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Further Results for Elliptic Biquaternions

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Kahraman Esen Özen[1,*], *Murat Tosun*[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey.*
ORCID: 0000-0002-3299-6709
[2] *Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey.*
ORCID: 0000-0002-4888-1412
* *Corresponding Author E-mail: kahraman.ozen1@ogr.sakarya.edu.tr*

Abstract: In this study, we show that the elliptic biquaternion algebra is algebraically isomorphic to the 2×2 total elliptic matrix algebra and so, we get a faithful 2×2 elliptic matrix representation of an elliptic biquaternion. Also, we investigate the similarity and the Moore-Penrose inverses of elliptic biquaternions by means of these matrix representations. Moreover, we establish universal similarity factorization equality (USFE) over the elliptic biquaternion algebra which reveals a deeper relationship between an elliptic biquaternion and its elliptic matrix representation. This equality and these representations can serve as useful tools for discussing many problems concerned with the elliptic biquaternions, especially for solving various elliptic biquaternion equations.

Keywords: Elliptic biquaternion, Generalized inverse, matrix representation, Universal similarity factorization equality.

1 Introduction

Sir W. R. Hamilton introduced the set of quaternions in 1843 [1], that was one of the his best contribution made to mathematical science. The set of quaternions can be represented as

$$H = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : \ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where the quaternion bases 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the multiplication laws

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

W.R. Hamilton introduced complex quaternion algebra ten years later from discovery of quaternions, in 1853 [2]. The set of complex quaternions is defined by

$$H_{\mathbb{C}} = \{Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} : \ Q_0, Q_1, Q_2, Q_3 \in \mathbb{C}\}$$

where 1, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are exactly the same in quaternions. There can be found some studies related to quaternions in [3–10].

A fundamental fact (see e.g., [3–6]) is that complex quaternion algebra is isomorphic to the 2×2 total complex matrix algebra $M_2(\mathbb{C})$ by means of the isomorphism

$$\psi : \ H_{\mathbb{C}} \to M_2(\mathbb{C}), \quad \psi(a_o + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = \begin{bmatrix} a_o + a_1 i & -a_2 - a_3 i \\ a_2 - a_3 i & a_o - a_1 i \end{bmatrix}.$$

Based on this isomorphism, any complex quaternion $x \in H_{\mathbb{C}}$ has a faithful complex matrix representation $\psi(x) \in M_2(\mathbb{C})$.

USFE over an algebra can serve as a precious material for investigating various problems concerned with this algebra and their applications. There can be found some studies which include USFE over various algebras in [11–15].

Recently, we have introduced the set of elliptic biquaternions and presented various studies related to elliptic biquaternions. We refer the readers to [16–20].

This article is organized as follows. In section 2, we recall the fundamental concepts of elliptic matrices and review the elliptic biquaternions and their matrices to disambiguate the ensuing sections. In section 3, 2×2 elliptic matrix representations of elliptic biquaternions are introduced. In section 4, the similarity of elliptic biquaternions is investigated and USFE for elliptic biquaternions is established. In section 5, the Moore-Penrose inverses of elliptic biquaternions are discussed with the aid of their aforementioned matrix representations.

Throughout this paper, the following notations are used. $\mathbb{C}$, $\mathbb{C}_p$, $H\mathbb{C}_p$, $M_{m\times n}(\mathbb{C})$, $M_{m\times n}(\mathbb{C}_p)$ and $M_{m\times n}(H\mathbb{C}_p)$ denote the complex number field, the elliptic number field, the elliptic biquaternion algebra, the set of all $m \times n$ complex matrices, the set of all $m \times n$ elliptic matrices and the set of all $m \times n$ elliptic biquaternion matrices, respectively. For convenience, the set of all square matrices on $\mathbb{C}_p$ is denoted by $M_n(\mathbb{C}_p)$.

2 Preliminaries

In this section, we recall some necessary properties of elliptic matrices. Also, we give some notions about elliptic biquaternions and their matrices. For more details see [16, 20, 21].

In the set of elliptic matrices $M_{m\times n}(\mathbb{C}_p)$ including $m\times n$ matrices with elliptic number entries, the scalar multiplication is defined as

$$\lambda A=\lambda\left[a_{ij}\right]=\left[\lambda a_{ij}\right]\in M_{m\times n}(\mathbb{C}_p)$$

where $\lambda\in\mathbb{C}_p$ and $A=\left[a_{ij}\right]\in M_{m\times n}(\mathbb{C}_p)$. Also, the ordinary matrix addition and multiplication are defined in this set. Let an elliptic matrix $A=\left[a_{ij}\right]\in M_{m\times n}(\mathbb{C}_p)$ be given. In that case, the complex conjugate of A is defined as $\overline{A}=\left[a_{ij}^*\right]\in M_{m\times n}(\mathbb{C}_p)$ where a_{ij}^* is the usual complex conjugation of $a_{ij}\in\mathbb{C}_p$. Also, the conjugate transpose of A is defined as $A^*=\left(\overline{A}\right)^T\in M_{n\times m}(\mathbb{C}_p)$.

