DEU FMD 21(61), 137-148, 2019

Dokuz Eylül Üniversitesi Mühendislik Fakültesi Fen ve Mühendislik Dergisi
Dokuz Eylul University Faculty of Engineering Journal of Science and Engineering

Basılı/Printed ISSN: 1302-9304. Elektronik/Online ISSN: 2547-958X

On the Moving Coordinate System and Euler-Savary Formula in Affine Cayley-Klein Planes

Afin Cayley-Klein Düzlemlerinde Hareketli Koordinat Sistemi ve Euler-Savary Formülü Üzerine

Nurten Gürses [1*], Salim Yüce[1]

[1] Yildiz Technical University, Faculty of Art and Sciences, Deparment of Mathematics, 34220, İstanbul, TURKEY
Sorumlu Yazar / Corresponding Author *: nbayrak@yildiz.edu.tr

Geliş Tarihi / Received: 12.07.2018
Kabul Tarihi / Accepted: 11.09.2018
DOI:10.21205/deufmd.2019216114
Araştırma Makalesi/Research Article

Atıf şekli/ How to cite: GÜRSES, N., YÜCE, S. (2019). On the Moving Coordinate System and Euler-Savary Formula in Affine Cayley-Klein Planes. DEUFMD, 21(61), 137-148.

Abstract

In this present paper, we will take three affine Cayley-Klein planes into consideration: $A_{\grave{o}}$, $\mathrm{P}_{\grave{o}}$ and $\mathrm{P}_{\grave{o}}^{\prime}$. The plane $\mathrm{P}_{\grave{o}}^{\prime}$ is a fixed plane relative to two other moving affine Cayley-Klein (CK)-planes. We will describe one-parameter motions $A_{\grave{o}} / \mathrm{P}_{\grave{o}}$, $A_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ and $\mathrm{P}_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ and discuss the relationship between the motions $A_{\grave{o}} / \mathrm{P}_{\grave{o}}$, $A_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ and $\mathrm{P}_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ by evaluating their derivative formulae, velocity vectors and pole points. Also, we will observe moving coordinate system and after that, we will examine the canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Moreover, Euler-Savary formula, which gives the relationship between the curvatures of trajectory curves, will be obtained with the help of canonical relative system for one-parameter motions in affine CK-planes planes by using the method given by H. R. Müller in 1956 [1].

Keywords: *Cayley-Klein planes, one-parameter planar motion, moving coordinate system, kinematics, Euler-Savary Formula.*

Öz

Bu çalışmada $A_{\grave{o}}$, $\mathrm{P}_{\grave{o}}$ ve $\mathrm{P}_{\grave{o}}^{\prime}$ üç afin Cayley-Klein düzlemi gözönüne alınmıştır. $\mathrm{P}_{\grave{o}}^{\prime}$ düzlemi diğer iki hareketli afin Cayley-Klein (CK)-düzlemine göre sabittir. Çalışmada bir parametreli $A_{\grave{o}} / \mathrm{P}_{\grave{o}}$, $A_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ ve $\mathrm{P}_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ hareketleri tarif edilecek; türev formülleri, hız vektörleri ve pol noktaları elde edilerek $A_{\grave{o}} / \mathrm{P}_{\grave{o}}$, $A_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ ve $\mathrm{P}_{\grave{o}} / \mathrm{P}_{\grave{o}}^{\prime}$ hareketleri arasındaki ilişki tartışılacaktır. Ayrıca afin (CK)-düzlemlerinde hareketli koordinat sistemi araştırılarak bu hareketli koordinat sisteminin kavramları ile kavramları bir parametreli hareketler için kanonik izafe sistemi incelenecektir. Bu ifadelere ek olarak, kanonik izafe sistemi yardımıyla afin (CK)-düzlemlerinde bir parametreli hareketler için yörünge eğrilerinin eğrilikleri arasındaki ilişkiyi veren Euler Savary formülü H. R. Müller tarafında 1956 yılında verilen metodla elde edilecektir [1].

