A numerical solution for advection-diffusion equation based on a semi-Lagrangian scheme

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Abstract

This paper describes a numerical solution for the advection-diffusion equation. The proposed method is based on the operator splitting method which helps to obtain accurate solutions. That is, instead of sum, the operators are considered separately for the physical compatibility. In the process, method of characteristics combined with cubic spline interpolation and Saulyev method are used in sub-operators, respectively. After guaranteeing the convergence of the method the efficiency is also tested on one-dimensional advection-diffusion problem for a wide range of Courant numbers which plays a crucial role on the convergence of the solution. The obtained results are compared with the analytical solution of the problem and other solutions which are available in the literature. It is revealed that the proposed method produces good approach not only for small Caurant numbers but also big ones even though it is explicit method.

Keywords: Advection-diffusion equation, method of characteristics, Saulyev scheme, numerical solution.

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Yarı-Lagrangian bir şema yaklaşımına dayalı adveksiyon-difüzyon denkleminin bir sayısal çözümü

Özet

Bu çalışmada, adveksiyon-difüzyon denklemi için sayısal bir çözüm tanıtılmaktadır. Önerilen yöntem, doğru çözümler elde edilmesine yardımcı olan operatör ayırma metoduna dayanmaktadır. Yani, toplam yerine, operatörler fiziksel uyumluluk için ayrı olarak ele alınmaktadır. Bu süreçte, alt operatörler için karakteristikler yöntemi ile bir araya getirilmiş kübik spline interpolasyonu ve Saulyev metodu sırası ile kullanılmıştır. Yöntemin yakınsamasını garanti altına aldıktan sonra, verimlilik de çözümün yakınsaması üzerinde önemli bir rol oynayan farklı Courant sayıları için tek boyutlu adveksiyon-difüzyon problemi üzerinde test edilmiştir. Elde edilen sonuçlar, problemin analitik çözümü ve literatürde mevcut olan diğer çözümlerle karşılaştırılmıştır. Önerilen yöntemin, açık bir yöntem olmasına rağmen sadece küçük Caurant sayıları için değil, büyük olanlar için de iyi bir yaklaşım oluşturduğu ortaya çıkmıştır.

Anahtar kelimeler: *Adveksiyon-difüzyon denklemi, karakteristikler metodu, Saulyev şeması.*

1. Introduction

Since the early days of civilization natural waters have been used as disposal places for human waste. At the beginning, there was no problem because the amount of the waste was not at a significant level and the content of the waste was simple. As the civilization progress and human population increase, the amount of the waste rises rapidly and the content of the waste is getting complex. The behavior of the pollutants in the water has been modeled by the advection-diffusion equation, see [1]. Developing a solution becomes important to interpret how the amount changes. The mathematical expression of three-dimensional advection-diffusion equation without the source term is given as follows:

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} + W\frac{\partial C}{\partial z} = D_x\frac{\partial^2 C}{\partial x^2} + D_y\frac{\partial^2 C}{\partial y^2} + D_z\frac{\partial^2 C}{\partial z^2} \tag{1}$$

where t is time, C is the concentration of the pollution or substance, x, y and z are the spatial directions in cartesian coordinates, U, V and W represent the velocity components of the water in each direction and D_x, D_y and D_z are the diffusivity coefficients in each direction.

In this paper, for the sake of clarity, one-dimensional advection-diffusion equation, which is defined in Eq. (2), is studied.

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = D_x\frac{\partial^2 C}{\partial x^2} \tag{2}$$

where U and D_x are constant. The spatial and temporal step sizes are denoted by Δx and Δt, respectively. Moreover, Courant number, Cr, is computed as $U\Delta t/\Delta x$ and the Peclet number, Pe, is obtained as $U\Delta x/D_x$.

In order to reduce the amount of existing wastes in natural waters and to manage wastes which are disposed, properly, Eq. (2) must be solved accurately. However, Eq. (2) consists of two different types of processes which are advection (hyperbolic) and diffusion (parabolic). Even if these processes occur simultaneously, they refer extremely different events. Advection process happens in flow direction while diffusion process happens in both directions. This situation creates challenging problem for solving such equations. For this reason, many researchers developed different methods to solve Eq. (2) accurately. Some of these methods are classical finite difference method [2], high-order finite element method [3], high-order finite difference methods [4, 5], green element method [6], cubic and extended B-spline collocation methods [7-9], cubic, quartic and quintic B-spline differential quadrature methods [10, 11], method of characteristics unified with splines [12-14], cubic trigonometric B-spline approach [15], Taylor collocation and Taylor-Galerkin methods [16], Lattice Boltzmann method [17]. Moreover, non-linear advection-diffusion equation is studied in [18].

The outline of the present paper is as follows: Section 2 is dedicated to derivation and the convergence analysis of the proposed method. To obtain the solution of Eq. (2) method of characteristics with cubic spline interpolation (MOC-CS) and Saulyev method are used for advection and diffusion parts, respectively. Section 3 presents the results of the proposed method on one-dimensional advection-diffusion problems by taking different diffusion constants. The effectiveness of the proposed method is tested for several Caurant numbers. Obtained results are compared with the analytical solution and also the available results of the other researchers in the literature.

