may be interpreted as polar moments of inertia T' for a path with closed homothetic motion. In this section we want to derive a formula combining T', m', n' and the area F' together.
A relation between the Steiner formula and the polar moment of inertia around the instantanous pole is given by [1]. Müller proved a relation to the polar moment of inertia around the origin [4]. Tölke investigated the same for the closed equiaffine mappings [7]. Furthermore Kuruoğlu et al. generalized Müller's results for homothetic motion [3].
If we use as α parametrization, we need to calculate

$$T' = \oint (x_1^2 + x_2^2) d\alpha \tag{2.26}$$

along the path of X. Then, using Eq. (2.4)

$$\begin{aligned} T' &= \left({x'_1}^2 + {x'_2}^2\right) m' + x'_1 \oint \left(-2\tfrac{1}{h^2} u'_1 d\alpha\right) + x'_2 \oint \left(-2\tfrac{1}{h^2} u'_2 d\alpha\right) \\ &+ \oint \tfrac{1}{h^2}\left({u'_1}^2 + {u'_2}^2\right) d\alpha \end{aligned} \tag{2.27}$$

is obtained.
We need calculate to the polar moments of inertia of the origin of the fixed system, therefore $T'_{o'} := T'\left(x'_1 = 0,\ x'_2 = 0\right)$ one obtains

$$T'_{o'} = \oint \frac{1}{h^2}\left({u'_1}^2 + {u'_2}^2\right) d\alpha. \tag{2.28}$$

If Eq. (2.28) is replaced in Eq. (2.27)

$$T' - T'_{o'} = \left({x'_1}^2 + {x'_2}^2\right) m' + x'_1 \oint (-2\frac{1}{h^2} u'_1 d\alpha) + x'_2 \oint (-2\frac{1}{h^2} u'_2 d\alpha) \tag{2.29}$$

can be written. Also if Eq. (2.12) is replaced in Eq. (2.29)

$$\begin{aligned} T' - T'_{o'} &= \left({x'_1}^2 + {x'_2}^2\right) m' \\ &+ x'_1\left(\oint -2\tfrac{1}{h^2} p'_1 d\alpha - 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2 + h(dh)^2}\left(dhdu'_1 + hd\alpha du'_2\right)\right) \\ &+ x'_2\left(\oint -2\tfrac{1}{h^2} p'_2 d\alpha - 2\oint \tfrac{d\alpha}{h^3(d\alpha)^2 + h(dh)^2}\left(dhdu'_2 - hd\alpha du'_1\right)\right) \end{aligned} \tag{2.30}$$

is found and by considering together with Eqs. (2.16) and (2.30), we arrive at the polar moments of inertia and the formula for the area below:

$$T' - T'_{o'} = -2(F' - F'_{o'}) + x'_1\left(-\oint \frac{1}{h^2} du'_2\right) + x'_2\left(\oint\left(\frac{1}{h^2} du'_1\right)\right) \tag{2.31}$$

3. Application: The inverse motion of crane

The preceding sections focused on three concepts: geometrical objects as the Steiner point or the Steiner line, the pole point and special closed paths with zero enclosed area. In this section, we intend to visualize these objects in terms of experimentally measured motion. Accordingly, we consider these orientations as characteristic for the inverse motion. Different motions will yield different characteristic values and statistical analysis should detect these differences. However, this could hardly be done by considering the kinematical input data. We will now illustrate how the previously introduced kinematical concepts can be applied to experimental data referring to the work of Dathe and Gezzi [2]. As an example, we have chosen the sagittal part of the movement of the crane during working. We have chosen the crane, because, the arm of the crane can extend and retract during one-parameter closed planar homothetic motion.
The motion of crane has a double hinge and 'a double hinge' is means that it has two systems, a fixed arm and a moving arm of the crane (Fig. 3.1). There is a control panel of the crane at the origin of the fixed system. 'L' arm can extend or retract by h parameter.

