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PARAFREE METABELIAN LIE ALGEBRAS WHICH ARE DETERMINED BY PARAFREE LIE ALGEBRAS

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Abstract. Let L be a Lie algebra. Denote by $\delta^k(L)$ the k-th term of the derived series of L and by $\Delta_w(L)$ the intersection of the ideals I of L such that L/I is nilpotent. We prove that if P is a parafree Lie algebra, then the algebra $Q = (P/\delta^k(P))/\Delta_w(P/\delta^k(P))$, $k \geq 2$ is a parafree solvable Lie algebra. Moreover we show that if Q is not free metabelian, then P is not free solvable for $k = 2$.

1. Introduction

A Lie algebra L is called parafree if, L is residually nilpotent and has the same lower central sequence as a free Lie algebra. The concept of parafree Lie algebras was introduced by Baur in 1978[6]. This class is of special interest since the algebras have many properties with a free Lie algebra. One can take this opportunity to obtain some results about parafree Lie algebras. The motivation of the studies of parafree Lie algebras is based on the studies in groups. In [1,2,3,4,5] Baumslag has investigated many properties of parafree groups. In [6,7] Baur has proved that some of these properties persist in class of parafree Lie algebras. There are very few studies about parafree Lie algebras. In [8] and [9] it is shown that the ascending union and the direct limit of parafree Lie algebras are again parafree. Is it the case to use known parafree Lie algebras to find their analogous in other varieties? In this work we show that we can construct parafree metabelian Lie algebras by using parafree Lie algebras.

2. Preliminaries

Let L be a Lie algebra over a field K. The lower central series

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq ... \supseteq \gamma_n(L) \supseteq ...$$

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is defined inductively by $\gamma_2(L) = [L, L]$, $\gamma_{n+1}(L) = [\gamma_n(L), L]$, $n \geq 2$. If n is the smallest integer satisfying $\gamma_n(L) = 0$, then L is called nilpotent of class n. Let $n_1, n_2, \ldots, n_k$ be a sequence of positive integers with $n_i \geq 1$ for $i = 1, 2, \ldots, k$. We define the polycentral series of L, relative to this sequence, inductively by

$$L_{n_1,n_2,\ldots,n_i} = \gamma_{n_i}\left(\gamma_{n_{i-1}}\left(\ldots\left(\gamma_{n_1}(L)\right)\ldots\right)\right)$$

for $k \geq i$. In case $n_1 = n_2 = \ldots = n_i = 2$ we write $\delta^i(L) = L_{n_1,n_2,\ldots,n_i}$ and call it the i-th term of the derived series of L. If m is the smallest integer satisfying $\delta^m(L) = \{0\}$ then L is called solvable of class m. We usually write $L^{'}$ and $L^{''}$ for $\gamma_2(L)$ and for $\delta^2(L)$ respectively. If $L^{''} = 0$ then L is called metabelian Lie algebra.

Definition 1. *The Lie algebra L is called residually nilpotent if*

$$\bigcap\nolimits_{n=1}^{\infty} \gamma_n(L) = \{0\}.$$

Equivalently, given any non-trivial element $u \in L$ there exists an ideal J of L such that $u \notin J$ with L/J nilpotent. We associate with the lower central series of L its lower central sequence:

$$L/\gamma_2(L), L/\gamma_3(L), \ldots$$

We say that two Lie algebras L and H have the same lower central sequence if $L/\gamma_n(L) \cong H/\gamma_n(H)$ for every $n \geq 1$.

Definition 2. *The Lie algebra L is called parafree over a set X, if*
i) L is residually nilpotent, and
ii) L has the same lower central sequence as a free Lie algebra generated by the set X.

We denote by $\Delta_w(L)$ the intersection of ideals J of the Lie algebra L such that L/J is nilpotent.

3. Parafree Lie algebras

In this section we consider all Lie algebras over a field of characteristic zero.

