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Various techniques to solve Blasius equation

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Abstract

This paper presents three distinct approximate methods for solving Blasius Equation. The first method can be regarded as an improvement to a series solution of Blasius by means of Padè approximation. The second method is a famous type of weighted residual technique which is called Galerkin method after the famous Russian engineer and mathematician Boris Galerkin. The last method is a simple discrete, numerical technique. Additionally, in order to show the power of the last method, the Thomas-Fermi problem is solved using the same technique. Results obtained by all three methods are highly accurate in comparison with the Howarth's solution and Bender's solution.

Keywords: *Blasius equation, perturbation technique, Padè approximation, weighted residual method, Galerkin method, Thomas Fermi equation.*

Blasius denkleminin çözümü için çeşitli teknikler

Özet

Bu makalede Blasius Denklemi'ni çözmek için üç farklı yaklaşık yöntem sunmaktadır. İlk yöntem Blasius'un seri çözümünün Padè yaklaşımı yardımı ile iyileştirilmesi olarak değerlendirilebilir. İkinci yöntem ünlü Rus mühendis ve matematikçi Boris Galerkin'e izafeten Galekin Metodu olarak adlandırılan bir ağırlıklı artık yöntemdir. Son yöntem ise basit, ayrık bir sayısal tekniktir. Ek olarak son yöntemin gücünü göstermek adına Thomas-Fermi Problemi de aynı teknik ile çözülmüştür. Her üç yöntem, sonuçlar Howarth'ın ve Bender'in çözümü ile kıyaslandığında, oldukça başarılı sonuç vermektedir.

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Anahtar Kelimeler: *Blasius denklemi, pertürbasyon tekniği, Padè yaklaşımı, ağırlıklı artık yöntemler, Galerkin metodu, Thomas Fermi denklemi.*

1. Introduction

The theory of boundary layer constitutes one of the most important branch of fluid dynamics since external flows with high Reynolds' numbers are common in both nature and many engineering applications. Solving these problems generally requires a challenging effort due to the non-linearity and multidimensional character of the governing equations. Although there is reasonable amount of exact solution found for the full Navier-Stokes equations in literature, they are only valid for some particular cases and geometries [1].

An effective approach to solve an external flow problem with high Reynolds number is known as boundary layer analysis technique which is first developed by Prandtl in 1904. One of his students Blasius, in 1908, introduced a technique to transform the well-known problem of laminar boundary layer flow over a flat plate into an ordinary differential equation (ODE). Blasius equation have great importance in many engineering applications since it provides very good approximations for boundary layer thickness and total drag force in laminar external flows [2]. For example, drag force acting on a thin airfoil in a laminar flow can be very well approximated by using Blasius equation. The equation is given as:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0 \tag{1}$$

where relative boundary conditions are defined as:

$$f(0) = 0, f'(0) = 0 \tag{2}$$

$$\lim_{\eta \to \infty} f'(\eta) = 1 \tag{3}$$

and where $f'(\eta)$ is the first derivative of f with respect to η. η is the similarity variable of the problem and defined in the Cartesian coordinates as:

$$\eta = y \sqrt{\frac{U}{\upsilon x}} \tag{4}$$

where U is free stream velocity and υ is kinematic viscosity of the fluid. The relationship between f and stream function (Ψ) is given by:

$$\Psi = f(\eta) \sqrt{U \upsilon x} \tag{5}$$

and velocity components (u, v) can be derived from stream function by:

$$u = \frac{\partial \Psi}{\partial y} \tag{6}$$

$$v = -\frac{\partial \Psi}{\partial x} \quad (7)$$

The first term of (1) represents the viscos diffusion, so it becomes dominant as η approaches to zero. The second term, on the other hand, is due to convective acceleration and is dominant for high values of η. Even though the equation looks very simple at first glance, there haven't been any exact analytical solution found for over 100 years, so all solutions suggested so far depend on some approximate techniques; some of them are very successful while some of them are not. In fact, the equation has been used as a tool to investigate the success of various approximate solution techniques.