2. Derivation of the proposed method

The purpose of this section is to introduce the proposed method in details. The method is mainly based on Lie-Trotter splitting method, or sequentially spilitting, which is a kind of operator splitting method. By the help of the Lie-Trotter splitting method Eq. (2) is divided into the two sub-problems. In order to obtain the solution of each sub-problems MOC-CS and Saulyev method are used, respectively. The convergence of the method is analyzed as well as the derivation.

2.1. Lie-Trotter operator splitting method
This method is a first-order splitting method and solves the problem sequentially. When its applied to Eq. (2), the problem will be split into two sub-problems such as advection and diffusion problems. The mathematical representation of it as follows [19]:

$$\frac{\partial \widehat{C}_1}{\partial t} + U\frac{\partial \widehat{C}_1}{\partial x} = 0, \quad \widehat{C}_1(t_n,x) = C(t_n,x), \quad t \in [t_n,t_{n+1}] \tag{3}$$

$$\frac{\partial \widehat{C}_2}{\partial t} = D_x\frac{\partial^2 \widehat{C}_2}{\partial x^2}, \quad \widehat{C}_2(t_n,x) = \widehat{C}_1(t_{n+1},x), \quad t \in [t_n,t_{n+1}] \tag{4}$$

where Eq. (3) and Eq. (4) represents advection and diffusion processes, respectively. These processes are solved sequentially. At the beginning of the solution of advection-diffusion problem, Eq. (3) is solved first by using MOC-CS with the initial condition of the general advection-diffusion problem for the temporal step size of Δt. The obtained result is used as the initial condition of the Eq. (4) and this equation is solved by using Saulyev method. The result of the Eq. (4) defines the solution of the advection-diffusion equation at Δt [20].

2.2. MOC-CS for advection process

MOC-CS is a method which uses Lagrangian point of view. For this reason, the trajectory line of the concentration should be determined. Multiplying both sides of Eq. (3) by dt the partial differential equation is turned into the following ordinary differential equations.

$$\frac{d\widehat{C}_1}{dt} = 0 \tag{5}$$

$$\frac{dx}{dt} = U \tag{6}$$

where Eq. (6) represents the characteristics line in the plane (x,t). Eq. (5) shows that the concentration value of the advection process is not changed, see Figure 1. Also, the solution of the advection process with MOC-CS is independent of time. Thus, the exact solution of Eq. (3) can be written as follows:

$$\widehat{C}_1(x_{i+1}, t_{n+1}) = \widehat{C}_1(\hat{x}_i, t_n) = \widehat{C}_1\left(x_{i+1} - \int_{t_n}^{t_{n+1}} U dt, t_n\right) \tag{7}$$
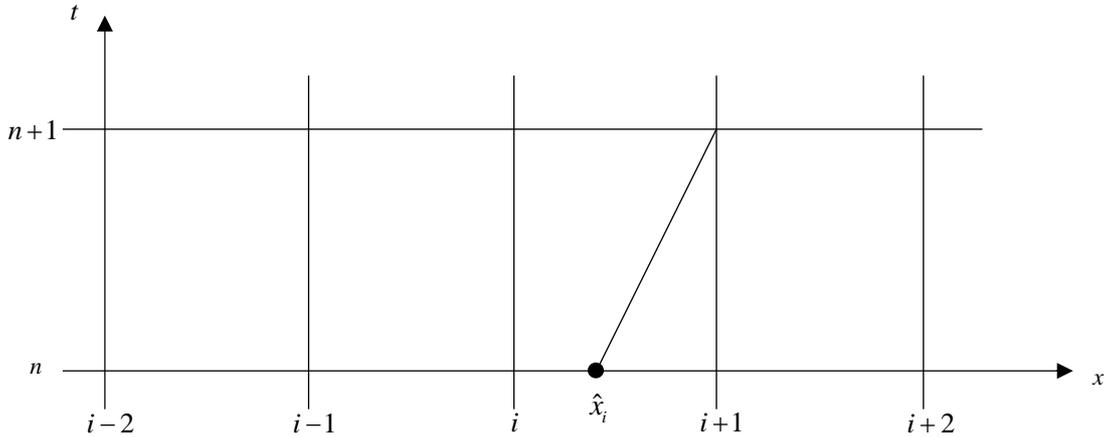

Figure 1: Finite difference grid and trajectory line of concentration in one-dimension.

To obtain the concentration value on the next time step in Eq. (7) the concentration value at the point $\hat{x}_i$ which is located between the nodal points should be found. To do so, interpolation method is used. As mentioned above, the solution of the advection process with the method of characteristics is independent of time. Therefore, there is no time discretization error, only an interpolation error in the solution. The magnitude of

this error depends on the order of the interpolation method. Taking into account this fact cubic spline polynomials method which has negligible interpolation error is used. Cubic spline polynomials can be written as follows:

$$C(x)=C_i+\alpha_i(x-x_i)+\beta_i(x-x_i)^2+\gamma_i(x-x_i)^3, \quad x_i \leq x \leq x_{i+1} \tag{8}$$

where C_i is the concentration value at the nodal point x_i and α_i, β_i, γ_i are the polynomial coefficients which have to determined by using known concentration values at time level n. The detailed discussion about construction of the cubic spline polynomials and calculation of coefficients in the polynomials is given [21].