Anahtar Kelimeler: *Cayley-Klein düzlemleri, bir parametreli düzlemsel hareket, hareketli koordinat sistemi, kinematik, Euler-Savary formülü*

1. Introduction

Cayley-Klein (CK) geometries, were originated in the 19th century, are number of geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevsikan, [2,3]. Following Cayley and Klein, Yaglom distinguished these geometries by choosing one of three ways of measuring length (parabolic, elliptic, or hyperbolic) between two points on a line and one of the three ways of measuring angles between two lines (parabolic, elliptic, or hyperbolic). This gives nine ways of measuring lengths and angles, [4].

Much recent research is conducted in CK-planes, [4-19]. There is a known (but not well-known) relationship between the plane geometries which have parabolic measure of distance: Euclidean, Galilean and Minkowskian (Lorentz) geometries. They are called *affine CK-plane geometries*, [4].

To observe one-parameter motion in plane geometries has a significant role in kinematics. In this aspect, many researchers have received considerable attention in the kinematic literature, [20-24]. In 1956, H. R. Müller defined one-parameter planar motion in the Euclidean plane E^2 and studied the relationship between absolute, relative and sliding velocities (accelerations) [1]. Then, one-parameter planar motions and the above same notions are investigated in Lorentzian (Minkowskian) plane L^2 and Galilean plane G^2 by [22] and [23], respectively. Besides, in [24] the one-parameter motions in the affine CK-planes P_δ are introduced by generalizing the concepts introduced by above scientists.

It is known that the moving coordinate systems are important because, no material body is at absolute rest. As we know, even galaxies are not stationary. In reality, we have the moving frames, major example being Earth itself. In the light of this truth; the researchers argued this notion by considering different plane geometries: Lorentzian and Galilean planes, [25,26].

Furthermore, the canonical relative system for one-parameter planar motions were studied in [1], [27] and [28] in the planes E^2, L^2 and G^2 by using the notions of moving coordinate system, respectively. Three Lorentzian planes moving with respect to one another and pole points are studied in [29].

Euler-Savary formula which gives the relationship between the curvature of trajectory curves, during one-parameter planar motions, was studied by [1]. This formula was studied in Lorentzian plane for the one-parameter Lorentzian motions by using two different ways: In 2002, I. Aytun studied the this formula for the one-parameter Lorentzian motions with using the Müller's Method [30]. In 2003, T. Ikawa gave this formula on Minkowski plane by taking a new aspect without using the Müller's Method [31]. Ikawa gave the relationship between the curvature of roulette and curvatures of these base curve and rolling curve, [31]. Euler-Savary formula is a well documented and an admitted formula in the literature and many scientists have contributed to the development of fundamental knowledge of Euler-Savary formula, [32-39].

In 1983, the kinematics in the isotropic plane was studied by O. Röschel. In [40], the fundamental properties of the point-paths are investigated, a formula analog to the well-known formula of Euler-Savary was developed and special motions: an isotropic elliptic motion and an isotropic four-bar-motion are studied. Besides, in 1985, the motions $\sum/\sum_0$ in the isotropic plane was studied in [41]. Given C^2-curve k in the moving frame $\sum$, Röschel found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondence between the isotropic curvatures A and A_0 of k and k_0. Then third-order properties of the point-paths are investigated.

In this present paper, we will consider three affine CK-planes into consideration: A_δ, P_δ and $P_\delta^{'}$. $P_\delta^{'}$ is a fixed plane relative to two other moving affine CK-planes. We will aim to examine the relationship between the motions A_δ / P_δ, $A_\delta / P_\delta^{'}$ and $P_\delta / P_\delta^{'}$ by evaluating their derivative formulae, velocity vectors and pole points. We will introduce canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Moreover, Euler-Savary formula is obtained with the help of canonical relative system for one-parameter motions in affine CK-planes by using the Müller's Method. We will establish a simple but effective method by unifying moving coordinate system and Euler-Savary formula in Euclidean, Lorentzian and Galilean planes.

2. Preliminaries

In this section, we will investigate the basic notations of affine CK-planes and one-parameter planar motions in affine CK-planes, [24].

2.1. Basic notations

In this subsection, we examine the basic notations of affine CK-planes [4,8,24] which are denoted by P_ε.

