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Nonexistence of Global Solutions for the Kirchhoff-Type Equation with Fractional Damped

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Abstract

In this work, we investigate the Kirchhoff-type equation with a fractional damping term in a bounded domain. The fractional damping term plays a quenching role, which is weaker than strong damping and stronger than weak damping term. We prove a nonexistence of global solutions with negative inital energy. This result extends and improves some results in the literature.

1. Introduction

In this work, we deal with the nonexistence of solutions following Kirchhoff-type equation:

$$\begin{cases} u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+\partial_t^{1+\alpha}u=|u|^{p-1}u, & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, and $M(s)=\beta_1+\beta_2 s^{\gamma}$. The constants $p>1$ real number, $\gamma\geq 0$, $\beta_1,\beta_2>0$ and $-1<\alpha<1$. Without loss of generality, we choose $\beta_1=\beta_2=1$ in (1.1) in this paper. The notation $\partial_t^{1+\alpha}$ stands for the Caputo's fractional derivative of order $1+\alpha$ with respect to the time variable [1, 2]. It is defined as follows

$$\partial_t^{1+\alpha}w(t)=\begin{cases} I^{-\alpha}\frac{d}{dt}w(t) & \text{for } -1<\alpha<0\\ I^{1-\alpha}\frac{d^2}{dt^2}w(t) & \text{for } \quad 0<\alpha<1\end{cases}$$

where $I^{\beta},\beta>0$ is fractional integral

$$I^{\beta}\frac{d}{dt}w(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}w_{\tau}(\tau)d\tau.$$

The fractional damping term plays a quenching role, which is weaker than strong damping and stronger than weak damping term [3]. The problem (1.1) is a generalization of a model introduced by Kirchhoff [4].
Ono [5] considered equation (1.1) for $\alpha=0$. He proved that the solution blows up with negative initial energy. Later, Wu and Tsai [6] proved the blow up of the solution with positive upper bounded initial energy.
In [7] Yang et al. studied the following equation

$$u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+(-\Delta)^{\alpha}u+f(u)=g(x).$$

They proved the attractors for $1/2<\alpha<1$.
There are many literatures on the nonexistence of solutions for the Kirchhoff-type equation.
This work is organized as follows. In Section 2, we give some notations and lemmas needed for our paper. In Section 3, we prove the nonexistence of the solution for the problem (1.1) with negative initial energy. We use improved the method of [8].

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2. Preliminaries

In this part, we give some notations and material needed in our main result. Without loss of generality, we get only the case $-1<\alpha<0$. We define the energy with problem (1.1) is

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Then,

$$E'(t)=-\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx.$$

Now, we define modified energy functional as

$$E_\varepsilon(t)=E(t)-\varepsilon\int_\Omega uu_t dx \tag{2.1}$$

where $0<\varepsilon<1$ is the constant which is specified later. Now a differentiation of $E_\varepsilon(t)$, with respect to time t gives

$$\begin{aligned}E'_\varepsilon(t) &= -\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\\ &-\varepsilon\int_\Omega|u_t|^2dx-\varepsilon\int_\Omega|u|^{p+1}dx+\varepsilon\int_\Omega|\nabla u|^2dx+\varepsilon\int_\Omega|\nabla u|^{2(\gamma+1)}dx\\ &+\frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx.\end{aligned} \tag{2.2}$$

Also, we define the following functionals

$$H(t)=-\left(e^{-\sigma\varepsilon t}E_\varepsilon(t)+\mu F(t)+d\right), \tag{2.3}$$

$$F(t)=\int_0^t\int_\Omega G(t-\tau)e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau \tag{2.4}$$

and

$$G(t)=e^{\beta t}\int_t^\infty e^{-\beta\tau}\tau^{-(\alpha+1)}d\tau$$

where $\sigma=\frac{p+1}{2}$ and β,μ,d are positive consants.

Lemma 2.1. *Let p be sufcifiently large and $E_\varepsilon(0)<0$. Then $H'(t)>0$ and $H(t)>0$.*

Proof. By taking a derivative of (2.3) and (2.4), we get

$$H'(t)=\sigma\varepsilon e^{-\sigma\varepsilon t}E_\varepsilon(t)-e^{-\sigma\varepsilon t}E'_\varepsilon(t)-\mu F'(t), \tag{2.5}$$

$$F'(t)=\beta^\alpha\Gamma(-\alpha)e^{-\sigma\varepsilon t}\int_\Omega u_t^2dx-\int_0^t\int_\Omega(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau+\beta F(t). \tag{2.6}$$