After the polynomials are constructed in Eq. (8), the concentration values at the time level $n+1$ can be calculated by equation as follows:

$$\widehat{C}_1(x_{i+1},t_{n+1})=\widehat{C}_1(x_i,t_n)+\alpha_i(\hat{x}_i-x_i)+\beta_i(\hat{x}_i-x_i)^2+\gamma_i(\hat{x}_i-x_i)^3 \tag{9}$$

The result of Eq. (9) gives the solution of the advection process.

2.3. Saulyev method for diffusion process

Even though Saulyev method is an explicit one, it has an the advantage of using a value at the next time level which improves the quality of the solution. Also, there are two ways of conducting Saulyev method such as from left to right and from right to left. In this study from left to right version is used. Discretization of the diffusion process given in Eq. (4) with the Saulyev method as follows:

$$\left.\frac{\partial^2 \widehat{C}_2}{\partial x^2}\right|_{i,n} \approx \frac{\left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i+1/2,n} - \left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i-1/2,n}}{\Delta x} \tag{10}$$

Due to the usage of the Saulyev method, the left-hand side derivative at time level n is replaced with the derivative at time level $n+1$. A similar procedure can be applied for from right to left version of the Saulyev method.

$$\left.\frac{\partial^2 \widehat{C}_2}{\partial x^2}\right|_{i,n} \approx \frac{\left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i+1/2,n} - \left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i-1/2,n+1}}{\Delta x} \tag{11}$$

The approximations used for the spatial discretizations of the derivatives in Eq. (11) and time discretization of the time derivative in Eq. (4) as follows:

$$\left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i+1/2,n} \approx \frac{\left.\widehat{C}_2\right|_{i+1,n} - \left.\widehat{C}_2\right|_{i,n}}{\Delta x} \tag{12}$$

$$\left.\frac{\partial \widehat{C}_2}{\partial x}\right|_{i-1/2,n+1} \approx \frac{\left.\widehat{C}_2\right|_{i,n+1} - \left.\widehat{C}_2\right|_{i-1,n+1}}{\Delta x} \tag{13}$$

$$\left.\frac{\partial \widehat{C}_2}{\partial t}\right|_{i,n} \approx \frac{\left.\widehat{C}_2\right|_{i,n+1} - \left.\widehat{C}_2\right|_{i,n}}{\Delta t} \tag{14}$$

By putting these equations together the solution of the Eq. (4) is attained, diffusion process, with the Saulyev method as follows:

$$\left.\widehat{C}_2\right|_{i,n+1} = \frac{\left.\theta \widehat{C}_2\right|_{i-1,n+1} + (1-\theta)\left.\widehat{C}_2\right|_{i,n} + \left.\theta \widehat{C}_2\right|_{i+1,n}}{(1+\theta)} \tag{15}$$

where $\theta = \Delta t / \Delta x^2$.

The term at time level $n+1$ at $i=1$ in the right-hand side of Eq. (15) is unknown. However, the first term, $\left.\widehat{C}_2\right|_{0,n+1}$, is known by the boundary condition of the problem. Then for $i>1$ the unknown term $\left.\widehat{C}_2\right|_{i-1,n+1}$ is calculated from Eq. (15), hence this method is an explicit method.

2.4. Convergence analysis of the method

To prove the convergence of the operator splitting method we have to show that the sequence which is obtained by the given method approaches to zero. By this sense, the notation of Δt represents a proportional value throughout the analysis. However, in the computational part for t_0 and t_{end} any value can be chosen any value. It means that in computational part Δt is taken from $[t_0, t_{end}]$ whereas in the analysis part it is considered in $[0, 1]$.

Due to the process of the Lie-Trotter splitting method the error obtained by solving $\widehat{C}_1$ is effecting the error of the solution of $\widehat{C}_2$. The error obtained in one-step solution is effecting the solution obtained from the next step. Thus, it is expected that the error for operator splitting methods cause a cummulative error at the end of the procedure. For this purpose, we shall give the error bound of both techniques defined in Section 2.2 and 2.3.

Lemma 1. The error bound for one-step method of characteristics given in Section 2.2 is

$$\widehat{C}_1 \approx \widehat{C}_1(x,t) + O\left(\frac{\Delta x^4}{\Delta t}\right).$$

Proof.
Using Eq. (7) and the relation $\Delta x = U\Delta t$

$$\frac{\Delta t}{\Delta t}\left(\frac{\partial \widehat{C}_1}{\partial t} + U\frac{\partial \widehat{C}_1}{\partial x}\right) = 0,$$

$$\frac{1}{\Delta t}\left(\Delta t\frac{\partial \widehat{C}_1}{\partial t} + U\Delta t\frac{\partial \widehat{C}_1}{\partial x}\right)$$

$$\frac{1}{\Delta t}\left(\underbrace{\Delta t\frac{\partial\widehat{C}_1}{\partial t}+\Delta x\frac{\partial\widehat{C}_1}{\partial x}}_{d\widehat{C}_1}\right)=0 \tag{16}$$

Due to the availability the nature of the advection equation, the former equation is solved by the method of characteristics. That is,

$$\frac{1}{\Delta t}d\widehat{C}_1=0, \tag{17}$$

$$\widehat{C}_1\approx\widehat{C}_1(x,t)+O\left(\frac{\Delta x^4}{\Delta t}\right). \tag{18}$$

Lemma 2. The error bound for one-step method of Saulyev method used for diffusion part of the Eq. (2) in Section 2.3 is

$$\left\|\widehat{C}_2^2-\widehat{C}_2^1\right\|\le O\left(\Delta t^2,\Delta x^2,\frac{\Delta t^2}{\Delta x}\right),$$

where $\widehat{C}_2^i$ represents the ith iteration.