Let us consider $\mathbb{R}^2$ with the bilinear form

$$\langle \mathbf{x},\mathbf{y}\rangle_\varepsilon = x_1y_1+\varepsilon x_2y_2,$$

where ε may be 1, 0 or -1 and $\mathbf{x}=(x_1,x_2), \mathbf{y}=(y_1,y_2)$. The matrix of this bilinear form can be given as below:

$$B=\begin{bmatrix}1 & 0\\ 0 & \varepsilon\end{bmatrix}.$$

For all $\mathbf{x}$ and $\mathbf{y}$ in P_ε we can write $\langle \mathbf{x},\mathbf{y}\rangle = \mathbf{x}^T B\mathbf{y}$. For ε=1 we have Euclidean plane E^2, for ε=0 we have Galilean plane G^2 and for ε=-1 we have Lorentzian plane L^2. If $\langle \mathbf{x},\mathbf{y}\rangle_\varepsilon = 0$, then the vectors $\mathbf{x}$ and $\mathbf{y}$ in P_ε are orthogonal. Self-orthogonal vectors are called *isotropic*. The norm of the vector $\mathbf{x}=(x_1,x_2)$ in P_ε is defined by

$$\|\mathbf{x}\|_\varepsilon = \sqrt{\left|\langle \mathbf{x},\mathbf{x}\rangle_\varepsilon\right|} = \sqrt{\left|x_1^2+\varepsilon x_2^2\right|}.$$

The distance between two points $A=(x_1,x_2)$ and $B=(y_1,y_2)$ is given by

$$\|\mathbf{AB}\| = \sqrt{\left|\langle \mathbf{AB},\mathbf{AB}\rangle_\varepsilon\right|} = d_{AB} = \sqrt{\left|(y_1-x_1)^2+\varepsilon(y_2-x_2)^2\right|}.$$

For ε=1 only the zero vector is isotropic, for $\varepsilon=0$ zero vector and vertical vectors are isotropic and for ε=-1 zero vector and vectors parallel to $(\pm 1,1)$ are isotropic, [8]. A circle is a locus of points equidistant from a given fixed point, namely the center of the circle. The unit circle in P_ε is the set of points with $\|\mathbf{P}\|=1$, for all $P\in \mathrm{P}_\varepsilon$. The equation of the unit circle in P_ε is $\mathbf{x}^2+\varepsilon\mathbf{y}^2=\pm 1$. The linear transformation $J:\mathrm{P}_\varepsilon \to \mathrm{P}_\varepsilon$ with matrix, also denoted by J and can be seen as below:

$$J=\begin{bmatrix}0 & -\varepsilon\\ 1 & 0\end{bmatrix}.$$

This linear transformation converts any vector $\mathbf{x}$ to an orthogonal vector $J\mathbf{x}$. If $\mathbf{x}$ is a nonisotropic vector and $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then we can write $\mathbf{y}=kJ\mathbf{x}$ for some real number k, [8].

It is not difficult to verify directly from the definition of the matrix exponential as

$$e^{J\varphi} = \sum_{n=0}^{\infty}\frac{(J\varphi)^n}{n!}$$

that

$$e^{J\varphi} = \cos_\varepsilon\varphi + J\sin_\varepsilon\varphi = \begin{bmatrix}\cos_\varepsilon\varphi & -\varepsilon\sin_\varepsilon\varphi\\ \sin_\varepsilon\varphi & \cos_\varepsilon\varphi\end{bmatrix}$$

where

$$\cos_\varepsilon\varphi = \sum_{n=0}^{\infty}\frac{(-\varepsilon)^n\varphi^{2n}}{(2n)!}\quad,\quad \sin_\varepsilon\varphi = \sum_{n=0}^{\infty}\frac{(-\varepsilon)^n\varphi^{2n+1}}{(2n+1)!}.$$

For ε=1 these are usual cosine and sine functions, for ε=-1 they are hyperbolic cosine and sine functions, and for ε=0 they are just $\cos_0\varphi=1$ and $\sin_0\varphi=\varphi$ for all φ. In all cases, we obtain

$$\cos_\varepsilon^2\varphi+\varepsilon\sin_\varepsilon^2\varphi=1$$

and

$$\partial_\varphi\cos_\varepsilon\varphi = -\varepsilon\sin_\varepsilon\varphi,\quad \partial_\varphi\sin_\varepsilon\varphi=\cos_\varepsilon\varphi.$$