Taking into account (2.6), (2.1) and (2.2) in (2.5), we obtain

$$\begin{aligned}H'(t) &= e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^\alpha\Gamma(-\alpha)\right)\int_\Omega|u_t|^2dx+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2}-1\right)\int_\Omega|\nabla u|^2dx\\ &+\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2(\gamma+1)}-1\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx+\varepsilon e^{-\sigma\varepsilon t}\left(1-\frac{\sigma}{p+1}\right)\int_\Omega|u|^{p+1}dx\\ &-\varepsilon\sigma\varepsilon e^{-\sigma\varepsilon t}\int_\Omega uu_tdx+\frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\\ &-\frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\\ &+\mu\int_0^t\int_\Omega(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2dxd\tau-\mu\beta F(t).\end{aligned} \tag{2.7}$$

Next, we estimate some terms in the right hand side of (2.7). For the sixth term on the right hand side of (2.7), using Young's inequality, we obtain

$$\begin{aligned} & e^{-\sigma\varepsilon t}\int_{\Omega} u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx \\ \leq\ & \delta_1 e^{-\sigma\varepsilon t}\int_{\Omega} u_t^2 dx + \frac{1}{4\delta_1} e^{-\sigma\varepsilon t}\int_{\Omega}\left[\int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau\right]^2 dx. \end{aligned}$$

Writing $-(\alpha+1) = -\frac{\alpha+1}{2} - \frac{\alpha+1}{2}$ and thanks to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & e^{-\sigma\varepsilon t}.\int_{\Omega} u_t \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx \\ \leq\ & \delta_1 e^{-\sigma\varepsilon t}\int_{\Omega} u_t^2 dx + \frac{(\sigma\varepsilon)^{\alpha}\Gamma(-\alpha)}{4\delta_1}\int_{\Omega}\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx. \end{aligned} \tag{2.8}$$

Smilarly, we have

$$\begin{aligned} & e^{-\sigma\varepsilon t}.\int_{\Omega} u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx \\ \leq\ & \delta_2 e^{-\sigma\varepsilon t}.\int_{\Omega} |u|^2 dx + \frac{1}{4\delta_2} e^{-\sigma\varepsilon t}\int_{\Omega}\left(\int_0^t (t-\tau)^{-(\alpha+1)} e^{\frac{-\sigma\varepsilon t}{2}} u_\tau(\tau) d\tau\right)^2 dx. \end{aligned}$$

Using Sobolev-Poincare's inequality, we arrive at

$$\begin{aligned} & e^{-\sigma\varepsilon t}\int_{\Omega} u \int_0^t (t-\tau)^{-(\alpha+1)} u_\tau(\tau) d\tau dx \\ \leq\ & \delta_2 e^{-\sigma\varepsilon t} C_{p_1}\int_{\Omega} |\nabla u|^2 dx \\ & + \frac{(\sigma\varepsilon)^{\alpha}\Gamma(-\alpha)}{4\delta_2}\int_{\Omega}\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx. \end{aligned} \tag{2.9}$$

Now, we estimate the fifth term in the right side of (2.7), thanks to the Young's and Sobolev-Poincare's inequalities, we have

$$\begin{aligned} \int_{\Omega} u u_t dx &\leq \delta_3\int_{\Omega}|u|^2 dx + \frac{1}{4\delta_3}\int_{\Omega}|u_t|^2 dx \\ &\leq \delta_3 C_{p_2}\int_{\Omega}|\nabla u|^2 dx + \frac{1}{4\delta_3}\int_{\Omega}|u_t|^2 dx. \end{aligned} \tag{2.10}$$

By (2.7), (2.8), (2.9) and (2.10), we have

$$\begin{aligned} H'(t) \geq\ & e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2} + \varepsilon - \mu\beta^{\alpha}\Gamma(-\alpha) - \frac{\varepsilon^2\sigma}{4\delta_3} - \frac{\delta_1}{\Gamma(-\alpha)}\right)\int_{\Omega}|u_t|^2 dx \\ & + \varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2} - 1 - \delta_3 C_{p_2}\varepsilon\sigma - \frac{\delta_2 C_{p_1}}{\Gamma(-\alpha)}\right)\int_{\Omega}|\nabla u|^2 dx \\ & + \varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2(\gamma+1)} - 1\right)\int_{\Omega}|\nabla u|^{2(\gamma+1)} dx + \varepsilon e^{-\sigma\varepsilon t}\left(1 - \frac{\sigma}{p+1}\right)\int_{\Omega}|u|^{p+1} dx \\ & + \left(\mu - \frac{(\sigma\varepsilon)^{\alpha}}{4\delta_1} - \frac{(\sigma\varepsilon)^{\alpha}\varepsilon}{4\delta_2}\right)\int_{\Omega}\int_0^t (t-\tau)^{-(\alpha+1)} e^{-\sigma\varepsilon\tau} u_\tau^2 d\tau dx - \mu\beta F(t). \end{aligned}$$