Theorem 3. *Let P be a parafree Lie algebra. Then the algebra $Q = (P/P^{''})/\Delta_w(P/P^{''})$ is a parafree metabelian Lie algebra.*

Proof. Let P be a parafree Lie algebra and H_{ik} be an ideal of P such that $P^{''} \subset H_{ik}$, $i, k \in I$. Then for each $i, k \in I$, $H_{ik}/P^{''}$ is an ideal of $P/P^{''}$. By the definition

$$\Delta_w(P/P^{''}) = \bigcap\nolimits_{k=1}^{\infty} (H_{ik}/P^{''}),$$

where $(P/P^{''})/(H_{ik}/P^{''})$ is nilpotent. Then for any $m \geq 1$

$$\gamma_m((P/P^{''})/(H_{ik}/P^{''})) = 0.$$

Therefore we obtain

$$\begin{aligned} 0 &= \gamma_k(P/P^{''})\diagup(H_{ik}/P^{''}) = \gamma_k(P/P^{''}) + (H_{ik}/P^{''})\diagup(H_{ik}/P^{''}) \\ &\cong \gamma_k(P/P^{''})\diagup(\gamma_k(P/P^{''}) \cap (H_{ik}/P^{''}). \end{aligned}$$

That means

$$\gamma_k(P/P^{''}) \subset \gamma_k(P/P^{''}) \cap (H_{ik}/P^{''})$$

for all $k \geq m$. Hence $\gamma_k(P/P^{''}) = \gamma_k(P/P^{''}) \cap (H_{ik}/P^{''})$. Therefore

$$\gamma_k(P/P^{''}) \subset H_{ik}/P^{''}. \tag{1}$$

We now prove that Q is a parafree Lie algebra.

$$\gamma_n(Q) = \gamma_n((P/P^{''})\diagup\Delta_w(P/P^{''})) \cong \gamma_n(P/P^{''}) + \Delta_w(P/P^{''})\diagup\Delta_w(P/P^{''}).$$

Since $\Delta_w(P/P^{''}) \subset \gamma_n(P/P^{''})$, then $\gamma_n(Q) \cong \gamma_n(P/P^{''})\diagup\Delta_w(P/P^{''})$.

By using inclusion we get

$$\gamma_n(Q) \subset (H_{in}/P^{''})\diagup\Delta_w(P/P^{''}).$$

Then

$$\begin{aligned} \bigcap\nolimits_{n=1}^{\infty}\gamma_n(Q) &\subset \bigcap\nolimits_{n=1}^{\infty}((H_{in}/P^{''})\diagup\Delta_w(P/P^{''})) \\ &= \bigcap\nolimits_{n=1}^{\infty}((H_{in}/P^{''})\diagup(\bigcap\nolimits_{n=1}^{\infty}(H_{in}/P^{''})) \\ &= (\bigcap\nolimits_{n=1}^{\infty}(H_{in}/P^{''}))\diagup(\bigcap\nolimits_{n=1}^{\infty}(H_{in}/P^{''})) = 0. \end{aligned}$$

Therefore

$$\bigcap\nolimits_{n=1}^{\infty}\gamma_n(Q) = 0.$$

Hence the algebra Q is residually nilpotent. Now we are going to show that Q has the same lower central sequence as a free Lie algebra. Consider the free Lie algebra F which has the same lower central sequence as P. Let $K = F/F^{''}$. K is free metabelian and $K/\gamma_n(K) \cong F/\gamma_n(F) + F^{''}$. Then we get

$$\begin{aligned}
Q/\gamma_n(Q) &= (P/P^{''})/\Delta_w(P/P^{''}) \big/ \gamma_n((P/P^{''})/\Delta_w(P/P^{''})) \\
&\cong (P/P^{''})/\Delta_w(P/P^{''}) \big/ (\gamma_n(P/P^{''}) + \Delta_w(P/P^{''}))/\Delta_w(P/P^{''}) \\
&\cong (P/P^{''})/\Delta_w(P/P^{''}) \big/ \gamma_n(P/P^{''})/\Delta_w(P/P^{''}) \\
&\cong (P/P^{''})/\gamma_n(P/P^{''}) \\
&\cong (P/P^{''})/(\gamma_n(P) + P^{''})/P^{''} \\
&\cong P/\gamma_n(P) + P^{''} \\
&\cong P/\gamma_n(P) \big/ (\gamma_n(P) + P^{''})/\gamma_n(P) \\
&\cong P/\gamma_n(P) \big/ (P/\gamma_n(P))^{''} \\
&\cong F/\gamma_n(F) \big/ (F/\gamma_n(F))^{''} \\
&\cong F/\gamma_n(F) \big/ (F^{''} + \gamma_n(F))/\gamma_n(F) \\
&\cong F/(F^{''} + \gamma_n(F)) \\
&\cong K/\gamma_n(K).
\end{aligned}$$