Proof. The proof can be found in [22]. We note that Δt^2 and Δx^2 can be neglected by considering $\frac{\Delta t^2}{\Delta x}$ because of the following assumption. Thus, throughout the analysis the truncation error of Saulyev method is considered as follows:

$$\left\|\widehat{C}_2^2-\widehat{C}_2^1\right\|\le O\left(\frac{\Delta t^2}{\Delta x}\right),$$

Proposition 1. Let Δt approach to zero faster than Δx. The Lie Trotter splitting method is convergent if the condition $\left|\frac{1}{(1+\rho)\Delta t}\right|<1$ is hold with the global error

$$||C_{n+1}-C_n||\le\beta(\Delta x)+K\Delta t,$$

where $\beta(\Delta x)=max\left\{O\left(\frac{\Delta x^6}{\Delta t^2}\right),O(\Delta x)\right\}$ and $\rho=D_x\frac{\Delta t}{\Delta x^2}$.

Proof:

To show the convergence of the Lie-Trotter operator splitting method we will use the induction technique. For the sake of brevity, we studied on $[0,\Delta t]$.approp

$$\widehat{C}_1^1\cong\widehat{C}_1^0(x,t)+O\left(\frac{\Delta x^4}{\Delta t}\right),\quad for\quad t\in[0,\Delta t] \tag{19}$$

$$\widehat{C}_2^1\cong\frac{1}{1+\rho}\left((1-\rho)\widehat{C}_1^1+\rho\left(\widetilde{w}+\widehat{C}_1^1\right)\right)+O\left(\frac{\Delta t^2}{\Delta x}\right)\quad for\quad t\in[0,\Delta t], \tag{20}$$

Substituting the Eq. (19) into Eq. (20) yields

$$\widehat{C}_2^1 = \underbrace{\frac{1}{1+\rho}\widehat{C}_1^0 + \frac{\rho}{1+\rho}\widetilde{w}}_{say\ w_0} + \frac{1}{1+\rho}O\left(\frac{\Delta x^4}{\Delta t}\right) + O\left(\frac{\Delta t^2}{\Delta x}\right). \tag{21}$$

After taking the one-step solution, w_0, as the initial condition of the second-step, that is on $[\Delta t, 2\Delta t]$, we have

$$\widehat{C}_1^2 \cong \widehat{C}_1^1(x,t) + \frac{1}{1+\rho}O\left(\frac{\Delta x^4}{\Delta t^2}\right) + \frac{1}{\Delta t}O\left(\frac{\Delta t^2}{\Delta x}\right) + O\left(\frac{\Delta x^4}{\Delta t}\right) \tag{22}$$

$$\widehat{C}_2^2 \cong \underbrace{\frac{1}{1+\rho}\widehat{C}_1^2 + \frac{\rho}{1+\rho}\widetilde{w}}_{say\ w_1} + O\left(\frac{\Delta t^2}{\Delta x}\right) \tag{23}$$

$$\widehat{C}_2^2 \cong \widehat{C}_2^1 + \frac{1}{(1+\rho)^2}O\left(\frac{\Delta x^4}{\Delta t^2}\right) + \frac{1}{1+\rho}\frac{1}{\Delta t}O\left(\frac{\Delta t^2}{\Delta x}\right) + \frac{1}{1+\rho}O\left(\frac{\Delta x^4}{\Delta t}\right) + O\left(\frac{\Delta t^2}{\Delta x}\right) \tag{24}$$

To get a general expression we use induction technique which leads to

$$\widehat{C}_2^{n+1} \cong \widehat{C}_2^n + \sum_{k=1}^{n}\frac{1}{(1+\rho)^k}O\left(\frac{\Delta x^4}{\Delta t^k}\right) + \sum_{k=0}^{n-1}\frac{1}{(1+\rho)^k \Delta t^k}O\left(\frac{\Delta t^2}{\Delta x}\right). \tag{25}$$

The convergence result of the Lie-Trotter method combined with method of characteristics and Saulyev method arises from the standard technique, that is

$$\widehat{C}_2^{n+1} - \widehat{C}_2^n = \sum_{k=1}^{n+1}\frac{1}{(1+\rho)^k}O\left(\frac{\Delta x^4}{\Delta t^k}\right) + \sum_{k=0}^{n}\frac{1}{(1+\rho)^k \Delta t^k}O\left(\frac{\Delta t^2}{\Delta x}\right). \tag{26}$$