Subtracting and adding $C_1H(t)$ on the right hand side of above inequality, we get

$$\begin{aligned} H'(t) \geq\ & C_1H(t)+e^{-\sigma\varepsilon t}\left(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{\alpha}\Gamma(-\alpha)-\frac{\varepsilon^2\sigma}{4\delta_3}-\frac{\delta_1}{\Gamma(-\alpha)}+\frac{C_1}{2}-\frac{C_1}{4\delta_3}\right)\int_\Omega u_t^2dx \\ & +\varepsilon e^{-\sigma\varepsilon t}\left(\frac{\sigma}{2}-1-\delta_3C_{p_2}\varepsilon\sigma-\frac{\delta_2C_{p_1}}{\Gamma(-\alpha)}+\frac{C_1}{2}-C_1\delta_3C_{p_2}\right)\int_\Omega|\nabla u|^2dx \\ & +e^{-\sigma\varepsilon t}\left(\frac{\varepsilon\sigma}{2(\gamma+1)}-\varepsilon+\frac{C_1}{2(\gamma+1)}\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx \\ & +e^{-\sigma\varepsilon t}\left(\varepsilon-\frac{\sigma\varepsilon}{p+1}-\frac{C_1}{p+1}\right)\int_\Omega|u|^{p+1}dx \\ & +\left(\mu-\frac{(\sigma\varepsilon)^{\alpha}}{4\delta_1}-\frac{(\sigma\varepsilon)^{\alpha}\varepsilon}{4\delta_2}\right)\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx \\ & +\mu(C_1-\beta)F(t)+C_1d. \end{aligned}$$

We choose $C_1=\frac{p+1}{2}\varepsilon,\ \delta_1=\delta_2=\frac{\Gamma(-\alpha)\varepsilon}{2},\ \delta_3=\frac{1}{2}$ and $\beta=1$, we obtain

$$\begin{aligned} H'(t) \geq\ & \frac{p+1}{2}\varepsilon H(t)+e^{-\sigma\varepsilon t}\left(\frac{p+1}{4}\varepsilon(1-\varepsilon)-\mu\Gamma(-\alpha)\right)\int_\Omega u_t^2dx \\ & +\varepsilon C_{p_3}e^{-\sigma\varepsilon t}\left(\frac{p-3+\varepsilon(p+1-C_p(2p+4))}{4}\right)\int_\Omega|\nabla u|^2dx \\ & +\varepsilon e^{-\sigma\varepsilon t}\left(\frac{p+1}{2(\gamma+1)}-1\right)\int_\Omega|\nabla u|^{2(\gamma+1)}dx \\ & +\left(\mu-\frac{(p+1)^{\alpha}\varepsilon^{\alpha-1}}{2^{\alpha+1}\Gamma(-\alpha)}(1+\varepsilon)\right)\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx \\ & +\mu\left(\frac{p+1}{2}\varepsilon-1\right)F(t)+\frac{p+1}{2}\varepsilon d. \end{aligned}$$

We choose

$$\varepsilon<\varepsilon_1=\min\left\{1,\frac{p-3}{2\left[2(p+2)C_p-(p+1)\right]}\right\}.$$

Where $C_p>\frac{p+1}{2(p+2)}$, it appears that the third coefficient is nonnegative. Observe if that $C_p<\frac{3}{4}$ ve $p\geq\frac{1+8C_p}{1-4C_p}$, than $\frac{p-3}{2[2(p+2)C_p-(p+1)]}\geq 1$ and this condition reduces to $\varepsilon<1$. We can take μ so that the second coefficient is nonnegative and the forth coefficient is greater than $\frac{(p+1)^{\alpha}}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}$. Also, if p is sufficiently large $\frac{p+1}{2}\varepsilon-1$ is positive. Consequently, we get

$$\begin{aligned} H'(t) \geq\ & \frac{p+1}{2}\varepsilon H(t)+\frac{p-3}{8}\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega|\Delta u|^2dx \\ & +\frac{(p+1)^{\alpha}}{2^{\alpha+1}\varepsilon^{1-\alpha}\Gamma(-\alpha)}\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx. \end{aligned} \tag{2.11}$$

If we select $E_\varepsilon(0)<-d$, then $H(0)>0$. This completes the proof. □

3. Nonexistence of global solutions

In this part, we obtain the nonexistence of global solutions of the problem (1.1).