Therefore Q is parafree. It remains to show that Q is metabelian. We calculate Q' and Q''.

$$\begin{aligned}
Q^{'} &= \gamma_2(Q) = \gamma_2((P/P^{''})/\Delta_w(P/P^{''})) \\
&\cong \gamma_2(P/P^{''}) + \Delta_w(P/P^{''}) \big/ \Delta_w(P/P^{''}) \\
&\cong \gamma_2(P/P^{''}) \big/ \Delta_w(P/P^{''}) \\
&\cong (\gamma_2(P) + P^{''})/P^{''} \big/ \Delta_w(P/P^{''}) \\
&\cong P^{'}/P^{'} \cap P^{''} \big/ \Delta_w(P/P^{''}) \\
&\cong P^{'}/P^{''} \big/ \Delta_w(P/P^{''}). \\
Q^{''} &= \gamma_2(Q^{'}) = \gamma_2((P^{'}/P^{''})/\Delta_w(P/P^{''})) \\
&\cong \gamma_2(P^{'}/P^{''}) + \Delta_w(P/P^{''}) \big/ \Delta_w(P/P^{''}) \\
&\cong \gamma_2(P^{'}/P^{''}) \big/ \Delta_w(P/P^{''})) \\
&\cong P^{''} + P^{''}/P^{''} \big/ \Delta_w(P/P^{''}) = 0.
\end{aligned}$$

□

Theorem 4. *Let P be a parafree Lie algebra and $Q = (P/\delta^k(P))/\Delta_w(P/\delta^k(P))$. Then Q is parafree solvable.*

Proof. Let P be a parafree Lie algebra and H_{im} be an ideal of P such that $\delta^k(P) \subset H_{im}$, $i, m \in I$. Then for each $i, m \in I$, $H_{im}/\delta^k(P)$ is an ideal of $P/\delta^k(P)$. By the definition

$$\Delta_w(P/\delta^k(P)) = \bigcap\nolimits_{m=1}^{\infty}(H_{im}/\delta^k(P)),$$

where $(P/\delta^k(P))/(H_{im}/\delta^k(P))$ is nilpotent. Then for any $m \geq 1$

$$\begin{aligned} 0 &= \gamma_m((P/\delta^k(P))\big/(H_{im}/\delta^k(P)) \cong \gamma_m(P/\delta^k(P)) + (H_{im}/\delta^k(P))\big/(H_{im}/\delta^k(P)) \\ &\cong \gamma_m(P/\delta^k(P))\big/\gamma_m(P/\delta^k(P)) \cap (H_{im}/\delta^k(P)). \end{aligned}$$

This means

$$\gamma_m(P/\delta^k(P)) \subset (\gamma_m(P)/\delta^k(P)) \cap (H_{im}/\delta^k(P)).$$

Hence

$$\gamma_m(P/\delta^k(P)) = (\gamma_m(P)/\delta^k(P)) \cap (H_{im}/\delta^k(P))$$

and

$$\gamma_m(P/\delta^k(P)) \subset H_{im}/\delta^k(P). \tag{2}$$

To show that Q is a parafree Lie algebra, we need to prove that Q is a residually nilpotent Lie algebra.

$$\gamma_n(Q) = \gamma_n((P/\delta^k(P))\big/\Delta_w(P/\delta^k(P)) \cong \gamma_n(P/\delta^k(P)) + \Delta_w(P/\delta^k(P))\big/\Delta_w(P/\delta^k(P)).$$