Blasius himself [3] suggested an approximate solution with an infinite series which is only convergent for small values of η. [4] reaches an estimation for shooting angle, $f''(0)$, with 8.6% relative error, using δ-perturbation method and Padè approximation. [5] gives a simple approach called iteration perturbation method and obtain $f''(0)$ with 0.73% relative error which can be considered as a very good result considering the simplicity of calculations. Using variational iteration method, [6 - 8] give valid solutions for whole domain. Solution with numerical transformations was presented by [9] while [10] uses an evaluation technique to find out Taylor coefficients. Reproducing Kernel Method was applied successfully by [11]. Amongst all of the numerical solutions, [12] is the most famous one with its extreme accuracy and is generally regarded as an exact result for comparison purposes. Some other solution techniques applied to Blasius equation are Sinc-collocation method [13], homotopy analisys method [14, 15], Laguerre-collocation method [16], homotopy perturbation method [17], parameter iteration method [18], differential transformation method [19, 20], Adomian's decomposition method [21, 22] and modified rational Legendre tau method [23].

In this paper approximate solutions for (1), under the boundary conditions (2) and (3), with three different methods are applied. First, series solution of Blasius (solution with perturbation technique) is considered and validity range of the series is increased making use of Padè approximation [24]. Secondly, a Galerkin-based weighted residual method [25] is applied with two different trial function. Finally, a simple numerical technique, which can be used for various non-linear problems, is introduced.

2. Perturbation Technique

In order to obtain a series solution, we put a perturbation parameter to (1) as:

$$f''' + \frac{1}{2}\epsilon f f'' = 0 \quad (8)$$

And assume that solution can be given with Poincare series:

$$f(\eta, \epsilon) = \sum_{i=0}^{N} \epsilon^i f_i \quad (9)$$

By putting (9) into (8) results in:

$$\sum_{i=0}^{N} \epsilon^i f_i''' + \frac{1}{2}\sum_{i=0}^{N} \epsilon^{i+1} f_i f_i'' = 0 \tag{10}$$

By considering the same powers of ϵ, we obtain a set of differential equations as:

$$f_0''' = 0 \tag{11}$$
$$f_{i+1}''' + f_i f_i'' = 0 \tag{12}$$

It is not possible to obtain a solution with boundary condition (3). Thus, we consider the initial value problem where:

$$f''(0) = \sigma \tag{13}$$

By solving (11) with (2) and (13):

$$f_0 = \frac{\sigma}{2}\eta^2 \tag{14}$$

Now solving (12) with homogenious initial conditions $(f_{i+1}''(0) = 0)$ and as $\epsilon = 1$, approximate solution can be obtained in following series form:

$$f = \sum_{i=0}^{N} \left(-\frac{1}{2}\right)^i \frac{\beta_i \sigma^{i+1}}{(3i+2)!} x^{3i+2} \tag{15}$$

where,

$$\beta_i = \sum_{j=0}^{i-1} \binom{3i-1}{3j} \beta_j \beta_{i-j-1} \tag{16}$$
$$\beta_0 = \beta_1 = 1 \tag{17}$$

This is the approximate series solution first given by Blasius [2]. However power series is only convergent for small values of η. In order to expand the range of validity, Padè approximation technique can be applied to the first n term of the series. For example, first five term of the (15) can be calculated as:

$$f = 0.5\sigma\eta^2 - 4.1667 \times 10^{-3}\sigma^2\eta^5 + 6.8204 \times 10^{-5}\sigma^3\eta^8 - 1.1743 \times 10^{-6}\sigma^4\eta^{11} + 2 \times 10^{-8}\sigma^5\eta^{14} \tag{18}$$

Padè approximant of (18) around $\eta = 0$, as degree of both numerator and denominator of the approximate rational function being seven, can be found as:

$$f = \frac{4.88204 \times 10^{-3}\sigma^2\eta^5 + 0.5\sigma\eta^2}{1.44037 \times 10^{-5}\sigma^2\eta^6 + 1.80974 \times 10^{-2}\sigma\eta^3 + 1} \tag{19}$$

and first derivative of f:

$$f' = \frac{\sigma\eta(2.44102 \times 10^{-2}\sigma\eta^3 + 1)}{1.44037 \times 10^{-5}\sigma^2\eta^6 + 1.80974 \times 10^{-2}\sigma\eta^3 + 1} + \frac{-4.21917 \times 10^{-7}\sigma^4\eta^{10} - 3.08268 \times 10^{-4}\sigma^3\eta^7 - 2.71461 \times \sigma^2\eta^4}{(1.44037 \times 10^{-5}\sigma^2\eta^6 + 1.80974 \times 10^{-2}\sigma\eta^3 + 1)^2} \quad (20)$$

