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Summability factors between the absolute Cesàro methods

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Abstract

If $\sum \varepsilon_n x_n$ is summable by the method Y whenever $\sum x_n$ is summable by the method X, then we say that the factor $\varepsilon = (\varepsilon_n)$ is of type (X, Y) and denote by (X, Y). In this study we characterize the sets $(|C, \alpha|_k, |C, -1|)$, $k > 1$ and $(|C, -1|, |C, \alpha|_k)$, $k \geq 1$ for $\alpha > -1$. Also, in the special case, we give some inclusion relations between methods, which completes some open problems in literature.

Keywords: Sequence spaces, Absolute Cesàro summability, Summability Factors.

1. INTRODUCTION

Let $\sum x_n$ be an infinite series with partial sum (s_n), and by (σ_n^α) and (u_n^α) we denote the n-th Cesàro means of order α with $\alpha > -1$ of the sequences (s_n) and (nx_n), respectively, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

and

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v x_v \qquad (1.1)$$

where $A_0^\alpha = 1$, $A_n^\alpha = \binom{\alpha+n}{n}$, $A_{-n}^\alpha = 0, n \geq 1$. The series $\sum x_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \qquad (1.2)$$

On the other hand, by the well known identity $u_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ [8], the condition (1.2) can be stated by

$$\sum_{n=1}^{\infty} \frac{1}{n} |u_n^\alpha|^k < \infty.$$

Note that the definition of Flett [4] doesn't include the case $\alpha = -1$, although the Cesàro summability (C, α) is studied usually for range $\alpha \geq -1$ (see [5]). Hence, Thorpe [22] gave the seperate definition for $\alpha = -1$ as follows. If the series to sequence transformation

$$T_n = \sum_{v=0}^{n-1} x_v + (n+1)x_n \qquad (1.3)$$

tends to a finite number s as n tends to infinity, then the series $\sum x_n$ is summable by Cesàro summability $(C, -1)$ to the number s [22].

Also, by the definition of Sarıgöl [16] and Thorpe [22], the series $\sum x_n$ is said to be summable $|C, -1|_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty.$$

In this context the series spaces $|C_\alpha|_k$, $k \geq 1$, have been defined as the set of all series summable by the absolute Cesàro summability method $|C, \alpha|_k$ in [14] and [6] for $\alpha > -1$ and $\alpha = -1$, respectively.

If $\sum \varepsilon_n x_n$ is summable by the method Y whenever $\sum x_n$ is summable by the method X, then the sequence $\varepsilon = (\varepsilon_n)$ is said to be a summability factor of type (X, Y) and we write it by $\varepsilon \in (X, Y)$. In the special case if it is taken as $\varepsilon = 1$, then $1 \in (X, Y)$ leads to the comparisons of these methods, where $1 = (1,1, \dots)$ i.e., $X \subset Y$.

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Such types of factors were investigated in detail by several authors [1-3, 10-13, 15, 17-21], and recently some well known results in [10-13, 15] have been extended by Sarıgöl [15] and Sarıgöl & Hazar [7].

In this study, we deal with the problem of absolute Cesàro summability factors. More precisely, we characterize the sets $(|C,\alpha|_k, |C,-1|)$, $k>1$ and $(|C,-1|, |C,\alpha|_k)$, $k \geq 1$ for $\alpha > -1$. So we give the inclusion relations between these methods, which completes some open problems in literature.

2. MAIN RESULTS

In this section we characterize the sets $(|C,\alpha|_k, |C,-1|)$, $k>1$ and $(|C,-1|, |C,\alpha|_k)$, $k \geq 1$ for $\alpha > -1$. Thus, in the special case, we give the inclusion relations between methods.

Now, we require the following lemmas for our investigations.

Throughout this paper, k^* denote the conjugate of $k>1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k=1$.

Lemma 2.1. Let $1<k<\infty$. Then, $A(x) \in \ell$ whenever $x \in \ell_k$ if and only if

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*} < \infty$$

where $\ell_k = \{x=(x_v) : \sum |x_v|^k < \infty\}$ [15].

Lemma 2.2. Let $1 \leq k < \infty$. Then, $A(x) \in \ell_k$ whenever $x \in \ell$ if and only if

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^k < \infty,$$

[9].