Theorem 3.1. *Suppose that* $-1<\alpha<0$,

$$E(0)<0 \text{ and } \int_\Omega u_1u_0dx\geq 0.$$

Then the solution of (1.1) blows up in finite time.

Proof. We define an auxiliary function

$$\Psi(t)=H^{1-\gamma}(t)+\varphi e^{-\sigma\varepsilon t}.\int_\Omega uu_tdx$$

where $\gamma=\frac{p-1}{2(p+1)}$ and φ is a positive constant to be specified later. Our aim is to show that $\Psi(t)$ satisfies the following differential inequality:

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq k\Psi'(t).$$

By differentiating $\Psi(t)$ with respect to t and using (1.1), we obtain

$$\begin{aligned}\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega uu_t dx+\varphi e^{-\sigma\varepsilon t}.\left(\int_\Omega u_t^2dx+\int_\Omega uu_{tt}dx\right)\\ &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}.\int_\Omega uu_t dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left[\int_\Omega |u|^{p+1}dx+\int_\Omega|\Delta u|^2dx-\frac{1}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-\tau)^{-(\alpha+1)}u_\tau(\tau)d\tau dx\right]\\ &\quad+\varphi e^{-\sigma\varepsilon t}\int_\Omega u_t^2dx.\end{aligned}$$

By using the inequalities (2.10) and (2.9) with the constant $\delta_4,\delta_5>0$, we get

$$\begin{aligned}\Psi'(t) &= (1-\gamma)H^{-\gamma}(t)H'(t)-\varphi\sigma\varepsilon\delta_4 e^{-\sigma\varepsilon t}.\int_\Omega|u|^2dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega|u|^{p+1}dx\\ &\quad-\frac{\varphi\sigma\varepsilon e^{-\sigma\varepsilon t}}{4\delta_4}\int_\Omega u_t^2dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega u_t^2dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega|\Delta u|^2dx\\ &\quad-\frac{\varphi\delta_5 e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega|u|^2dx-\frac{\varphi(\sigma\varepsilon)^\alpha}{4\delta_5}\int_\Omega\int_0^t(t-\tau)^{-(\alpha+1)}e^{-\sigma\varepsilon\tau}u_\tau^2d\tau dx.\end{aligned}$$

By (2.11), we have

$$\begin{aligned}\Psi'(t) &\geq \left((1-\gamma)H^{-\gamma}(t)-\frac{\varphi\varepsilon\Gamma(-\alpha)}{2\delta_5}\right)H'(t)\\ &\quad+\frac{\varphi(p+1)\Gamma(-\alpha)\varepsilon^2}{4\delta_5}H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1+\frac{(p-3)\Gamma(-\alpha)}{8\delta_5}\varepsilon^2-\left(\sigma\varepsilon\delta_4+\frac{\delta_5}{\Gamma(-\alpha)}\right)C_p\right)\int_\Omega|\Delta u|^2dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)\int_\Omega u_t^2dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega|u|^{p+1}dx.\end{aligned}$$

If we take $\delta_5=L\Gamma(-\alpha)H^\gamma(t)$, we get

$$\begin{aligned}\Psi'(t) &\geq \left((1-\gamma)-\frac{\varphi\varepsilon}{2L}\right)H^{-\gamma}(t)H'(t)+\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1+\frac{(p-3)H^{-\gamma}(t)}{8L}\varepsilon^2-(\sigma\varepsilon\delta_4+LH^\gamma(t))C_p\right)\int_\Omega|\Delta u|^2dx\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left(1-\frac{\sigma\varepsilon}{4\delta_4}\right)\int_\Omega u_t^2dx+\varphi e^{-\sigma\varepsilon t}.\int_\Omega|u|^{p+1}dx.\end{aligned}$$

If we substitute and add $H(t)$ to the right side of the equation, we arrive at

$$\begin{aligned}\Psi'(t) &\geq \left(1-\gamma-\frac{\varphi\varepsilon}{2L}\right)H^{-\gamma}(t)H'(t)\\ &\quad+\left(\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)+1\right)H(t)\\ &\quad+\varphi e^{-\sigma\varepsilon t}\left[\varphi+\frac{(p-3)H^{-\gamma}(t)\varphi}{8L}\varepsilon^2-\varphi(\varepsilon\sigma\delta_4+LH^\gamma(t))C_p\right.\\ &\quad\left.-C_p\left(\varepsilon\delta_6C_*+\frac{1}{2}\right)\right]\int_\Omega|\Delta u|^2dx\\ &\quad+e^{-\sigma\varepsilon t}\left(\varphi-\frac{\varphi(p+1)\varepsilon}{8\delta_4}+\frac{1}{2}-\frac{\varepsilon}{4\delta_6}\right)\int_\Omega u_t^2dx\\ &\quad+e^{-\sigma\varepsilon t}\left(\varphi-\frac{1}{p+1}\right).\int_\Omega|u|^{p+1}dx+\frac{e^{-\sigma\varepsilon t}}{2(\gamma+1)}\int_\Omega|\nabla u|^{2(\gamma+1)}dx+\mu F(t)+d. \qquad (3.1)\end{aligned}$$