By writing corresponding entries of the matrix equation $e^{J(\varphi+\theta)} = e^{J\varphi}e^{J\theta}$, we can find the sum formulae [4] as follows:

$$\cos_{\grave{o}}(\varphi+\theta) = \cos_{\grave{o}}\varphi\cos_{\grave{o}}\theta - \grave{o}\sin_{\grave{o}}\varphi\sin_{\grave{o}}\theta$$
$$\sin_{\grave{o}}(\varphi+\theta) = \sin_{\grave{o}}\varphi\cos_{\grave{o}}\theta + \cos_{\grave{o}}\varphi\sin_{\grave{o}}\theta.$$

2.2. One-parameter planar motions in affine CK-planes

The main purpose of this subsection is to argue the one-parameter planar motions in affine CK-planes, [24].

Let $P_{\grave{o}}$ and $P'_{\grave{o}}$ be moving and fixed affine CK-planes and $\{O;\mathbf{c_1},\mathbf{c_2}\}$ and $\{O';\mathbf{c}'_1,\mathbf{c}'_2\}$ be their orthonormal coordinate systems, respectively. Let us take the vector

$$\mathbf{OO}' = \mathbf{u} = u_1\mathbf{c_1} + u_2\mathbf{c_2} \text{ for } u_1, u_2 \in \mathbb{R}. \tag{1}$$

Let us define a transformation as given below:

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}, \tag{2}$$

where $\mathbf{x},\mathbf{x}'$ are coordinate vectors with respect to the moving and fixed rectangular coordinate system of a point $X=(x_1,x_2)\in P_{\grave{o}}$, respectively. By the equation (2), *one-parameter planar motions in affine CK-planes* are defined. These motions denoted by $P_{\grave{o}}/P'_{\grave{o}}$, [24].

Moreover, φ, the angle between the vectors $\mathbf{c_1}$ and $\mathbf{c}'_1$, is the *rotation angle* of the motions $P_{\grave{o}}/P'_{\grave{o}}$ and $\mathbf{x},\mathbf{x}',\mathbf{u}$ are continuously differentiable functions of the time parameter $t\in I\subset\mathbb{R}$. For $t=0$, the coordinate systems are coincident. By taking $\varphi=\varphi(t)$, we can write

$$\begin{cases}\mathbf{c_1} = \cos_{\grave{o}}\varphi\mathbf{c}'_1 + \sin_{\grave{o}}\varphi\mathbf{c}'_2\\ \mathbf{c_2} = -\grave{o}\sin_{\grave{o}}\varphi\mathbf{c}'_1 + \cos_{\grave{o}}\varphi\mathbf{c}'_2\end{cases} \tag{3}$$

We assume that $\dot{\varphi}(t) = d\varphi/dt \neq 0$. In this case $\dot{\varphi}(t)$ is called the *angular velocity* of the motions $P_{\grave{o}}/P'_{\grave{o}}$. By differentiating the equations (1) and (3) with respect to the parameter t, the *derivative formulae* of the motions $P_{\grave{o}}/P'_{\grave{o}}$ are obtained as follows:

$$\begin{cases}\dot{\mathbf{c}}_1 = \dot{\varphi}\mathbf{c_2},\\ \dot{\mathbf{c}}_2 = -\grave{o}\dot{\varphi}\mathbf{c_1},\\ \dot{\mathbf{u}} = (\dot{u}_1 - \grave{o}\dot{\varphi}u_2)\mathbf{c_1} + (\dot{u}_2 + \dot{\varphi}u_1)\mathbf{c_2}.\end{cases} \tag{4}$$