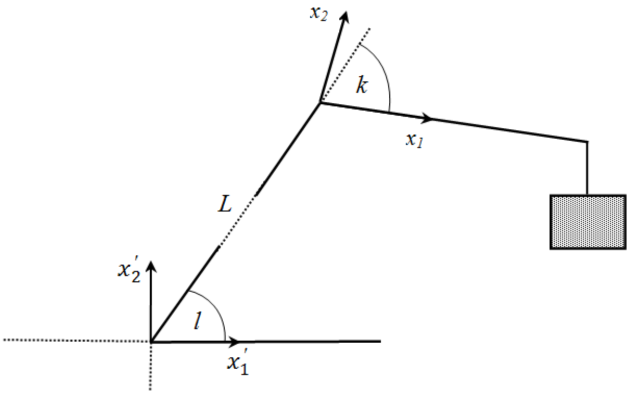

Fig. 3.1: The arms of crane as a double hinge

3.1. The mathematical model by inverse motion

We start by writing the equations of the double hinge in cartesian coordinates. Then we define using the condition $m' = 0$, the Steiner line and the total angle in relation to the double hinge. Finally, we will use the resulting values of the resulting values to characterize the experimental data.
By taking

$$R(t) = \begin{pmatrix} \cos(l(t) - k(t)) & -\sin(l(t) - k(t)) \\ \sin(l(t) - k(t)) & \cos(l(t) - k(t)) \end{pmatrix}, \; U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -L\cos(k(t)) \\ -L\sin(k(t)) \end{pmatrix} \tag{3.1}$$

we have Eq. (2.1), namely

$$X(t) = \frac{1}{h(t)} (R(t))^T \left(X' - U'(t) \right).$$

Also we know that $U' = -RU$ and

$$U'(t) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} L\cos(l(t)) \\ L\sin(l(t)) \end{pmatrix} \tag{3.2}$$

can be written. So the double hinge may be written as

$$\left. \begin{array}{l} x_1(t) = \frac{1}{h(t)} \left[cos(l(t) - k(t)) \left(x_1' - L\cos(l) \right) + \sin(l(t) - k(t)) (x_2' - L\sin(l)) \right], \\ x_2(t) = \frac{1}{h(t)} \left[-\sin(l(t) - k(t)) \left(x_1' - L\cos(l) \right) + \cos(l(t) - k(t)) (x_2' - L\sin(l)) \right]. \end{array} \right\} \tag{3.3}$$

where $\alpha = l - k$ is the resulting total angle. We begin by calculating the time derivative of Eq. (3.3). In this way, we obtain the velocities $\dot{x}_1(t), \dot{x}_2(t)$ which have to be inserted into Eq. (2.5).

$$\begin{array}{rcl} x_1\dot{x}_2 - x_2\dot{x}_1 & = & -\left({x_1'}^2 + {x_2'}^2 \right) \frac{1}{h^2} \left(\dot{l} - \dot{k} \right) + \frac{1}{h^2} L^2 \dot{k} \\ & + & x_1' \left(2\frac{1}{h^2} L\cos l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}\cos l \right) \\ & + & x_2' \left(2\frac{1}{h^2} L\sin l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}\sin l \right) \end{array} \tag{3.4}$$

We now integrate the previous equation using periodic boundary conditions by assuming the integrands as periodic functions. The periodicity of f implies that integrals of the following types vanish
$\oint df = \oint_1^F \dot{f} dt = f \,|_1^F = 0$. As a result of this, some of the integrals of Eq. (3.4) are not equal to zero and we finally obtain a simplified expression for the area namely,

$$F' = \frac{1}{2} \int_{t_1}^{t_2} (x_1\dot{x}_2 - x_2\dot{x}_1)\, dt,$$

$$\begin{array}{rcl} 2F' & = & x_1' \int_{t_1}^{t_2} \left(2\frac{1}{h^2} L\cos l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}cosl \right) dt \\ & + & x_2' \int_{t_1}^{t_2} \left(2\frac{1}{h^2} L\sin l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}sinl \right) dt. \end{array} \tag{3.5}$$

We may have the following expressions from Eq. (3.5)

$$\left. \begin{array}{l} \int_{t_1}^{t_2} \left(2\frac{1}{h^2} L\cos l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}cosl \right) dt = a'^{*}, \\ \int_{t_1}^{t_2} \left(2\frac{1}{h^2} L\sin l \left(\dot{l} - \dot{k} \right) - \frac{1}{h^2} L\dot{l}sinl \right) dt = b'^{*}. \end{array} \right\} \tag{3.6}$$