On the other hand, the square elliptic matrices A and B with the same dimension over $\mathbb{C}_p$ are said to be similar, if there exists an invertible elliptic matrix P satisfying $P^{-1}AP=B$, [21].

The set of elliptic biquaternions is represented as

$$H\mathbb{C}_p=\{Q=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}\ :\ A_0,A_1,A_2,A_3\in\mathbb{C}_p\}$$

where $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$ are the quaternionic units which satisfy

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-1,\ \ \mathbf{ij}=-\mathbf{ji}=\mathbf{k},\ \ \mathbf{jk}=-\mathbf{kj}=\mathbf{i},\ \ \mathbf{ki}=-\mathbf{ik}=\mathbf{j}.$$

The operations of addition, multiplication and scalar multiplication are given as

$$\begin{aligned}
Q+R=&\ (A_0+B_0)\ +\ (A_1+B_1)\ \mathbf{i}\ +\ (A_2+B_2)\ \mathbf{j}\ +\ (A_3+B_3)\ \mathbf{k}\\
QR=&\ [(A_0B_0)-(A_1B_1)-(A_2B_2)-(A_3B_3)]\\
&+\ [(A_0B_1)+(A_1B_0)+(A_2B_3)-(A_3B_2)]\ \mathbf{i}\\
&+\ [(A_0B_2)-(A_1B_3)+(A_2B_0)+(A_3B_1)]\ \mathbf{j}\\
&+[(A_0B_3)+(A_1B_2)-(A_2B_1)+(A_3B_0)]\ \mathbf{k}\\
\lambda Q\ =&\ (\lambda A_0)+(\lambda A_1)\,\mathbf{i}+(\lambda A_2)\,\mathbf{j}+(\lambda A_3)\,\mathbf{k}
\end{aligned}$$

where $\lambda\in\mathbb{C}_p$ and $Q=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$, $R=B_0+B_1\mathbf{i}+B_2\mathbf{j}+B_3\mathbf{k}\in H\mathbb{C}_p$. Also, the following equations

$$\begin{aligned}
Q^*\ =&\ {A_0}^*+{A_1}^*\mathbf{i}+{A_2}^*\mathbf{j}+{A_3}^*\mathbf{k}\\
\overline{Q}\ =&\ A_0-A_1\mathbf{i}-A_2\mathbf{j}-A_3\mathbf{k}\\
Q^\dagger=&\left(\overline{Q}\right)^*={A_0}^*-{A_1}^*\mathbf{i}-{A_2}^*\mathbf{j}-{A_3}^*\mathbf{k}
\end{aligned}$$

state the complex conjugate, quaternion conjugate and Hermitian conjugate of Q, respectively. Here the stars given as superscript on $A_0,\ A_1,\ A_2$ and A_3 indicate the usual complex conjugation. If $Q^\dagger=Q$, Q is said to be Hermitian, [16].

As can be seen easily, the meanings of the symbols; star and dagger given as superscript and over bar vary according to terms which they are applied to. We need to warn the readers about these cases for the rest of the paper.

Another thing that can be of importance is the inner product of two elliptic biquaternions. The inner product of Q and R is defined in the following way:

$$\langle Q,R\rangle=\frac{1}{2}\left(\overline{Q}R+\overline{R}Q\right)=\frac{1}{2}\left(Q\overline{R}+R\overline{Q}\right)=A_0B_0+A_1B_1+A_2B_2+A_3B_3.$$

On the other hand, the semi-norm of Q is expressed as follows:

$$N_Q=\langle Q,Q\rangle={A_0}^2+{A_1}^2+{A_2}^2+{A_3}^2=Q\,\overline{Q}=\overline{Q}\,Q\in\mathbb{C}_p.$$

When $N_Q\neq 0$, Q has a multiplicative inverse such that $Q^{-1}=\overline{Q}/N_Q$, [16].