We begin with the characterization of the set $(|C,\alpha|_k, |C,-1|)$ for $k>1$ and $\alpha > -1$.

Theorem 2.3. Let $k>1$ and $\alpha>-1$. Then, $\varepsilon \in (|C,\alpha|_k, |C,-1|)$ if and only if

$$\sum_{r=1}^{\infty}\left(\sum_{n=r}^{\infty}\left|\left(\frac{(n+1)\varepsilon_n A_{n-r}^{-\alpha-1}}{n} - \varepsilon_{n-1}A_{n-1-r}^{-\alpha-1}\right) r^{1/k} A_r^{\alpha}\right|\right)^{k^*} < \infty. \tag{2.1}$$

Proof. Let define u_n^α and T_n by (1.1) and

$$T_n = \sum_{v=0}^{n-1} \varepsilon_v x_v + (n+1)\varepsilon_n x_n$$

respectively. Using the definitions of u_n^α and T_n , we define the sequences $y=(y_n)$ and $\tilde{y} = (\tilde{y}_n)$ by

$$y_n = \frac{u_n^\alpha}{n^{1/k}} = \frac{1}{n^{1/k}A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v x_v \text{ , } n \geq 1$$
$$and\ y_0 = x_0 \tag{2.2}$$

and

$$\tilde{y}_n = T_n - T_{n-1} = (n+1)\varepsilon_n x_n - (n-1)\varepsilon_{n-1}x_{n-1},$$
$$n \geq 1\ and\ \tilde{y}_0 = \varepsilon_0 x_0 \tag{2.3}$$

respectively. Then, $\varepsilon \in (|C,\alpha|_k, |C,-1|)$ iff $\tilde{y} \in \ell$ whenever $y \in \ell_k$. By inversion of (2.2), we write for $n \geq 1$

$$x_n = \frac{1}{n}\sum_{v=1}^{n} A_{n-v}^{-\alpha-1} v^{1/k} A_v^\alpha y_v \tag{2.4}$$

Hence, by (2.4) we get for $n \geq 1$

$$\begin{aligned}\tilde{y}_n &= (n+1)\varepsilon_n x_n - (n-1)\varepsilon_{n-1}x_{n-1}\\ &= (n+1)\varepsilon_n \frac{1}{n}\sum_{r=1}^{n} A_{n-r}^{-\alpha-1} r^{1/k} A_r^\alpha y_r\\ &\quad - (n-1)\,\varepsilon_{n-1}\frac{1}{n-1}\sum_{r=1}^{n-1} A_{n-1-r}^{-\alpha-1} r^{1/k} A_r^\alpha y_r\\ &= \sum_{r=1}^{n}\left(\frac{(n+1)\varepsilon_n A_{n-r}^{-\alpha-1}}{n} - \varepsilon_{n-1}A_{n-1-r}^{-\alpha-1}\right) r^{1/k} A_r^\alpha y_r\\ &= \sum_{r=1}^{n} c_{nr} y_r\end{aligned}$$

where

$$c_{nr} = \begin{cases}\left(\frac{(n+1)\varepsilon_n A_{n-r}^{-\alpha-1}}{n} - \varepsilon_{n-1}A_{n-1-r}^{-\alpha-1}\right) r^{1/k} A_r^\alpha \text{ ,} 1 \leq r \leq n\\ 0 \text{ , } r > n.\end{cases}$$

So $\tilde{y} \in \ell$ whenever $y \in \ell_k$ if and only if

$$\sum_{r=1}^{\infty}\left(\sum_{n=r}^{\infty}|c_{nr}|\right)^{k^*} < \infty,$$

by Lemma 2.1 or, equivalently, (2.1) holds. Thus the proof is completed.

Since $1 \in (|C,\alpha|_k, |C,-1|)$ leads us to a comparison of summability fields of methods $|C,\alpha|_k$ and $|C,-1|$, where $1 = (1,1,\ldots)$, that is $|C,\alpha|_k \subset |C,-1|$, taking $\varepsilon_n = 1$ for all $n \geq 1$ in Theorem 2.3 we get the following result.