Since $\Delta_w(P/\delta^k(P)) \subset \gamma_n(P/\delta^k(P))$, then

$$\gamma_n(Q) = \gamma_n(P/\delta^k(P))\big/\Delta_w(P/\delta^k(P)).$$

By using inclusion in (2), we get $\gamma_n(Q) \subset (H_{in}/\delta^k(P))\big/\Delta_w(P/\delta^k(P))$. Then

$$\begin{aligned} \bigcap\nolimits_{n=1}^{\infty}\gamma_n(Q) &\subset \bigcap\nolimits_{n=1}^{\infty}((H_{in}/\delta^k(P))\big/\Delta_w(P/\delta^k(P)) \\ &= \bigcap\nolimits_{n=1}^{\infty}((H_{in}/\delta^k(P)\big/(\bigcap\nolimits_{n=1}^{\infty}(H_{in}/\delta^k(P))) \\ &= (\bigcap\nolimits_{n=1}^{\infty}(H_{in}/\delta^k(P)))\big/(\bigcap\nolimits_{n=1}^{\infty}(H_{in}/\delta^k(P))) = 0. \end{aligned}$$

Therefore

$$\bigcap\nolimits_{n=1}^{\infty}\gamma_n(Q) = 0.$$

That is, Q is residually nilpotent. Now we are going to show that Q has the same lower central sequence as a free Lie algebra. Since P is parafree, then there is a free Lie algebra F such that $P/\gamma_n(P) \cong F/\gamma_n(F)$. Let $K = F/\delta^k(F)$. Then

$$K/\gamma_n(K) = (F/\delta^k(F))\big/\gamma_n(F/\delta^k(F)) \cong F/\delta^k(F)\big/(\gamma_n(F) + \delta^k(F))/\delta^k(F)$$

and

$$K/\gamma_n(K) \cong F/\gamma_n(F) + \delta^k(F). \tag{3}$$

On the other hand, we get

$$\begin{aligned} Q/\gamma_n(Q) &= ((P/\delta^k(P))/\Delta_w(P/\delta^k(P)))\big/\gamma_n((P/\delta^k(P))/\Delta_w(P/\delta^k(P))) \\ &\cong ((P/\delta^k(P))/\Delta_w(P/\delta^k(P)))\big/(\gamma_n(P/\delta^k(P)) \\ &\quad +\Delta_w(P/\delta^k(P)))/\Delta_w(P/\delta^k(P)) \\ &\cong (P/\delta^k(P))\big/(\gamma_n(P) + \delta^k(P))/\delta^k(P) \\ &= P\big/(\gamma_n(P) + \delta^k(P)) \\ &\cong P/\gamma_n(P)\big/(\gamma_n(P) + \delta^k(P))/\gamma_n(P). \end{aligned}$$

By using the fact that P is parafree and (3), we have

$P/\gamma_n(P)\diagup\delta^k(P/\gamma_n(P)) \cong F/\gamma_n(F)\diagup\delta^k(F/\gamma_n(F)) \cong F/\delta^k(F)+\gamma_n(F) \cong K/\gamma_n(K)$.

Therefore,

$$Q/\gamma_n(Q) \cong K/\gamma_n(K).$$

Hence Q is parafree. It remains to show that Q is solvable.

$$\begin{aligned}\delta_k(Q) &= \delta_k((P/\delta^k(P))/\Delta_w(P/\delta^k(P))) \cong \delta_k((P/\delta^k(P)) \\ &\quad +\Delta_w(P/\delta^k(P))/\Delta_w(P/\delta^k(P)) \\ &\cong \delta_k((P/\delta^k(P))/\delta_k((P/\delta^k(P)) \cap \Delta_w(P/\delta^k(P)) = 0.\end{aligned}$$

This completes the proof. □

Theorem 5. *Let P be a parafree Lie algebra and $Q = (P/P^{''})/\Delta_w(P/P^{''})$. If P is free solvable then Q is free metabelian.*

Proof. Suppose that P is a free solvable parafree Lie algebra. Then there is a free Lie algebra F such that $P \cong F/\delta^k(F)$. Therefore

$$\begin{aligned}P/P''^k(F)\diagup(F/\delta^k(F))'' &\cong F/\delta^k(F)\diagup F''^k(F)/\delta^k(F) \\ &\cong F/\delta^k(F)\diagup F^{''}/\delta^k(F) \cong F/F^{''}.\end{aligned}$$

Then, $Q \cong F/F^{''}\diagup\Delta_w(F/F^{''})$. It is clear that $\Delta_w(F/F^{''}) = 0$. Then $Q \cong F/F^{''}$. Hence Q is free metabelian. □

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