The set of all $m\times n$ type matrices with elliptic biquaternion entries is denoted by $M_{m\times n}(H\mathbb{C}_p)$. The ordinary matrix addition and multiplication are defined in this matrix set. Also, the scalar multiplication is expressed as in the following:

$$QA=Q\left[a_{ij}\right]=\left[Qa_{ij}\right]\in M_{m\times n}(H\mathbb{C}_p)$$

where $Q\in H\mathbb{C}_p$ and $A=\left[a_{ij}\right]\in M_{m\times n}(H\mathbb{C}_p)$. For $A=\left[a_{ij}\right]\in M_{m\times n}(H\mathbb{C}_p)$, the Hermitian conjugate of A is defined as $A^\dagger=\left[{a_{ji}}^\dagger\right]\in M_{n\times m}(H\mathbb{C}_p)$ where ${a_{ji}}^\dagger$ is the Hermitian conjugate of $a_{ji}\in H\mathbb{C}_p$, [20].

3 Elliptic Matrix Representations of Elliptic Biquaternions

In this section, we get 2×2 elliptic matrix representations of elliptic biquaternions and give some properties which are satisfied by these representations and elliptic biquaternions.

Let us consider the matrix set $M_2(\mathbb{C}_p)$ which can be represented as

$$M_2(\mathbb{C}_p)=\left\{\begin{bmatrix} x & y \\ z & t \end{bmatrix}:\ x,y,z,t\in\mathbb{C}_p\right\}.$$

In the following lemma, we show that this matrix set can be represented as in a somewhat different form which are used to define the required isomorphism.

Result 1. *The set of* 2×2 *elliptic matrices can be represented as*

$$M_2(\mathbb{C}_p)=\left\{\begin{bmatrix} X_0+\frac{1}{\sqrt{|p|}}IX_1 & -X_2-\frac{1}{\sqrt{|p|}}IX_3 \\ X_2-\frac{1}{\sqrt{|p|}}IX_3 & X_0-\frac{1}{\sqrt{|p|}}IX_1 \end{bmatrix}: X_i=x_i+I{x_i}'\in\mathbb{C}_p,\ 0\le i\le 3\right\}. \tag{1}$$

Proof: Let $A=\begin{bmatrix} z_1+Iz_1{}^{\#} & z_2+Iz_2{}^{\#} \\ z_3+Iz_3{}^{\#} & z_4+Iz_4{}^{\#} \end{bmatrix}$ be an arbitrary 2×2 elliptic matrix where $z_1,\ z_1{}^{\#},\ z_2,\ z_2{}^{\#},\ z_3,\ z_3{}^{\#}$, z_4 and $z_4{}^{\#}$ are real numbers. Then, we can write

$$A=\begin{bmatrix} \left(x_0-\sqrt{|p|}\,{x_1}'\right)+I\left({x_0}'+\frac{x_1}{\sqrt{|p|}}\right) & \left(-x_2+\sqrt{|p|}\,{x_3}'\right)+I\left(-{x_2}'-\frac{x_3}{\sqrt{|p|}}\right) \\ \left(x_2+\sqrt{|p|}\,{x_3}'\right)+I\left({x_2}'-\frac{x_3}{\sqrt{|p|}}\right) & \left(x_0+\sqrt{|p|}\,{x_1}'\right)+I\left({x_0}'-\frac{x_1}{\sqrt{|p|}}\right) \end{bmatrix} \tag{2}$$

such that

$$x_0=\frac{z_1+z_4}{2},\quad {x_0}'=\frac{z_1{}^{\#}+z_4{}^{\#}}{2},\quad x_1=\frac{\sqrt{|p|}\left(z_1{}^{\#}-z_4{}^{\#}\right)}{2},\quad {x_1}'=\frac{z_4-z_1}{2\sqrt{|p|}}\in\mathbb{R}$$

$$x_2=\frac{z_3-z_2}{2},\quad {x_2}'=\frac{z_3{}^{\#}-z_2{}^{\#}}{2},\quad x_3=-\frac{\sqrt{|p|}\left(z_2{}^{\#}+z_3{}^{\#}\right)}{2},\quad {x_3}'=\frac{z_2+z_3}{2\sqrt{|p|}}\in\mathbb{R}.$$

It can be easily seen that the arbitrary 2×2 elliptic matrix in (2) is equal to the matrix

$$\begin{bmatrix} X_0+\frac{1}{\sqrt{|p|}}IX_1 & -X_2-\frac{1}{\sqrt{|p|}}IX_3 \\ X_2-\frac{1}{\sqrt{|p|}}IX_3 & X_0-\frac{1}{\sqrt{|p|}}IX_1 \end{bmatrix}$$

where $X_i=x_i+I{x_i}'\in\mathbb{C}_p$, $0\le i\le 3$.