It is of course impossible to satisfy the boundary condition at infinity for any finite value of σ. However, we can investigate the behaviour of f' for various values of σ and determine the necessary σ in order to approximate the exact solution of f. For this purpose, the range at which f' is monotone increasing, for positive values of η, is calculated. By using a shooting technique and forcing the $f' = 1$ at local maxima, very accurate solution is obtained for σ. In order to do that, we start with two initial guess for σ; 0 and 1 respectively:

$$\sigma = 0, f'_{max} = 0 \quad (21)$$

$$\sigma = 1, f'_{max} = 2.09866 \; where \; \eta = 4.60438 \quad (22)$$

Now more and more accurate σ values can be estimated by linear interpolation. Table 1 shows σ_n and $f'_{max,n}$ values where n denotes the number of iteration.

Table 1**.** σ and f'_{max} with respect to iteration number.

n	σ	f'_{max}	η
1	0.4765	1.28032	5.8950
2	0.3722	1.08591	6.4009
3	0.3428	1.02795	6.5790
4	0.3335	1.00927	6.6395
5	0.3304	1.00300	6.6602
6	0.3294	1.00098	6.6670
7	0.3291	1.00037	6.6690
8	0.3290	1.00017	6.6697
9	0.3289	0.99997	6.6704

This indicates that solution is not valid as $\eta > 6.67$. As a result, $f''(0)$ is estimated with only 0.96% relative error. Accuracy can be increased by adding higher order terms of the series, but we do not go further under this section. Table 2. shows calculated results for $f, f' \; and \; f''$ and their comparison to result of [12].

Table 2. Results and comparison of solution with perturbation technique.

η	f	Howarth [12]	f'	Howarth [12]	f''	Howarth [12]
0	0.0000	0.0000	0.0000	0.0000	0.3289	0.3321
1	0.1640	0.1656	0.3267	0.3298	0.3200	0.3230
2	0.6440	0.6500	0.6241	0.6298	0.2648	0.2668
3	1.3843	1.3969	0.8392	0.8461	0.1609	0.1614
4	2.2864	2.3059	0.9489	0.9555	0.0652	0.0642
5	3.2581	3.2834	0.9869	0.9916	0.0189	0.0159
6	4.2516	4.2798	0.9979	0.9990	0.0058	0.0024
7	5.2510	5.2794	0.9993	0.9999	-0.004	0.0002

This results are amazing considering the simplicity of the methodology however, since the domain is infinite and problem has a strong non-linearity, technique can be used for limited range of the domain. This is the general restriction of perturbation techniques as it is pointed out by [26].

3. Weighted residual method

In this section, an approximate solution technique using well-known Galerkin method is presented. In this method, first a trial function which satisfies all boundary condition should be introduced such as:

$$F_t(\eta, c_i) = g(\eta) + \sum_{i=1}^{N} c_i H(\eta) \tag{23}$$

Where F_t is trial function, $g(\eta)$ is a function of η which satisfies all boundary conditions, $H(\eta)$ is a function of η which satisfies the homogenious form of the given boundary conditions, and c_is are unknown coefficients to be determined.