Taking appropriate norm of both sides and using the Cauchy-Schwartz inequality

$$\left\|\widehat{C}_2^{n+1} - \widehat{C}_2^n\right\| = \left\|\sum_{k=1}^{n+1}\frac{1}{(1+\rho)^k \Delta t^k}O(\Delta x^4) + \sum_{k=0}^{n}\frac{1}{(1+\rho)^k \Delta t^k}O\left(\frac{\Delta t^2}{\Delta x}\right)\right\| \tag{27}$$

$$\left\|\widehat{C}_2^{n+1} - \widehat{C}_2^n\right\| \leq \left\|\sum_{k=1}^{n+1}\frac{1}{(1+\rho)^k \Delta t^k}O(\Delta x^4)\right\| + \left\|\sum_{k=0}^{n}\frac{1}{(1+\rho)^k \Delta t^k}O\left(\frac{\Delta t^2}{\Delta x}\right)\right\| \tag{28}$$

Using the condition $\left|\frac{1}{(1+\rho)\Delta t}\right| \leq 1$ which also holds $D_x\frac{\Delta t}{\Delta x^2} < -\frac{1}{2\Delta t}$. These inequalities guaranteed that

$$\left\|\widehat{C}_2^{n+1}-\widehat{C}_2^n\right\|\leq O(\Delta x^4)\left(\frac{1}{1-\frac{1}{(1+\rho)\Delta t}}\right)+O\left(\frac{\Delta t^2}{\Delta x}\right)\left(\frac{1}{1-\frac{1}{(1+\rho)\Delta t}}\right) \tag{29}$$

$$\left\|\widehat{C}_2^{n+1}-\widehat{C}_2^n\right\|\leq O(\Delta x^4)\left(\frac{\Delta x^2\Delta t+D_x\Delta t^2}{\Delta t\Delta x^2+D_x\Delta t^2-\Delta x^2}\right)+O\left(\frac{\Delta t^2}{\Delta x}\right)\left(\frac{\Delta x^2\Delta t+D_x\Delta t^2}{\Delta t\Delta x^2+\Delta t^2-\Delta x^2}\right), \tag{30}$$

$$\left\|\widehat{C}_2^{n+1}-\widehat{C}_2^n\right\|\leq O(\Delta x^4)\left(1+\frac{\Delta x^2}{\Delta t\Delta x^2+3D_x\Delta t^2}\right)+O\left(\frac{\Delta t^2}{\Delta x}\right)\left(1+\frac{\Delta x^2}{\Delta t\Delta x^2+3D_x\Delta t^2}\right). \tag{31}$$

Due to the condition $0<\Delta t<\Delta x<1$, which has to be satisfied for the stability of the Saulyev method, we have

$$1+\frac{\Delta x^2}{\Delta t\Delta x^2+3D_x\Delta t^2}\approx 1+M\frac{\Delta x^2}{\Delta t\Delta x^2}$$

By taking $C_{n+1}:=\widehat{C}_2^{n+1}$ the proof is concluded with the following relation

$$||C_{n+1}-C_n||\leq O\left(\frac{\Delta x^4}{\Delta t}\right)+O\left(\frac{\Delta t}{\Delta x}\right). \tag{32}$$

3. Numerical applications

In this section, the proposed method is applied to one-dimensional advection-diffusion problems which have the different types of characteristics such as sharp gradient and smooth behavior. For the sake of clarity we note that the values of Δt in this section are different from those in analysis part. Also, the efficiency of the MOC-CS-Saulyev is tested for different Courant numbers. Comparisons of obtained results with the analytical solution and other solutions which are available in the literature are discussed. In these comparisons computed concentration values and error norms are used which are defined as follows:

$$L_\infty=\max_i\left|C_i^{exact}-C_i^{numerical}\right| \tag{33}$$

$$L_2=\sqrt{\sum_{i=1}^M\left|C_i^{exact}-C_i^{numerical}\right|^2} \tag{34}$$

***Example* 1.** In this example, flow in a channel with the velocity $U=0.01$ *m/s* and diffusion coefficient $D=0.002$ *m²/s* are considered. The length of the channel $L=100$ *m* is taken. Analytical solution of this problem is given as follows [23]:

$$C(x,t) = \frac{1}{2} \, erfc\left(\frac{x\text{-}Ut}{\sqrt{4Dt}}\right) + \frac{1}{2} \, exp\left(\frac{Ux}{D}\right) erfc\left(\frac{x+Ut}{\sqrt{4Dt}}\right) \tag{35}$$