Conversely, it is clear that the matrix given in (1) is a 2×2 elliptic matrix. □

Let us take into account the function

$$\begin{aligned} \sigma\ :\ \ & H\mathbb{C}_p\ \rightarrow\ M_2(\mathbb{C}_p) \\ Q=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}\rightarrow\sigma(Q)&=\begin{bmatrix} A_0+\frac{1}{\sqrt{|p|}}IA_1 & -A_2-\frac{1}{\sqrt{|p|}}IA_3 \\ A_2-\frac{1}{\sqrt{|p|}}IA_3 & A_0-\frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}. \end{aligned}$$

The function σ comprises the properties

$$\sigma(Q+R)=\sigma(Q)+\sigma(R),\ \ \sigma(QR)=\sigma(Q)\,\sigma(R)$$

where Q and R are any elliptic biquaternions. Also it is bijection. So, σ is a linear isomorphism.

Corollary 1. *For an arbitrary* 2×2 *elliptic matrix* A*,* $Q\in H\mathbb{C}_p$ *satisfying the equality* $\sigma(Q)=A$ *is existence and uniqueness.*

Proof: The proof is obvious from the linear isomorphism σ and Lemma 1. □

Definition 1. *Let* $Q=A_0+A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}\in H\mathbb{C}_p$ *be an arbitrary elliptic biquaternion where* A_0*,* A_1*,* A_2*,* $A_3\in\mathbb{C}_p$*, in that case the elliptic matrix*

$$\sigma(Q)=\begin{bmatrix} A_0+\frac{1}{\sqrt{|p|}}IA_1 & -A_2-\frac{1}{\sqrt{|p|}}IA_3 \\ A_2-\frac{1}{\sqrt{|p|}}IA_3 & A_0-\frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}$$

which corresponds to Q *is called* 2×2 *elliptic matrix representation of* Q*.*

Next two theorems include some properties which are satisfied by elliptic biquaternions and their 2×2 elliptic matrix representations.

Theorem 1. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $R = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \in H\mathbb{C}_p$ and $\lambda \in \mathbb{C}_p$ be given. In this case*

1. $\det(\sigma(Q)) = N_Q = {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2$,
2. *Q is invertible if and only if $\sigma(Q)$ is invertible, then $\sigma\left(Q^{-1}\right) = (\sigma(Q))^{-1}$ and $Q^{-1} = \frac{1}{4}E_2(\sigma(Q))^{-1}{E_2}^{\dagger}$,*
3. $Q = R \Leftrightarrow \sigma(Q) = \sigma(R)$,
4. $\sigma(Q+R) = \sigma(Q) + \sigma(R)$, $\sigma(QR) = \sigma(Q)\sigma(R)$, $\sigma(\lambda Q) = \sigma(Q\lambda) = \lambda\sigma(Q)$, $\sigma(1) = I_2$,
5. $Q = \frac{1}{4}E_2\sigma(Q){E_2}^{\dagger}$,

where $E_2 = \begin{bmatrix} 1 - \frac{1}{\sqrt{|p|}}I\mathbf{i} & \mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k} \end{bmatrix} \in M_{1\times 2}(H\mathbb{C}_p)$.

Proof: The proof of 3 and 4 are obvious due to the aforementioned linear isomorphism σ. On the other hand, the proof of 5 can be completed by direct calculation. Now, we will prove 1 and 2.

1. We know that $\sigma(Q) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}}IA_1 & -A_2 - \frac{1}{\sqrt{|p|}}IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}}IA_3 & A_0 - \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}$. Then we obtain

$$\begin{aligned}\det(\sigma(Q)) &= {A_0}^2 - {A_1}^2\frac{I^2}{|p|} + {A_2}^2 - {A_3}^2\frac{I^2}{|p|} \\ &= {A_0}^2 - {A_1}^2\frac{p}{(-p)} + {A_2}^2 - {A_3}^2\frac{p}{(-p)} \\ &= {A_0}^2 + {A_1}^2 + {A_2}^2 + {A_3}^2.\end{aligned}$$

2. For an elliptic biquaternion Q, we know that Q is invertible if and only if $N_Q \neq 0$. Therefore, by means of the first property in this theorem, we can write

$$Q\, is\, invertible \Leftrightarrow N_Q \neq 0 \Leftrightarrow \det(\sigma(Q)) \neq 0 \Leftrightarrow \sigma(Q)\ is\, invertible.$$

Suppose that Q and $\sigma(Q)$ are invertible. In this case, from the inverse property, the equality

$$QQ^{-1} = Q^{-1}Q = 1$$

is satisfied. Then, by means of the third and fourth properties in this theorem, the equalities

$$\sigma(Q)\sigma\left(Q^{-1}\right) = \sigma\left(QQ^{-1}\right) = \sigma(1) = I_2$$

and

$$\sigma\left(Q^{-1}\right)\sigma(Q) = \sigma\left(Q^{-1}Q\right) = \sigma(1) = I_2$$

are obtained. It means that $(\sigma(Q))^{-1} = \sigma\left(Q^{-1}\right)$. Therefore, by considering the fifth property in this theorem, we obtain $Q^{-1} = \frac{1}{4}E_2(\sigma(Q))^{-1}{E_2}^{\dagger}$. □