We take $1-\gamma-\frac{\varphi\varepsilon}{2L}\geq 0$ and $\varepsilon\leq\varepsilon_2=\frac{2L(1-\gamma)}{\varphi}$, we have

$$\frac{\varphi(p+1)\varepsilon^2}{4L}H^{-\gamma}(t)\geq 0.$$

Also, we take $\varphi=\frac{p+3}{p+1}$, $\delta_4=\delta_6=\frac{1}{2}$ and $\varepsilon<\varepsilon_3=\frac{4(p+3)}{(p+1)(p+5)}$, we have

$$\varphi-\frac{\varphi(p+1)\varepsilon}{8\delta_4}-\frac{\varepsilon}{4\delta_6}\geq 0,$$

The fifth coefficient is nonnegative as soon as ε and C_p is chosen small enough, we have

$$\varphi+\frac{(p-3)H^{-\gamma}(t)\varphi}{8L}\varepsilon^2-\varphi\left(\varepsilon\sigma\delta_4+LH^{\gamma}(t)\right)C_p-C_p\left(\varepsilon\delta_6C_*+\frac{1}{2}\right)\geq 0$$

and

$$\begin{aligned}&\frac{p+3}{p+1}+\frac{(p-3)(p+3)H^{-\gamma}(t)}{8L(p+1)}\varepsilon^2\\&-\frac{1}{2}C_p\left(\frac{p+3}{p+1}\left(\varepsilon\frac{p+1}{2}+2LH^{\gamma}(t)\right)+\varepsilon C_*+1\right)\\\geq\;&0.\end{aligned}$$

Therefore (3.1) takes the form

$$\Psi'(t)\geq H(t)+\frac{1}{2}\int_{\Omega}u_t^2dx+\frac{p+2}{p+1}\int_{\Omega}|u|^{p+1}dx. \tag{3.2}$$

By the defination $\Psi(t)$, we deduce that

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}\left(\int_{\Omega}uu_tdx\right)^{\frac{1}{1-\gamma}}\right].$$

By the Cauchy-Schwarz and Hölder's inequalities, we arrive at

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq 2^{\frac{\gamma}{1-\gamma}}\left[H(t)+\varphi^{\frac{1}{1-\gamma}}b\left(\int_{\Omega}|u_t|^2dx+\int_{\Omega}|u|^{p+1}dx\right)\right]. \tag{3.3}$$

If we take k is large enough

$$2^{\frac{\gamma}{1-\gamma}}\leq k,$$

$$2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}b\leq\frac{k}{2},$$

$$2^{\frac{\gamma}{1-\gamma}}\varphi^{\frac{1}{1-\gamma}}b\leq\frac{p+2}{p+1}k.$$

That is k has to be chosen so that

$$k\geq 2^{\frac{\gamma}{1-\gamma}}\max\left\{1,2\varphi^{\frac{1}{1-\gamma}}b,\frac{p+1}{p+2}\varphi^{\frac{1}{1-\gamma}}b\right\}.$$

Combining (3.2) and (3.3), we have

$$\Psi^{\frac{1}{1-\gamma}}(t)\leq k\Psi'(t). \tag{3.4}$$

From (3.2) it is clear that $\Psi'(t)\geq 0$. Therefore, by the definition of $\Psi(t)$ and the hypotheses on the initial data, we get

$$\Psi(t)\geq\Psi(0)>\varphi\int_{\Omega}u_1u_0dx\geq 0.$$

Thus $\Psi(t)>0$. Integrating (3.4) over $(0,t)$, we get

$$\Psi^{\frac{\gamma}{1-\gamma}}(t)\geq\frac{1}{\Psi^{\frac{-\gamma}{1-\gamma}}(0)-\frac{\gamma}{k(1-\gamma)}t}. \tag{3.5}$$

Therefore (3.5) shows that $\Psi(t)$ blows up in finite time

$$T^*\leq\frac{k(1-\gamma)\Psi^{\frac{-\gamma}{1-\gamma}}(0)}{\gamma}.$$

This completes the proof. □

Remark 3.2. *The larger $\Psi(0)$ is the quicker the blow up takes place.*

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