By using these derivative formulae, we will determine velocities of a point $X=(x_1,x_2)\in P_{\grave{o}}$ where

$$X = \mathbf{x} = x_1\mathbf{c_1} + x_2\mathbf{c_2}.$$

The velocity of the point X with respect to $P_{\grave{o}}$ is called the *relative velocity vector* denoted by $\mathbf{V}_r = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}$ and it is described by

$$\mathbf{V}_r = \dot{x}_1\mathbf{c_1} + \dot{x}_2\mathbf{c_2}. \tag{5}$$

Besides, the *absolute velocity* of the X with respect to $P_{\grave{o}}$ is obtained by differentiating the equation (2) with respect to t and using derivative formulae. It is denoted by $\mathbf{V}_a = \frac{d\mathbf{x}'}{dt}$ and obtained as follows:

$$\begin{aligned}\mathbf{V}_a &= \{-\dot{u}_1 + \grave{o}\dot{\varphi}(u_2 - x_2)\}\mathbf{c_1}\\ &\quad + \{-\dot{u}_2 + \dot{\varphi}(-u_1 + x_1)\}\mathbf{c_2} + \mathbf{V}_r.\end{aligned} \tag{6}$$

By using equation (6), we get the *sliding velocity vector* as described below:

$$\begin{aligned}\mathbf{V}_f &= \{-\dot{u}_1 + \grave{o}\dot{\varphi}(u_2 - x_2)\}\mathbf{c_1}\\ &\quad + \{-\dot{u}_2 + \dot{\varphi}(-u_1 + x_1)\}\mathbf{c_2}.\end{aligned} \tag{7}$$

From equations (5), (6) and (7) the following theorem can be given.

Theorem 2.1.

Let X be a moving point on the plane $P_{\grave{o}}$ and $\mathbf{V}_r,\mathbf{V}_a$ and $\mathbf{V}_f$ be the relative, absolute and sliding velocity vectors of X under the one-parameter planar CK-motions $P_{\grave{o}}/P'_{\grave{o}}$, respectively. Then, the relationship between the velocities as indicated below:

$$\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r.$$

Now, we will investigate the points that do not move during the motions $P_\varepsilon / P'_\varepsilon$. At this point, the sliding velocity vector $\mathbf{V}_f$ is equal to zero for every $t \in [t_0, t_1]$. These points are called the *pole points* or the *instantaneous rotation pole centers*. If we use the equation (7) for a pole point $P = (p_1, p_2) \in P_\varepsilon$ of the motions $P_\varepsilon / P'_\varepsilon$, we have

$$\begin{cases} -\dot{u}_2 + \dot{\varphi}(-u_1 + x_1) = 0 \\ -\dot{u}_1 + \varepsilon\dot{\varphi}(u_2 - x_2) = 0. \end{cases} \tag{8}$$

So, we obtain the pole point from the solution of the system (8) as follows:

$$\begin{cases} p_1(t) = x_1(t) = u_1(t) + \dfrac{\dot{u}_2(t)}{\dot{\varphi}(t)} \\ \varepsilon p_2(t) = \varepsilon x_2(t) = \varepsilon u_2(t) - \dfrac{\dot{u}_1(t)}{\dot{\varphi}(t)}. \end{cases} \tag{9}$$

Therefore, the point P is fixed in the plane P_ε. Let us rearrange the sliding velocity vector (7) by using the equation (9):

$$\mathbf{V}_f = \{-\varepsilon(x_2 - p_2)\mathbf{c}_1 + (x_1 - p_1)\mathbf{c}_2\}\dot{\varphi}. \tag{10}$$

With reference to the above equation, we can give the following corollaries:

Corollary 2.1.

During the one-parameter planar motions $P_\varepsilon / P'_\varepsilon$ in affine CK-planes, the pole ray $\mathbf{PX}$ and the sliding velocity vector $\mathbf{V}_f$ are perpendicular vectors in the sense of affine CK-geometry, i.e, $\langle \mathbf{PX}, \mathbf{V}_f \rangle_\varepsilon = 0$. Then, the focus of the point X of the motions $P_\varepsilon / P'_\varepsilon$ is an orbit that its normal pass through the rotation pole P.

Corollary 2.2.

Under the motions $P_\varepsilon / P'_\varepsilon$, the affine CK-norm of the sliding velocity vector $\mathbf{V}_f$ is written below:

$$\|\mathbf{V}_f\|_\varepsilon = \|\mathbf{PX}\|_\varepsilon |\dot{\varphi}|.$$

3. Moving Coordinate System and Pole Points in Affine CK-Planes

In this first original section, we will introduce the one-parameter motions $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P'_\varepsilon$ in affine CK-planes. Let A_ε and P_ε be moving and P'_ε be fixed affine CK-planes and $\{B; \mathbf{a}_1, \mathbf{a}_2\}, \{O; \mathbf{c}_1, \mathbf{c}_2\}$ and $\{O'; \mathbf{c}'_1, \mathbf{c}'_2\}$ be their coordinate systems, respectively.