Theorem 2. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given. In this case*

1. $\sigma\left(\overline{Q}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(\sigma(Q))^T\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *where $\overline{Q}$ is the quaternion conjugate of Q,*
2. $\sigma(Q^*) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\overline{\sigma(Q)}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *where Q^* is the complex conjugate of Q,*
3. $\sigma\left(Q^{\dagger}\right) = \left(\overline{\sigma(Q)}\right)^T = (\sigma(Q))^*$ *where $Q^{\dagger}$ is the Hermitian conjugate of Q.*

Proof: 2 and 3 can be easily shown, Now, we will prove 1.

1. For $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, we can write $\overline{Q} = A_0 - A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k}$. In this case, we get

$$\sigma\left(\overline{Q}\right) = \begin{bmatrix} A_0 - \frac{1}{\sqrt{|p|}}IA_1 & A_2 + \frac{1}{\sqrt{|p|}}IA_3 \\ -A_2 + \frac{1}{\sqrt{|p|}}IA_3 & A_0 + \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}.$$

On the other hand, it is clear that

$$\left(\sigma\left(Q\right)\right)^T = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}}IA_1 & A_2 - \frac{1}{\sqrt{|p|}}IA_3 \\ -A_2 - \frac{1}{\sqrt{|p|}}IA_3 & A_0 - \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix}.$$

Then, by directly multiplying we obtain

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\sigma\left(Q\right)\right)^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} A_0 - \frac{1}{\sqrt{|p|}}IA_1 & A_2 + \frac{1}{\sqrt{|p|}}IA_3 \\ -A_2 + \frac{1}{\sqrt{|p|}}IA_3 & A_0 + \frac{1}{\sqrt{|p|}}IA_1 \end{bmatrix} = \sigma\left(\overline{Q}\right).$$

□

4 Similarity of Elliptic Biquaternions and USFE for Elliptic Biquaternions

In this section, we investigate the similarity of elliptic biquaternions with the aid of their elliptic matrix representations and establish universal similarity factorization equality for elliptic biquaternions.

4.1 Similarity of elliptic biquaternions

One of the natural questions concerned with elliptic biquaternions is the similarity of two elliptic biquaternions. By analogy with the classic quaternion case, the next definition is given.

Definition 2. *For $Q, R \in H\mathbb{C}_p$, if there exists an invertible elliptic biquaternion X such that $X^{-1}QX = R$, Q and R are called similar elliptic biquaternions. This case is denoted by $Q \sim R$.*

By considering Definition 2, a simple result, which characterizes the similarity of two elliptic biquaternions, can be given as follows.

Theorem 3. *Let $Q, R \in H\mathbb{C}_p$ be given. In this case,*

$$Q \sim R \Leftrightarrow \sigma\left(Q\right) \sim \sigma\left(R\right). \tag{3}$$

Proof: $Q \sim R$ if and only if there is an invertible elliptic biquaternion X such that $X^{-1}QX = R$. Then, we have

$$\begin{aligned} Q \sim R &\Leftrightarrow \sigma\left(X^{-1}QX\right) = \sigma\left(R\right) \\ &\Leftrightarrow \sigma\left(X^{-1}\right)\sigma\left(Q\right)\sigma\left(X\right) = \sigma\left(R\right) \\ &\Leftrightarrow \left(\sigma\left(X\right)\right)^{-1}\sigma\left(Q\right)\sigma\left(X\right) = \sigma\left(R\right) \\ &\Leftrightarrow \sigma\left(Q\right) \sim \sigma\left(R\right) \end{aligned}$$

from Theorem 1 (2), (3) and (4). □

As a consequence of Theorem 3, we can give the following theorem.