Boundary conditions of the problem are taken as follows

$$C(0,t) = 1 \tag{36}$$

$$-D\left(\frac{\partial C}{\partial x}\right)(L,t) = 0 \tag{37}$$

The initial condition of the problem can be obtained from Eq. (35) by using $t=0$. Comparison of obtained result with MOC-CS-Saulyev and analytical solution of the problem can be seen from Figure 2. Also, the sharp behavior of the problem is clearly shown.
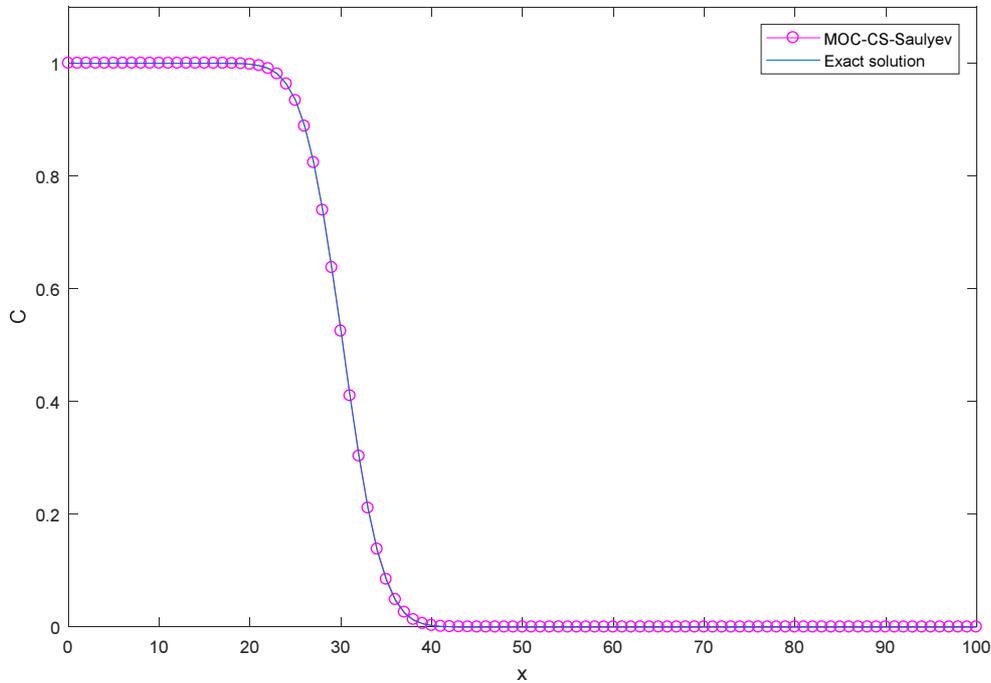
Figure 2: Comparison of the exact solution and the numerical solution obtained with MOC-CS-Saulyev method for $\Delta x=1$ *m* and $\Delta t=10$ *s*.

The spatial step size $\Delta x=1$ *m* is taken for all the calculations conducted for this problem throughout the study. Also, maximum calculation time is picked as 3000 *s*. As it can be seen from Figure 2 the coordinates of the critical concentration values of the problem are between 18 *m* to 42 *m*. This makes sense when we consider the fact that the maximum computational time is 3000 *s*.

For the obtained results in Table 1 temporal step size is taken as $\Delta t=1$ *s* because of the advection dominance of the problem where *Pe=5*. Even though MOC-CS-Saulyev has a lower order of accuracy than the sixth order compact finite difference methods, the results are so close to each other.

Table 2 shows the results for temporal step size $\Delta t=10\ s$. With this relatively small change in the temporal step size is enough to diverge the solutions of the numerical methods from the analytical solution. MOC-CS-Saulyev produced the best results in the comparison. Also, this can be seen by checking the error norm values in which MOC-CS-Saulyev has the smallest.

Table 3, the temporal step size is taken $\Delta t=60\ s$. Table 3 indicates that the proposed method is in a perfect agreement with the analytical solution.

Table 1: Comparison of numerical solutions and exact solution (Cr=0.01).

$x\ (m)$	[5] MC-CD6	[4] RK4-CD6	[15] CuTBSM	MOC-CS Saulyev	Analytical Solution
0	1.000	1.000	1.000	1.000	1.000
18	1.000	1.000	1.000	1.000	1.000
19	0.999	0.999	0.999	0.999	0.999
20	0.998	0.998	0.998	0.998	0.998
21	0.996	0.996	0.996	0.996	0.996
22	0.991	0.991	0.991	0.991	0.991
23	0.982	0.982	0.982	0.981	0.982
24	0.964	0.964	0.964	0.963	0.964
25	0.935	0.934	0.934	0.933	0.934
26	0.889	0.889	0.888	0.888	0.889
27	0.824	0.823	0.822	0.823	0.823
28	0.739	0.738	0.736	0.738	0.738
29	0.637	0.636	0.635	0.635	0.636
30	0.523	0.523	0.522	0.522	0.523
31	0.408	0.408	0.408	0.408	0.408
32	0.301	0.301	0.301	0.301	0.301
33	0.208	0.208	0.208	0.209	0.208
34	0.135	0.135	0.136	0.137	0.135
35	0.082	0.082	0.082	0.084	0.082
36	0.047	0.046	0.046	0.048	0.046
37	0.025	0.024	0.024	0.026	0.024
38	0.012	0.012	0.012	0.013	0.012
39	0.005	0.005	0.005	0.006	0.005
40	0.002	0.002	0.002	0.003	0.002
41	0.001	0.001	0.001	0.001	0.001
42	0.000	0.000	0.000	0.000	0.000
L_2	0.0017	0.0017	-	0.0047	-
L_∞	0 .0008	0.00 08	-	0.0019	-

Table 2: Comparison of numerical solutions and exact solution (Cr=0.10).