Let us take the vectors $\mathbf{BO}$ and $\mathbf{BO'}$ as follows:

$$\begin{cases} \mathbf{BO} = \mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 \\ \mathbf{BO}' = \mathbf{b}' = b'_1\mathbf{a}_1 + b'_2\mathbf{a}_2. \end{cases}$$

We assume that φ, the angle between the vectors $\mathbf{a}_1$ and $\mathbf{c}_1$, is the rotation angle of one-parameter planar motions $A_\varepsilon / P_\varepsilon$. Similarly, φ', the angle between the vectors $\mathbf{a}_1$ and $\mathbf{c}'_1$, is the rotation angle of one-parameter planar motions $A_\varepsilon / P'_\varepsilon$.

By taking $\varphi = \varphi(t)$ and $\varphi' = \varphi'(t)$ into account to avoid the cases of pure translation, we can define the derivative formulae and velocities of these motions. For

$$\dot{\varphi}(t) = d\varphi / dt \neq 0$$

and

$$\varphi'(t) = d\varphi' / dt \neq 0,$$

we can write the above equations by using the equation (3):

$$\begin{cases} \mathbf{a}_1 = \cos_\varepsilon \varphi \mathbf{c}_1 + \sin_\varepsilon \varphi \mathbf{c}_2 \\ \mathbf{a}_2 = -\varepsilon \sin_\varepsilon \varphi \mathbf{c}_1 + \cos_\varepsilon \varphi \mathbf{c}_2, \end{cases} \tag{11}$$

and

$$\begin{cases} \mathbf{a}_1 = \quad \cos_\varepsilon \varphi' \mathbf{c}_1' + \sin_\varepsilon \varphi' \mathbf{c}_2' \\ \mathbf{a}_2 = -\varepsilon \sin_\varepsilon \varphi' \mathbf{c}_1' + \cos_\varepsilon \varphi' \mathbf{c}_2', \end{cases} \tag{12}$$

for $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P_\varepsilon'$, respectively.

Additionally, $\dot{\varphi}(t)$ and $\varphi'(t)$ are called the angular velocities of the motions $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P_\varepsilon'$, respectively. Assume that "d..." denotes the differential with respect to P_ε and "d'..." denotes the differential with respect to P_ε'. The derivative formulae of the motions $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P_\varepsilon'$ (taking $d'b = d'b'$) can be calculated from the equation (11) and (12) as follows:

$$\begin{cases} d\mathbf{a}_1 = d\varphi \mathbf{a}_2 \\ d\mathbf{a}_2 = -\varepsilon d\varphi \mathbf{a}_1 \\ d\mathbf{b} = (db_1 - \varepsilon d\varphi b_2)\mathbf{a}_1 + (db_2 + d\varphi b_1)\mathbf{a}_2 \end{cases} \tag{13}$$

and

$$\begin{cases} d'\mathbf{a}_1 = d\varphi' \mathbf{a}_2 \\ d'\mathbf{a}_2 = -\varepsilon d\varphi' \mathbf{a}_1 \\ d'\mathbf{b} = (db_1' - \varepsilon d\varphi' b_2')\mathbf{a}_1 + (db_2' + d\varphi' b_1')\mathbf{a}_2. \end{cases} \tag{14}$$

For the sake of shortness, we use the following equalities:

$$\begin{cases} d\varphi = \tau \\ db_1 - \varepsilon d\varphi b_2 = \sigma_1 \\ db_2 + d\varphi b_1 = \sigma_2 \end{cases}, \quad \begin{cases} d\varphi' = \tau' \\ db_1' - \varepsilon d\varphi' b_2' = \sigma_1' \\ db_2' + d\varphi' b_1' = \sigma_2' \end{cases}. \tag{15}$$

Definition 3.1.