Theorem 4. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given where $Q \notin \mathbb{C}_p$.*

1. *If ${A_1}^2 + {A_2}^2 + {A_3}^2 \neq 0$, in that case $Q \sim A_0 + \gamma\left(Q\right)\mathbf{i}$ where $\gamma\left(Q\right)$ is an elliptic number satisfying the equality $\gamma^2\left(Q\right) = {A_1}^2 + {A_2}^2 + {A_3}^2$.*
2. *If ${A_1}^2 + {A_2}^2 + {A_3}^2 = 0$, in that case $Q \sim A_0 - \frac{1}{2}\mathbf{j} + \frac{1}{2\sqrt{|p|}}I\mathbf{k}$.*

Proof: For a given elliptic biquaternion $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$, we have its 2×2 elliptic matrix representation $\sigma(Q)$. We can calculate its characteristic polynomial as follows:

$$|\lambda I_2 - \sigma(Q)| = \begin{vmatrix} \lambda - A_0 - \frac{I}{\sqrt{|p|}}A_1 & A_2 + \frac{I}{\sqrt{|p|}}A_3 \\ -A_2 + \frac{I}{\sqrt{|p|}}A_3 & \lambda - A_0 + \frac{I}{\sqrt{|p|}}A_1 \end{vmatrix} = (\lambda - A_0)^2 + {A_1}^2 + {A_2}^2 + {A_3}^2.$$

For ${A_1}^2 + {A_2}^2 + {A_3}^2 \neq 0$, we can get the roots of the above characteristic polynomial as $\lambda_{1,2} = A_0 \pm \frac{1}{\sqrt{|p|}} I\gamma(Q)$. Thus, we immediately have

$$\sigma(Q) \sim \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I\gamma(Q) & 0 \\ 0 & A_0 - \frac{1}{\sqrt{|p|}} I\gamma(Q) \end{bmatrix} = \sigma(A_0 + \gamma(Q)\,\mathbf{i}). \tag{4}$$

For ${A_1}^2 + {A_2}^2 + {A_3}^2 = 0$, we can get the roots of the characteristic polynomial of $\sigma(Q)$ as $\lambda_{1,2} = A_0$. Then, considering the Jordan canonical form of $\sigma(Q)$, we can write the following

$$\sigma(Q) \sim \begin{bmatrix} A_0 & 1 \\ 0 & A_0 \end{bmatrix} = \sigma\left(A_0 - \frac{1}{2}\mathbf{j} + \frac{1}{2\sqrt{|p|}} I\mathbf{k}\right). \tag{5}$$

If we apply Theorem 3 to (4) and (5), we can easily prove the first part and second part of this theorem, respectively. □

4.2 USFE for elliptic biquaternions

There is a deeper relationship between an elliptic biquaternion Q and its elliptic matrix representation $\sigma(Q)$ which appears with USFE over the elliptic biquaternion algebra.

In [11], Tian presents a general result on the universal similarity factorization of elements over any algebra as follows:
Let A be an algebra over an arbitrary field F and $M_n(A)$ be the matrix algebra which includes all $n \times n$ matrices with elements in A. Also, let $\{\tau_{ij}\}$ be the basis of A that satisfies the following rules

$$\tau_{ij}\tau_{st} = \begin{cases} \tau_{it} & j = s \\ 0 & j \neq s \end{cases}, \quad i, j, s, t = 1, ..., n. \tag{6}$$

In this case, any $Q = \sum_{i,j=1}^{n} a_{ij}\tau_{ij} \in A \ \ (a_{ij} \in F)$ satisfies the following USFE

$$P \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} P^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{7}$$

where P has the following independent form

$$P = P^{-1} = \begin{bmatrix} \tau_{11} & \tau_{21} & \cdots & \tau_{n1} \\ \tau_{12} & \tau_{22} & \cdots & \tau_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{1n} & \tau_{2n} & \cdots & \tau_{nn} \end{bmatrix}. \tag{8}$$

By basing on the general result indicated above, we establish USFE for elliptic biquaternions as follows.

Theorem 5. *Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be given. In this case, the elliptic biquaternion matrix $\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}$ satisfies the following USFE*

$$P \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} P^{-1} = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} IA_1 & -A_2 - \frac{1}{\sqrt{|p|}} IA_3 \\ A_2 - \frac{1}{\sqrt{|p|}} IA_3 & A_0 - \frac{1}{\sqrt{|p|}} IA_1 \end{bmatrix} = \sigma(Q) \in M_2(\mathbb{C}_p) \tag{9}$$

where P is in the following independent form:

$$P = P^{-1} = \frac{1}{2}\begin{bmatrix} 1 - \frac{1}{\sqrt{|p|}} I\mathbf{i} & \mathbf{j} + \frac{1}{\sqrt{|p|}} I\mathbf{k} \\ -\mathbf{j} + \frac{1}{\sqrt{|p|}} I\mathbf{k} & 1 + \frac{1}{\sqrt{|p|}} I\mathbf{i} \end{bmatrix} \in M_2(H\mathbb{C}_p). \tag{10}$$