Unless F_t has not the form of exact solution of the problem, there is always a residual function of η and c_i. In all weighted residual methods, the goal is to determine the $c_i s$, which somewhat minimize the residual, by using suitable weight functions and integral relations. Specifically, in Galerkin method, the weight functions are chosen with having same form of the trial function, for another words:

$$w_j = \frac{\partial F_t}{\partial c_j} \tag{24}$$

Where w_js are weight functions. The integral relations to find out c_is are:

$$\int_0^{\infty} w_j \left(F_t''' + \frac{1}{2} F_t'' F_t \right) d\eta = 0 \tag{25}$$

As one can notice (25) provides N equations for N c_is. In order to solve (1) with boundary conditions (2) and (3), we first introduce following trial function.

$$F_t = \eta + e^{-\eta} - 1 + \sum_{i=1}^{N} c_i \left[e^{-(i+1)\eta} - (i+1)e^{-\eta} + i \right] \tag{26}$$

And relative weight functions in this case:

$$w_j = e^{-(j+1)\eta} - (j+1)e^{-\eta} + j \tag{27}$$

By substituting trial function (26) into (1), residual function (R) is found as:

$$R = -e^{-\eta} + \sum_{i=1}^{N} c_i \left[-(i+1)^3 e^{-(i+1)\eta} + (i+1)e^{-\eta} \right] + \tag{28}$$

$$\frac{1}{2}\left[\eta + e^{-\eta} - 1 + \sum_{i=1}^{N} c_i\left[e^{-(i+1)\eta} - (i+1)e^{-\eta} + i\right]\right]$$
$$\left[e^{-\eta} + \sum_{i=1}^{N} c_i\left[(i+1)^2 e^{-(i+1)\eta} - (i+1)e^{-\eta}\right]\right]$$

The simplest case occurs when $N = 1$ which results in only one quadratic equation for one unknown. By multiplying (28) and (27) and substituting into (25), and with some algebra we get:

$$48c_1^2 + 65c_1 + 55 = 0 \tag{29}$$

The difficilty due to the non-linearity arises one more time since (29) does not have a solution for c_1 in real numbers. Although using only real part of the solution of (29) gives very good results for f, this is nothing to do with the method. However, by taking $N = 2$ and following the same process we get two quadratic equations for c_1, and c_2:

$$4720c_1^2 + 19680c_2^2 + 19170c_1c_2 + 7826c_1 + 9630c_2 + 7875 = 0 \tag{30}$$
$$3360c_1^2 + 13815c_2^2 + 13560c_1c_2 + 4550c_1 + 4599c_2 + 4900 = 0 \tag{31}$$

These equations have two distinct real solutions for c_1 and c_2 but we are interested in only one of them which minimize the residual. The relevant solution for f can be found as:

$$c_1 = -2.33236, c_2 = 0.7607 \tag{32}$$

Thus, the approximate solution becomes:

$$f_2 = \eta + e^{-\eta} - 1 - 2.33236(e^{-2\eta} - 2e^{-\eta} + 1) + 0.7607(e^{-3\eta} - 3e^{-\eta} + 2) \tag{33}$$

Where f_2 denotes the approximate solution of f when $N = 2$. f_3 and f_4 can be calculated similarly and given as:

$$f_3 = \eta + e^{-\eta} - 1 + 1.5796(e^{-2\eta} - 2e^{-\eta} + 1) + 0.2864(e^{-3\eta} - 3e^{-\eta} + 2) + 0.1093(e^{-4\eta} - 4e^{-\eta} + 3) \tag{34}$$
$$f_4 = \eta + e^{-\eta} - 1 - 1.7237(e^{-2\eta} - 2e^{-\eta} + 1) + 0.0008(e^{-3\eta} - 3e^{-\eta} + 2) + 0.8121(e^{-4\eta} - 4e^{-\eta} + 3) - 0.3547(e^{-5\eta} - 5e^{-\eta} + 4) \tag{35}$$

Comparative results for f, f' and f'' are shown by Table 3, 4 and 5 respectively.

Table 3. Comparative Results for f

η	f_2	f_3	f_4	Howarth [12]
0	0.0000	0.0000	0.0000	0.0000
1	0.1556	0.1768	0.1674	0.1656
2	0.6060	0.6805	0.6661	0.6500
3	1.3518	1.4599	1.4391	1.3969
4	2.2502	2.3731	2.3494	2.3059
5	3.2117	3.3405	3.3155	3.2834
6	4.1974	4.3284	4.3129	4.2798
7	5.1921	5.3238	5.2983	5.2794

Table 4. Comparative Results for f'