σ_1, σ_2, τ and $\sigma_1', \sigma_2', \tau'$ are called CK-*Pfaffian forms* of the one-parameter CK-motions $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P_\varepsilon'$ with respect to t, respectively.

With reference to the above definition, the derivative formulae of the motions $A_\varepsilon / P_\varepsilon$ and $A_\varepsilon / P_\varepsilon'$ can be rearranged as follows:

$$\begin{cases} d\mathbf{a}_1 = \tau \mathbf{a}_2 \\ d\mathbf{a}_2 = -\varepsilon \tau \mathbf{a}_1 \\ d\mathbf{b} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 \end{cases}, \quad \begin{cases} d'\mathbf{a}_1 = \tau' \mathbf{a}_2 \\ d'\mathbf{a}_2 = -\varepsilon \tau' \mathbf{a}_1 \\ d'\mathbf{b} = \sigma_1' \mathbf{a}_1 + \sigma_2' \mathbf{a}_2. \end{cases} \tag{16}$$

Let us consider a point X with the coordinates of (x_1, x_2) in moving plane A_ε. Since $\mathbf{BX} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2$ is a vector on the moving system of A_ε, we have

$$\begin{cases} \mathbf{x} = \mathbf{OX} = \mathbf{OB} + \mathbf{BX} = -\mathbf{b} + x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 \\ \mathbf{x}' = \mathbf{O'X} = \mathbf{O'B} + \mathbf{BX} = -\mathbf{b}' + x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2, \end{cases}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are coordinate vectors of the point X with respect to P_ε and P_ε', respectively.

The differential of X with respect to P_ε is

$$\begin{aligned} d\mathbf{x} = & \left(dx_1 - \sigma_1 - \varepsilon \tau x_2\right)\mathbf{a}_1 \\ & + \left(dx_2 - \sigma_2 + \tau x_1\right)\mathbf{a}_2. \end{aligned} \tag{17}$$

Hence, the *relative velocity vector* of X with respect to P_ε is as follows:

$$\mathbf{V}_r = \frac{d\mathbf{x}}{dt}.$$

Similarly, differential of X with respect to P_ε' is

$$\begin{aligned} d'\mathbf{x}' = d'\mathbf{x} = & \left(dx_1 - \sigma_1' - \varepsilon \tau' x_2\right)\mathbf{a}_1 \\ & + \left(dx_2 - \sigma_2' + \tau' x_1\right)\mathbf{a}_2. \end{aligned} \tag{18}$$

Thus, the *absolute velocity vector* of X with respect to P_ε' is

$$\mathbf{V}_a = \frac{d'\mathbf{x}}{dt}.$$

If $\mathbf{V}_r = 0$ and $\mathbf{V}_a = 0$ then the point X is fixed in the planes $\mathbb{P}_\epsilon$ and $\mathbb{P}_\epsilon^{'}$, respectively. So, the conditions become

$$dx_1 = \sigma_1 + \epsilon\tau x_2, \quad dx_2 = \sigma_2 - \tau x_1 \tag{19}$$

and

$$dx_1 = \sigma_1^{'} + \epsilon\tau^{'} x_2,\; dx_2 = \sigma_2^{'} - \tau^{'} x_1, \tag{20}$$

respectively. By using the equations (19) and (20) and considering that the sliding velocity vector of the point X is $\mathbf{V}_f = \dfrac{d_f\mathbf{x}}{dt}$, we have

$$\begin{aligned} d_f\mathbf{x} &= [(\sigma_1 - \sigma_1^{'}) - \epsilon(\tau^{'} - \tau)x_2]\mathbf{a_1} \\ &\quad + [(\sigma_2 - \sigma_2^{'}) + (\tau^{'} - \tau)x_1]\mathbf{a_2}. \end{aligned} \tag{21}$$

In this manner, from (17), (18) and (21) we can give the following theorem.

Theorem 3.1.

Let X be a fixed point on the plane $\mathbb{P}_\epsilon$ under the one-parameter planar CK-motions $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$. Then, there is a relation between the differentials as noted below:

$$d^{'}\mathbf{x} = d_f\mathbf{x} + d\mathbf{x}. \tag{22}$$

The above equation enables us to write the relationship between the velocities: $\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r$. Hence, the above theorem is implemented.