Proof: Let $Q = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in H\mathbb{C}_p$ be an arbitrary elliptic biquaternion and let us consider the elliptic biquaternions

$$\tau_{11} = \frac{1}{2} - \frac{I}{2\sqrt{|p|}}\mathbf{i},\ \ \tau_{12} = -\frac{1}{2}\mathbf{j} + \frac{I}{2\sqrt{|p|}}\mathbf{k},\ \ \tau_{21} = \frac{1}{2}\mathbf{j} + \frac{I}{2\sqrt{|p|}}\mathbf{k},\ \ \tau_{22} = \frac{1}{2} + \frac{I}{2\sqrt{|p|}}\mathbf{i}. \tag{11}$$

It is clear that the system $\{\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}\}$ is a base of elliptic biquaternion algebra from the equalities

$$\langle \tau_{st}\,,\, \tau_{pq} \rangle = \begin{cases} 1\,,\ \ (s = p) \wedge\ (t = q)\,,\ \ s, t, p, q = 1, 2 \\ \\ 0\,,\ \ (s \neq p)\ \vee\ (t \neq q)\,,\ \ s, t, p, q = 1, 2 \end{cases}$$

and

$$Q = \left(A_0 + \frac{A_1 I}{\sqrt{|p|}} \right) \tau_{11} + \left(-A_2 - \frac{A_3 I}{\sqrt{|p|}} \right) \tau_{12} + \left(A_2 - \frac{A_3 I}{\sqrt{|p|}} \right) \tau_{21} + \left(A_0 - \frac{A_1 I}{\sqrt{|p|}} \right) \tau_{22}.$$

For the case $n = 2$, it is easy to verify that these new bases in (11) satisfy the multiplication rules in (6). Then, if we consider the last equality above and (11) in (7) and in (8) by keeping the case $n = 2$ in mind, we get (9) and (10). □

If Lemma 1 is considered, by means of USFE for elliptic biquaternions, it can be said that every 2×2 elliptic matrix is uniformly similar to the diagonal matrix $diag\,(Q, Q)$ where Q is the elliptic biquaternion which corresponds to this 2×2 elliptic matrix.

5 Moore-Penrose Inverses of Elliptic Biquaternions

In this section, we define the Moore-Penrose inverse of any elliptic matrix and show that it always exists uniquely. Afterwards, we give the similar definition for elliptic biquaternions as well. Then, the existence and uniqueness of the Moore Penrose inverse for an elliptic biquaternion Q are determined by the matrix $\sigma(Q) \in M_2(\mathbb{C}_p)$.

Definition 3. *Let an arbitrary elliptic matrix $A \in M_{m\times n}(\mathbb{C}_p)$ be given. If the equations*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA \tag{12}$$

have a common solution $X \in M_{n\times m}(\mathbb{C}_p)$, in this case this solution is called Moore-Penrose inverse of A. It is showed with $X = A^+$.

Theorem 6. *Let $A \in M_{m\times n}(\mathbb{C}_p)$ be given. In this case the Moore-Penrose inverse of A is existence and uniqueness.*

Proof: We define a function between the space of $m \times n$ elliptic matrices and the space of $m \times n$ complex matrices as follows:

$$\delta : M_{m\times n}(\mathbb{C}_p) \to M_{m\times n}(\mathbb{C})$$

$$\begin{bmatrix} a_{11} + Ib_{11} & \dots & a_{1n} + Ib_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + Ib_{m1} & \cdots & a_{mn} + Ib_{mn} \end{bmatrix} \to \begin{bmatrix} a_{11} + i\left(b_{11}\sqrt{|p|}\right) & \dots & a_{1n} + i\left(b_{1n}\sqrt{|p|}\right) \\ \vdots & \ddots & \vdots \\ a_{m1} + i\left(b_{m1}\sqrt{|p|}\right) & \cdots & a_{mn} + i\left(b_{mn}\sqrt{|p|}\right) \end{bmatrix}.$$

As can be seen easily, the function δ is bijection and so we can write

$$A = B \ \Leftrightarrow\ \delta(A) = \delta(B) \tag{13}$$

for $A,\ B \in M_{m\times n}(\mathbb{C}_p)$. Also, it comprises the following properties

$$\delta(A + B) = \delta(A) + \delta(B)\,,\quad \delta(AB) = \delta(A)\,\delta(B) \tag{14}$$

and

$$\delta\left(A^*\right) = (\delta(A))^* \tag{15}$$

where A^* is the conjugate transpose of $A \in M_{m\times n}(\mathbb{C}_p)$ and $(\delta(A))^*$ is the conjugate transpose of $\delta(A) \in M_{m\times n}(\mathbb{C})$. From (13) and (14), it is clear that δ is an isomorphism.