η	f_2'	f_3'	f_4'	Howarth [12]
0	0.0000	0.0000	0.0000	0.0000
1	0.2733	0.3237	0.3263	0.3298
2	0.6220	0.6682	0.6602	0.6298
3	0.8429	0.8652	0.8606	0.8461
4	0.9396	0.9486	0.9468	0.9555
5	0.9774	0.9809	0.9801	0.9916
6	0.9916	0.9929	0.9927	0.9990
7	0.9969	0.9974	0.9973	0.9999

Table 5. Comparative Results for f''

η	f_2''	f_3''	f_4''	Howarth [12]
0	0.8995	0.8715	0.2094	0.3321
1	0.3227	0.3584	0.3381	0.3230
2	0.3039	0.2787	0.2796	0.2668
3	0.1461	0.1272	0.1309	0.1614
4	0.0589	0.0503	0.0521	0.0642
5	0.0224	0.019	0.0197	0.0159
6	0.0083	0.0071	0.0073	0.0024
7	0.0031	0.0026	0.0027	0.0002

It appears that the results are very good for f and f'. However method is not successful, with this trial function, at predicting f'' for small values of η, especially when $\eta < 1$. In order to obtain better results for $\eta < 1$ we can force the trial function to include additional information where $\eta = 0$. By satisfying (1) at $\eta = 0$, we obtain following condition:

$$f'''(0) = 0 \tag{36}$$

Now we introduce following trial function which satisfies (35) besides (2) and (3).

$$f_t = \eta - \frac{e^{-2\eta}}{6} + \frac{4e^{-\eta}}{3} - \frac{7}{6} + \sum_{i=1}^{N} c_i \left[e^{-(i+2)\eta} + \frac{i+2-(i+2)^3}{6} e^{-2\eta} + \frac{(i+2)^3 - 4(i+2)}{3} e^{-\eta} - \frac{(i+2)^3 - 7(i+2) + 6}{6} \right] \tag{37}$$

This time, there are no real solutions available for $N = 1$ and $N = 2$. However, by taking $N = 3$ and applying the same procedure we obtain;

$$f_3 = \eta - \frac{e^{-2\eta}}{6} + \frac{4e^{-\eta}}{3} - \frac{7}{6} + 0.75(e^{-3\eta} - 4e^{-2\eta} + 5e^{-\eta} - 2) - 0.05(e^{-4\eta} - 10e^{-2\eta} + 16e^{-\eta} - 7) - 0.04(e^{-5\eta} - 20e^{-2\eta} + 35e^{-\eta} - 16) \tag{38}$$

Results are tabulated below.

Table 6. Results for Second Trial Function

η	f	Howarth [12]	f'	Howarth [12]	f''	Howarth [12]
0	0.0000	0.0000	0.0000	0.0000	0.3667	0.3321
1	0.1676	0.1656	0.3375	0.3298	0.3649	0.3230
2	0.6812	0.6500	0.6727	0.6298	0.2699	0.2668
3	1.4624	1.3969	0.8654	0.8461	0.1259	0.1614
4	2.3755	2.3059	0.9484	0.9555	0.0503	0.0642
5	3.3427	3.2834	0.9807	0.9916	0.0191	0.0159
6	4.3305	4.2798	0.9929	0.9990	0.0071	0.0024
7	5.3260	5.2794	0.9974	0.9999	0.0026	0.0002

Inclusion of additional information did not increase the solution accuracy in general. On the other hand, new trial function predicts $f''(0)$ far better than the previous solution. Even though this method is more complicated than the perturbation technique, it is useful since it gives good results for whole domain.

4. Discrete solution based on numerical integration

In this section, a simple numerical procedure is presented. Even though common methods such as Runge Kutta Fehlberg algorithm are successful enough to solve Blasius equation with extreme accuracy, trying less common or new solution techniques is important since all numerical methods have their own advantages and disadvantages when solving various type of differential equations. The method which is discussed in this section is based on numerical integration and quite simple to adopt many type of differential equations. We first integrate (1) from η to $\eta + h$ where h is considered to be small.

$$\int_{\eta}^{\eta+h} f''' d\varphi + \frac{1}{2}\int_{\eta}^{\eta+h} ff'' d\varphi = 0 \tag{39}$$