Differentiating Eq. (3.2) with respect to t and then replacing both of them in Eq. (3.6), Eq. (2.7) is found for application.
In Section (2.1.1), using Eq. (2.12)

$$\left.\begin{array}{l} a'^{*}=\underbrace{\int_{t_1}^{t_2}\left(2\frac{1}{h^2}p_1'd\alpha\right)}_{a'}+\underbrace{\int_{t_1}^{t_2}\left(\frac{2d\alpha}{h^3(d\alpha)^2+h(dh)^2}\left(dhdu_1'+hd\alpha du_2'\right)-\frac{1}{h^2}du_2'\right)}_{\mu_1'} \\ a'^{*}=\underbrace{\int_{t_1}^{t_2}\left(2\frac{1}{h^2}p_2'd\alpha\right)}_{b'}+\underbrace{\int_{t_1}^{t_2}\left(\frac{2d\alpha}{h^3(d\alpha)^2+h(dh)^2}\left(dhdu_2'-hd\alpha du_1'\right)-\frac{1}{h^2}du_1'\right)}_{\mu_2'} \end{array}\right\} \tag{3.7}$$

are found and we have a straight line below:

$$2F'=\left(a'+\mu_1'\right)x_1'+\left(b'+\mu_2'\right)x_2' \tag{3.8}$$

In this case, we have the Steiner normal

$$n'=\begin{pmatrix} a'+\mu_1' \\ b'+\mu_2' \end{pmatrix}=L\begin{pmatrix} \int_{t_1}^{t_2}\left(2\frac{1}{h^2}\cos l\left(\dot{l}-\dot{k}\right)\right)-\frac{1}{h^2}\dot{l}\cos l\right)dt \\ \int_{t_1}^{t_2}\left(2\frac{1}{h^2}\sin l\left(\dot{l}-\dot{k}\right)\right)-\frac{1}{h^2}\dot{l}\sin l\right)dt \end{pmatrix}. \tag{3.9}$$

3.2. The fixed pole point of the inverse motion of crane

If Eq. (3.2) is replaced in Eq. (2.24), the pole point $P'=\begin{pmatrix} p_1' \\ p_2' \end{pmatrix}$ with the components

$$\left.\begin{array}{l} p_1'=\dfrac{h}{(dh)^2+h^2(\dot{l}-\dot{k})^2}\left(dhL\dot{l}\sin l-h(\dot{l}-\dot{k})L\dot{l}\cos l\right)+L\cos l \\ p_2'=-\dfrac{h}{(dh)^2+h^2(\dot{l}-\dot{k})^2}\left(dhL\dot{l}\cos l+h(\dot{l}-\dot{k})L\dot{l}\sin l\right)+L\sin l \end{array}\right\} \tag{3.10}$$

is found and also using Eqs. (3.9) and (3.10), we reach the relation between the Steiner normal and the Pole point Eq. (2.25).

3.3. The polar moments of inertia of the inverse motion of crane

Using Eqs. (2.26) and (3.3), if Eq. (3.2) is replaced in Eq. (2.27)

$$T'=x_1'\oint\left(-2\frac{1}{h^2}L\cos l(\dot{l}-\dot{k})\right)dt+x_2'\oint\left(-2\frac{1}{h^2}L\sin l(\dot{l}-\dot{k})\right)dt \tag{3.11}$$

is found. By considering together with Eqs. (3.7), (3.8) and (3.11), we arrive at the relation between the polar moments of inertia and the formula for the area below:

$$T'=-2F'-x_1'L\oint\frac{1}{h^2}\dot{l}\cos l dt-x_2'L\oint\frac{1}{h^2}\dot{l}\sin l dt \tag{3.12}$$

References

[1] Blaschke W., Müller H.R., Ebene Kinematik, R. Oldenbourg, München, (1956).
[2] Dathe H., Gezzi R., Characteristic directions of closed planar motions, Zeitschrift für Angewandte Mathematik und Mechanik, 92 (2012), 731-748.
[3] Kuruoğlu N., Düldül M., Tutar A., Generalization of Steiner formula for the homothetic motions on the planar kinematics, Applied Mathematics and Mechanics (English Edition), 24 (2003), 945-949.
[4] Müller H.R., Verall gemeinerung einer Formel von Steiner, Abh. Braunschweig. Wiss. Ges., 29 (1978), 107-113.
[5] Müller H.R., Über Trägheitsmomente bei Steinerscher Massenbelegung, Abh. Braunschweig. Wiss. Ges., 29 (1978), 115-119.
[6] Steiner J., Von dem Krümmungs-Schwerpuncte ebener Curven, Journal für die reine und angewandte Mathematik, 21 (1840), 33-63.
[7] Tölke J. Steiner-Formein für die Bahnflachen geschlossener Aquiaffinbewegungen, Sitzungsber Österr. Akad. Wiss, Math.-Nat. Klasse, 187 (1978), 325-337.
[8] Tutar A., Kuruoğlu N., The Steiner formula and the Holditch theorem for the homothetic motions on the planar kinematics, Mechanism and Machine Theory, 34 (1999), 1-6.