Thanks to (13), (14) and (15), the elliptic matrix equation system (12) is equivalent to the following complex matrix equation system:

$$\begin{aligned} \delta(A)\,\delta(X)\,\delta(A) = \delta(A)\,, &\qquad \delta(X)\,\delta(A)\,\delta(X) = \delta(X)\,, \\ (\delta(A)\,\delta(X))^* = \delta(A)\,\delta(X)\,, &\quad (\delta(X)\,\delta(A))^* = \delta(X)\,\delta(A)\,. \end{aligned} \tag{16}$$

According to the complex matrix theory (see [22] for more details) the four equations

$$\delta(A)\,Y\delta(A) = \delta(A)\,,\quad Y\delta(A)\,Y = Y,\quad (\delta(A)\,Y)^* = \delta(A)\,Y,\quad (Y\delta(A))^* = Y\delta(A)$$

have a unique common solution $Y = (\delta(A))^+$ which is called the Moore-Penrose inverse of $\delta(A)$. Thus, if we take into account the system (16) we can immediately obtain $\delta(X) = (\delta(A))^+$. From the definition of isomorphism δ, it is clear that the matrix $X \in M_{m\times n}(\mathbb{C}_p)$ which satisfies $\delta(X) = (\delta(A))^{+}$ is existence and uniqueness. In this case, with the aid of the equalities (13), (14) and (15), we conclude that the elliptic matrix X, which is indicated above, is the unique solution of the elliptic matrix equation system (12). $\square$

Definition 4. *Let an elliptic biquaternion $Q \in H\mathbb{C}_p$ be given. If the equations*

$$QXQ = Q, \quad XQX = X, \quad (QX)^{\dagger} = QX, \quad (XQ)^{\dagger} = XQ \tag{17}$$

have a common solution $X \in H\mathbb{C}_p$, in this case this solution is called Moore-Penrose inverse of Q. It is showed with $X = Q^{+}$.

Theorem 7. *Let $Q \in H\mathbb{C}_p$. In that case, its Moore-Penrose inverse Q^{+} is existence and uniqueness. Also Q^{+} satisfies the following equalities*

$$\sigma\left(Q^{+}\right) = (\sigma(Q))^{+}, \quad Q^{+} = \frac{1}{4}E_2(\sigma(Q))^{+}{E_2}^{\dagger}$$

where $E_2 = \left[1 - \frac{1}{\sqrt{|p|}}I\mathbf{i} \quad \mathbf{j} + \frac{1}{\sqrt{|p|}}I\mathbf{k}\right] \in M_{1\times 2}(H\mathbb{C}_p)$.

Proof: If we consider Theorem 1 (3), (4) and Theorem 2 (3), we can easily see that the elliptic biquaternion equation system (17) is equivalent to the following elliptic matrix equation system:

$$\begin{aligned} &\sigma(Q)\,\sigma(X)\,\sigma(Q) = \sigma(Q)\,, &&\sigma(X)\,\sigma(Q)\,\sigma(X) = \sigma(X)\,, \\ &(\sigma(Q)\,\sigma(X))^{*} = \sigma(Q)\,\sigma(X)\,, &&(\sigma(X)\,\sigma(Q))^{*} = \sigma(X)\,\sigma(Q)\,. \end{aligned} \tag{18}$$

According to Definition 3 and Theorem 6, the four equations

$$\begin{aligned} &\sigma(Q)\,Y\sigma(Q) = \sigma(Q)\,, && Y\sigma(Q)\,Y = Y, \\ &(\sigma(Q)\,Y)^{*} = \sigma(Q)\,Y, && (Y\sigma(Q))^{*} = Y\sigma(Q) \end{aligned}$$

have a unique common solution $Y = (\sigma(Q))^{+} \in M_2(\mathbb{C}_p)$ which is called the Moore-Penrose inverse of $\sigma(Q)$. Thus, if we take into account the system (18) we can write $\sigma(X) = (\sigma(Q))^{+}$. From Corollary 1, we know that the elliptic biquaternion $X \in H\mathbb{C}_p$ which satisfies $\sigma(X) = (\sigma(Q))^{+}$ is existence and uniqueness. In this case, with the aid of Theorem 1 (3), (4) and Theorem 2 (3), we conclude that the elliptic biquaternion X, which is indicated above, is the unique solution of the system (17). According to Definition 4, we denote this X by $X = Q^{+}$. Thus, it is clear that $\sigma\left(Q^{+}\right) = (\sigma(Q))^{+}$. From this last equality and Theorem 1 (5), $Q^{+} = \frac{1}{4}E_2(\sigma(Q))^{+}{E_2}^{\dagger}$ can be easily obtained. $\square$

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