$x\ (m)$	[5] MC-CD6	[4] RK4-CD6	[15] CuTBSM	MOC-CS Saulyev	Analytical Solution
0	1.000	1.000	1.000	1.000	1.000
18	1.000	1.000	1.000	1.000	1.000
19	0.999	0.999	0.999	0.999	0.999
20	0.998	0.998	0.998	0.998	0.998
21	0.996	0.996	0.996	0.996	0.996
22	0.991	0.992	0.991	0.991	0.991
23	0.982	0.982	0.982	0.981	0.982
24	0.965	0.965	0.963	0.964	0.964
25	0.936	0.936	0.933	0.934	0.934
26	0.891	0.891	0.885	0.889	0.889
27	0.827	0.827	0.818	0.824	0.823
28	0.743	0.743	0.732	0.739	0.738
29	0.642	0.641	0.631	0.636	0.636
30	0.529	0.528	0.517	0.524	0.523
31	0.414	0.413	0.404	0.409	0.408
32	0.306	0.306	0.298	0.302	0.301
33	0.213	0.212	0.207	0.211	0.208
34	0.138	0.138	0.134	0.138	0.135
35	0.084	0.084	0.081	0.085	0.082
36	0.048	0.048	0.045	0.049	0.046
37	0.025	0.025	0.023	0.027	0.024
38	0.012	0.012	0.011	0.014	0.012
39	0.006	0.006	0.005	0.006	0.005
40	0.002	0.002	0.002	0.003	0.002
41	0.001	0.001	0.001	0.001	0.001
42	0.000	0.000	0.000	0.000	0.000
L_2	0.0148	0.0142	-	0.0071	-
L_∞	0.0060	0.0055	-	0.0031	-

Table 3: Comparison of numerical solutions and exact solution (Cr=0.60)

x (m)	[12] MOCS	[23] MOCG	[25] CBSG	[26] FEMLSF	[26] FEMQSF	[16] TC	[16] TG	MOC-CS Saulyev	Analytical Solution
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
19	1.000	0.999	1.000	1.000	1.000	0.999	0.999	1.000	0.999
20	1.000	0.998	0.999	0.999	1.000	0.999	0.998	0.999	0.998
21	1.000	0.996	0.996	0.997	0.999	0.999	0.996	0.997	0.996
22	1.000	0.990	0.991	0.993	0.996	0.998	0.991	0.993	0.991
23	1.000	0.978	0.981	0.985	0.989	0.994	0.980	0.985	0.982
24	1.000	0.957	0.961	0.970	0.974	0.987	0.960	0.969	0.964
25	1.000	0.922	0.927	0.943	0.946	0.972	0.926	0.942	0.934
26	0.996	0.870	0.874	0.902	0.900	0.945	0.874	0.898	0.889
27	1.013	0.799	0.800	0.842	0.832	0.902	0.800	0.834	0.823
28	1.047	0.708	0.706	0.763	0.743	0.838	0.705	0.750	0.738
29	0.897	0.602	0.596	0.666	0.638	0.755	0.595	0.648	0.636
30	0.457	0.488	0.479	0.556	0.524	0.653	0.479	0.535	0.523
31	0.067	0.375	0.366	0.442	0.411	0.541	0.366	0.420	0.408
32	-0.036	0.272	0.265	0.332	0.306	0.427	0.264	0.313	0.301
33	-0.010	0.185	0.181	0.235	0.218	0.320	0.181	0.220	0.208
34	0.002	0.118	0.118	0.156	0.147	0.227	0.117	0.146	0.135
35	0.000	0.070	0.072	0.096	0.095	0.152	0.072	0.092	0.082
36	0.000	0.038	0.042	0.055	0.058	0.096	0.041	0.055	0.046
37	0.000	0.020	0.023	0.030	0.034	0.057	0.023	0.031	0.024
38	0.000	0.009	0.012	0.015	0.019	0.032	0.012	0.016	0.012
39	0.000	0.004	0.006	0.007	0.010	0.017	0.006	0.008	0.005
40	0.000	0.002	0.003	0.003	0.005	0.008	0.002	0.004	0.002
41	0.000	0.001	0.001	0.001	0.003	0.004	0.001	0.002	0.001
42	0.000	0.000	0.001	0.000	0.001	0.001	0.000	0.001	0.000
L_2	-	-	-	-	-	-	-	0.0396	-
L_∞	-	-	-	-	-	-	-	0.0123	-

Example 2. In this problem flow velocity and diffusion coefficient picked as U=0.8 *m/s* and D=0.005 *m²/s*, respectively. Also length of the channel takes as L=9 *m*. Exact solution of this problem given by [24]

$$C(x,t)=\frac{1}{\sqrt{4t+1}}\, exp\left[-\frac{(x\text{-}1\text{-}Ut)^2}{D(4t+1)}\right] \tag{38}$$

subject to the following boundary conditions

$$C(0,t) = \frac{1}{\sqrt{4t+1}} exp \left[-\frac{(-1-Ut)^2}{D(4t+1)} \right], \tag{39}$$

$$C(L,t) = \frac{1}{\sqrt{4t+1}} exp \left[-\frac{(8-Ut)^2}{D(4t+1)} \right]. \tag{40}$$

The concentration values at the boundaries should be updated in each temporal step size in the solution algorithm. Initial condition of the problem can be calculated by taking *t*=0 *s* in Eq. (38).

MOC-CS-Saulyev, cubic B-spline collocation method and extended cubic B-spline collocation method are compared for different temporal step sizes in L_∞-norm. And, Table 4 emphasizes that MOC-CS-Saulyev has produced better concentration value results compare to other methods as the value of Δ*t* increases.