Corollary 2.4. If $k>1$ and $\alpha > -1$, then, $|C,\alpha|_k \subset |C,-1|$ if and only if

$$\sum_{r=1}^{\infty}\left(\sum_{n=r}^{\infty}\left|\left(\frac{(n+1)A_{n-r}^{-\alpha-1}}{n}-A_{n-1-r}^{-\alpha-1}\right)r^{1/k}A_r^{\alpha}\right|\right)^{k^*} < \infty.$$

Theorem 2.5. Let $k \geq 1$ and $\alpha > -1$. Then the necessary and sufficient condition for $\varepsilon \in (|C,-1|,|C,\alpha|_k)$, is

$$\sup_r \sum_{n=r}^{\infty}\left|\frac{r}{n^{1/k}A_n^{\alpha}}\sum_{v=r}^{n}\frac{A_{n-v}^{\alpha-1}\varepsilon_v}{v+1}\right|^k < \infty. \qquad (2.5)$$

Proof. As in proof of Theorem 2.3, we define sequences $y = (y_n)$ and $\tilde{y} = (\tilde{y}_n)$ by

$$y_n = \frac{1}{n^{1/k}A_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}\, v\varepsilon_v x_v\,, n \geq 1$$
$$and\ y_0 = \varepsilon_0 x_0$$

and

$$\tilde{y}_n = (n+1)x_n - (n-1)x_{n-1}, n \geq 1\ and$$
$$\tilde{y}_0 = x_0 \qquad (2.6)$$

respectively.

Then, $\varepsilon \in (|C,-1|,|C,\alpha|_k)$ if and only if $y \in \ell_k$ whenever $\tilde{y} \in \ell$. On the other hand, from (2.6) we write

$$x_n = \frac{1}{n(n+1)}\sum_{v=1}^{n} v\,\tilde{y}_v\,, n \geq 1\ and\ x_0 = \tilde{y}_0 \qquad (2.7)$$

Hence, by (2.7) we get for $n \geq 1$

$$y_n = \frac{1}{n^{1/k}A_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}\, v\varepsilon_v x_v$$
$$= \frac{1}{n^{1/k}A_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}\, v\varepsilon_v \frac{1}{v(v+1)}\sum_{r=1}^{v} r\,\tilde{y}_r$$
$$= \frac{1}{n^{1/k}A_n^{\alpha}}\sum_{r=1}^{n} r\left(\sum_{v=r}^{n}\frac{A_{n-v}^{\alpha-1}\varepsilon_v}{v+1}\right)\tilde{y}_r = \sum_{r=1}^{n} c_{nr}\tilde{y}_r$$

where

$$c_{nr} = \begin{cases} \dfrac{r}{n^{1/k}A_n^{\alpha}}\displaystyle\sum_{v=r}^{n}\frac{A_{n-v}^{\alpha-1}\varepsilon_v}{v+1}, 1 \leq r \leq n \\ 0, r > n. \end{cases}$$

Then, $y \in \ell_k$ whenever $\tilde{y} \in \ell$ if and only if

$$\sup_r \sum_{n=r}^{\infty}\left|\frac{r}{n^{1/k}A_n^{\alpha}}\sum_{v=r}^{n}\frac{A_{n-v}^{\alpha-1}\varepsilon_v}{v+1}\right|^k < \infty$$

by Lemma 2.2, which is the same as the condition (2.5). This completes the proof.

Since $1 \in (|C,-1|,|C,\alpha|_k)$ leads us to a comparison of summability fields of methods $|C,\alpha|_k$ and $|C,-1|$, where $1 = (1,1,\dots)$, that is $|C,-1| \subset |C,\alpha|_k$, taking $\varepsilon_n = 1$ for all $n \geq 1$ in Theorem 2.5 we get the following result.

Corollary 2.6. If $k \geq 1$ and $\alpha > -1$, then, $|C,-1| \subset |C,\alpha|_k$ if and only if

$$\sup_r \sum_{n=r}^{\infty}\left|\frac{r}{n^{1/k}A_n^{\alpha}}\sum_{v=r}^{n}\frac{A_{n-v}^{\alpha-1}}{v+1}\right|^k < \infty.$$

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