With some arrangement, we obtain following equation:

$$f''(\eta + h) = f''(\eta) - \frac{1}{2}\int_{\eta}^{\eta+h} ff'' d\varphi \tag{40}$$

Now we can attack this problem with two different approaches. First, we can directly apply a suitable numerical integration method, such as left end point (starting point) method, to the last term of (39). Secondly, we can apply integration by parts before numerical integration. So we get:

$$f''(\eta + h) = f''(\eta) - \frac{h}{2} f''(\eta) f(\eta) \tag{41}$$

Or:

$$f''(\eta + h) = f''(\eta) - \frac{1}{2}[f'(\eta + h)f(\eta + h) - f'(\eta)f(\eta)] - \frac{h}{2}[f'(\eta)]^2 \tag{42}$$

As it is shown in previous section, $f''(0)$ can be obtained by shooting technique, so we can assume that the only unknown at (40) is $f''(\eta + h)$. Thus we get a simple linear initial value problem in each step, which can be solved easily by direct integration.

$$f_{i+1} = \frac{\sigma}{2}h^2 + f_i'h + f_i \tag{43}$$

where f_i denotes $f(\eta)$ and f_{i+1} is $f(\eta + h)$ and σ is given by:

$$\sigma = f_i'' - \frac{h}{2}f_i f_i'' \tag{44}$$

(42) and (43) provide an incredibly simple algorithm, and should give quite good results for small values of h, since it depends only on one approximation. (41) on the other hand, includes two additional unknowns which can be determined by Taylor series expansion, or for more accurate estimate a predictor, corrector type algorithm can be applied. For this algorithm, we first try to guess f_{i+1} and f_{i+1}' by Taylor series:

$$f_{T,i+1} = f_i + hf_i' + \frac{h^2}{2}f_i'' - \frac{h^3}{12}f_i f_i'' + O(h^4) \tag{45}$$

$$f_{T,i+1}' = f_i' + hf_i'' - \frac{h^2}{4}f_i f_i'' + O(h^3) \tag{46}$$

Where T implies that, value is estimated by Taylor series while f_i''' term is calculated from (1). As it is mentioned above, these values can directly be used to determine σ as well as, in order to obtain further accuracy, calculated value of σ can be treated as a prediction and used to calculate more accurate values, than Taylor series, for f_{i+1} and f_{i+1}'. Then a correction can be applied by calculating σ with these new values.
Even though second method is far more complicated than the first one, it is important since it can be generalized for more accurate numerical integration methods such as trapezoidal method, Simpson's methods etc. Additionally, it should be noted that the first approximation cannot be used for equations which have singularities at any point while integration by parts may be used in some cases to get rid of singularities. In order to guarantee an adequately accurate result, we take the integration at (39) with trapezoidal method and obtain following relation for σ.

$$\sigma = \frac{f_i'' - \frac{h}{4}f_i f_i''}{1 + \frac{h}{4}f_{i+1}} \tag{47}$$

Extreme accuracy can be obtained by applying more accurate integration techniques, however we do not go further. By taking $h = 0.01$ and solving (42) and (46) as f_{42} denoting results for (42) and f_{46} denoting results for (46), obtained results are summarized at Table 7, 8 and 9.

Table 7. Comparative Results of Numerical Methods for f

η	f_{42}	f_{46}	Howarth [12]
0	0.0000	0.0000	0.0000
1	0.1658	0.1659	0.1656
2	0.6510	0.6513	0.6500
3	1.3987	1.3990	1.3969
4	2.3084	2.3086	2.3059
5	3.2864	3.2863	3.2834
6	4.2828	4.2827	4.2798
7	5.2825	5.2823	5.2794

Table 8. Comparative Results of Numerical Methods for f'

η	f'_{42}	f'_{46}	Howarth [12]
0	0.0000	0.0000	0.0000
1	0.3303	0.3305	0.3298
2	0.6307	0.6308	0.6298
3	0.8470	0.8469	0.8461
4	0.9561	0.9559	0.9555
5	0.9918	0.9916	0.9916
6	0.9990	0.9990	0.9990
7	0.9999	0.9999	0.9999

Table 9. Comparative Results of Numerical Methods for f''

η	f''_{42}	f''_{46}	Howarth [12]
0	0.3326	0.3328	0.3321
1	0.3236	0.3237	0.3230
2	0.2674	0.2672	0.2668
3	0.1618	0.1615	0.1614
4	0.0642	0.0642	0.0642
5	0.0158	0.0159	0.0159
6	0.0023	0.0024	0.0024
7	0.0002	0.0002	0.0002

Results are incredibly interesting since the simplest approach provides amazingly accurate results. It seems there is no used to carry all that calculations with the trapezoidal rule.