Table 4: Comparison of L_∞ error norms (Δ*x*=1 *m*).

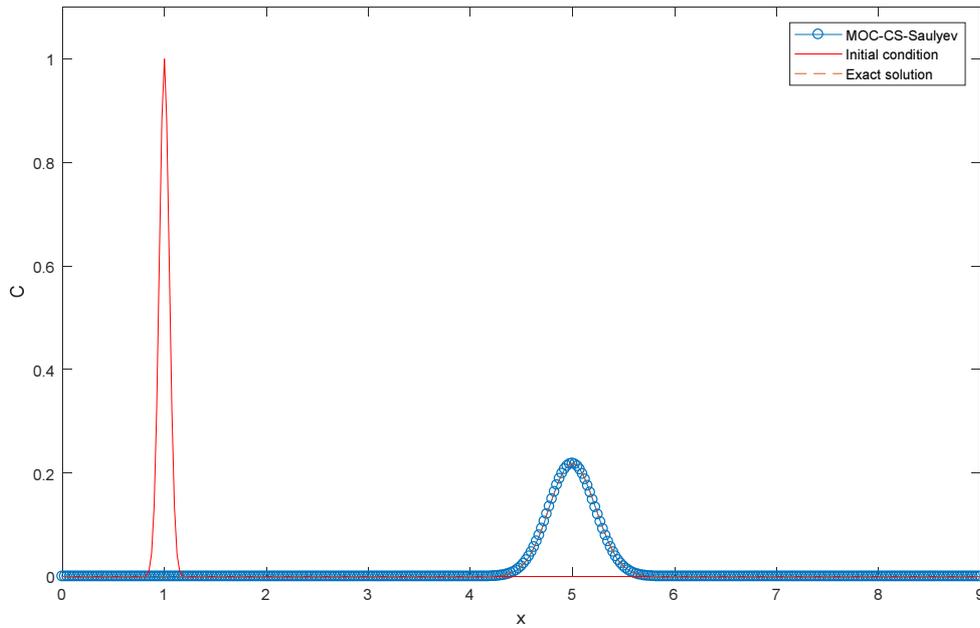| Δ*t* (s)	[9] BSCM	[9] ECuBSCM	MOC-CS Saulyev
60	0.04330	0.04250	0.01235
30	0.01962	0.01961	0.00635
20	0.01270	0.01260	0.00471
10	0.00685	0.00608	0.00314
5	0.00409	0.00307	0.00243
1	0.00224	0.00127	0.00193

Figure 3: Comparison of the exact solution and the numerical solution obtained with MOC-CS-Saulyev method for Δ*x*=0.025 *m* and Δ*t*=0.005 *s*.

The exhibited figure, Figure 3, shows the smooth behavior of the exact equation and harmony of MOC-CS-Saulyev. While the advection process moves the peak concentration along the channel, the diffusion process spreads the concentration around.

In Table 5, absolute peak error values for a wide range of Courant numbers are compared with RK4-CD6 method. RK4-CD6 has smaller absolute peak errors for $Cr<1$. This is expected because RK4-CD6 has sixth-order in space and fourth-order in time discretizations with a stability condition. Therefore, it can not produce results for $Cr\geq1$. Although MOC-CS-Saulyev has bigger absolute peak errors, it is unconditionally stable. Thanks to that it can produce solutions even when Cr is really big. When the results are examined it is seen that MOC-CS-Saulyev method has provided comparable low errors and acceptable results for all Courant numbers.

Table 5 Comparison of absolute peak errors ($\Delta x=0.025$ *m*).

Cr	Δt (s)	[4] RK4-CD6	MOC-CS Saulyev
0.016	0.0005	5.64E-09	3.01E-04
0.032	0.001	5.64E-09	2.92E-04
0.064	0.002	5.78E-09	2.75E-04
0.08	0.0025	5.99E-09	2.68E-04
0.16	0.005	1.10E-08	2.38E-04
0.32	0.01	8.18E-08	2.12E-04
0.64	0.02	9.02E-07	2.41E-04
0.8	0.025	1.79E-06	2.63E-04
1.6	0.05	-	1.77E-04
3.2	0.1	-	1.46E-04
6.4	0.2	-	1.53E-03
8	0.25	-	2.59E-03

4. Conclusions

This paper deals with a numerical solution of the advection-diffusion problem which contains two different type of processes such as advection and diffusion. Advection and diffusion processes are sequentially solved by method of characteristics with cubic-spline interpolation (MOC-CS) and Saulyev method, respectively. These two methods are combined with the help of Lie-Trotter operator splitting method. The convergence analysis of the proposed method is studied as well as computational results. After the convergence issue is guaranteed the effectiveness of the proposed method is tested on two different types of one-dimensional advection-diffusion problems. The first problem has sharp gradient which makes it quite hard to solve accurately whereas the second problem has smooth behavior. Obtained results with the MOC-CS-Saulyev method are compared with the analytical solutions of the problems and the results of other researchers available in the literature. It is seen that MOC-CS-Saulyev method, which is explicit one, has low error norm values and produces acceptable results even when the Cr is really big. As a conclusion, MOC-CS-Saulyev method can be adapted in a simple way to solve different types of advection-diffusion problems efficiently.

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