Now in order to show the power of second approach on equations which includes singularity, we consider well known Thomas-Fermi equation [27] which is given as:

$$y'' = \frac{y^{\frac{3}{2}}}{\sqrt{x}} \tag{48}$$

Where relative boundary conditions are:

$$y(0) = 1 \tag{49}$$

$$y(\infty) = 0 \tag{50}$$

As one can notice, equation has a singularity at $x = 0$ which cause additional difficulty to solve this equation numerically. In fact, this equation has high reputation with its difficulty not only since it is non-linear or has a singularity, but also equation is so sensitive to initial slope, $y'(0)$, that we cannot solve it with common techniques. We give a relatively simple technique which is developed with a similar idea. By integrating the whole equation, we obtain:

$$y'_{i+1} = y'_i + 2\sqrt{x+h}(y_{i+1})^{\frac{3}{2}} - 2\sqrt{x}(y_i)^{\frac{3}{2}} - \int_{x}^{x+h} 3\sqrt{xy}y'd\varphi \tag{51}$$

Now applying the trapezoidal method and similar procedure, we obtain following equation.

$$y'_{i+1} = \frac{y'_i + 2\sqrt{x+h}(y_{i+1})^{\frac{3}{2}} - 2\sqrt{x}(y_i)^{\frac{3}{2}} + 3\sqrt{x}\sqrt{y_i}y'_i}{1 + 3\sqrt{x+h}\sqrt{y_{i+1}}} \tag{52}$$

Following a similar procedure presented above, and using bisection method, assuming that relative $y'(0)$ value is in between -1.5 and -2, an accurate solution can be obtained. Solution of (51) as $h = 0.001$ is summarized at Table 10.

Table 10. Comparative Results for Thomas Fermi Equation

x	$y(x)$	Bender [27]
0.0	1.0000	1.0000
0.2	0.7932	0.7931
0.5	0.6071	0.6070
1.0	0.4241	0.4240
5.0	0.0788	0.0788
10	0.0243	0.0243
20	0.0058	0.0058
50	0.0006	0.0006
100	0.0001	0.0001

Additionally, initial slope, $y'(0)$, is calculated as -1.58927 with 0.076% relative error comparing to [28].

5. Conclusion

Several approximate solution techniques were applied to Blasius Equation. Obtained results reveals that all three approaches are useful and highly effective. The first approach which depends on simple perturbation technique is so easy that solution can be obtained with less than 1% error without even using any computer programme. It should be kept in mind that the results are not valid for whole domain though.

Galerkin method is more complicated than the first approach but it provides an approximate analytical solution which is valid for whole domain. The only weakness of the method is that it cannot give highly accurate results for second derivative at small values of η. In order to get rid of this weakness, a new trial function which includes an additional information was introduced. Even though weakness diminishes when using this technique, calculation became more complicated than before.

The last approach is a simple numerical method which discretize the domain into small sections and calculates the values at the end points by solving an initial value problem at each section. Most accurate results were obtained with this approach. In order to show the power of this approach, famous Thomas Fermi problem was also solved with the same technique, and a highly accurate solution was obtained one more time.

These three techniques have some benefits and disadvantages. The fist technique gave an approximated continuous curve which the discrete solutions such as finite difference techniques cannot provide. However, solution with this approach is only valid for small values of η. Galerkin method overcomes that issue while it cannot estimate the drag force on the plate correctly. Discrete solution gives the most accurate results though it is not capable of providing analytical curves as first two methods does. As a consequence, all three approaches presented in this paper can be effectively used to obtain highly accurate solutions for Blasius Flow problem.

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