Now, by considering the planes $\mathbb{P}_\epsilon$ and $\mathbb{P}_\epsilon^{'}$ are fixed, we give the following theorem:

Theorem 3.2.

Let $\tilde{\mathbf{V}}_a, \tilde{\mathbf{V}}_f$ and $\tilde{\mathbf{V}}_r$ be absolute, relative and sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}_\epsilon$, respectively. Similarly, let $\tilde{\mathbf{V}}_a^{'}, \tilde{\mathbf{V}}_f^{'}$ and $\tilde{\mathbf{V}}_r^{'}$ be absolute, relative and sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}_\epsilon^{'}$, respectively. Then, the sliding velocity vector of the motions $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$ can be given as below:

$$\begin{aligned} \mathbf{V}_f &= \tilde{\mathbf{V}}_f^{'} - \tilde{\mathbf{V}}_f \\ &= [(\sigma_1 - \sigma_1^{'}) - \epsilon(\tau^{'} - \tau)x_2]\mathbf{a_1} \\ &\quad + [(\sigma_2 - \sigma_2^{'}) + (\tau^{'} - \tau)x_1]\mathbf{a_2}. \end{aligned} \tag{23}$$

Proof 3.1.

By taking into consideration the conditions (19) and (20), we have the sliding velocity vectors of the motions $A_\epsilon / \mathbb{P}_\epsilon$ and $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$, respectively:

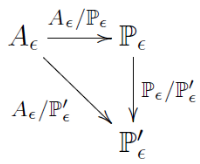
$$\begin{cases} \tilde{\mathbf{V}}_f = (-\sigma_1 - \epsilon\tau x_2)\mathbf{a_1} + (-\sigma_2 + \tau x_1)\mathbf{a_2} \\ \tilde{\mathbf{V}}_f^{'} = (-\sigma_1^{'} - \epsilon\tau^{'} x_2)\mathbf{a_1} + (-\sigma_2^{'} + \tau^{'} x_1)\mathbf{a_2} \end{cases}$$

Accordingly, the sliding velocity vector of the motions $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$ can be calculated as follows:

$$\begin{aligned} \mathbf{V}_f &= \tilde{\mathbf{V}}_f^{'} - \tilde{\mathbf{V}}_f \\ &= [(\sigma_1 - \sigma_1^{'}) - \epsilon(\tau^{'} - \tau)x_2]\mathbf{a_1} \\ &\quad + [(\sigma_2 - \sigma_2^{'}) + (\tau^{'} - \tau)x_1]\mathbf{a_2}. \end{aligned}$$

Finally, It is quite obvious that we get the equation (21) again.

Corollary 3.1.



The motions $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$ is characterized by the composition of the inverse motions $\mathbb{P}_\epsilon / A_\epsilon$ and the motions $A_\epsilon / \mathbb{P}_\epsilon^{'}$ as follows:

$$\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'} = (\mathbb{P}_\epsilon / A_\epsilon) \circ (A_\epsilon / \mathbb{P}_\epsilon^{'}). \tag{24}$$

4. Moving Planes with Respect to the Another and Rotation Poles

In this original section, we will aim to find out the rotation poles during the motions $A_\epsilon / \mathbb{P}_\epsilon$, $A_\epsilon / \mathbb{P}_\epsilon^{'}$ and $\mathbb{P}_\epsilon / \mathbb{P}_\epsilon^{'}$ with the different perspective to be a moving or fixed plane. It is indicated that in the one-parameter planar motions in affine CK-planes, the rotation pole is characterized by vanishing sliding velocity. In the consideration of this statement, the rotation poles can be found as follows:

The pole point $Q=(q_1,q_2)$ of the motions A_{o}/P_{o} is calculated with $\tilde{\mathbf{V}}_f=0$ as below:

$$\begin{cases} q_1=\dfrac{\sigma_2}{\tau}, \\ q_2=-\dfrac{\sigma_1}{\tau}. \end{cases} \quad (25)$$

The pole point $Q^{'}=(q_1^{'},q_2^{'})$ of the motions $A_{o}/\mathrm{P}_{o}^{'}$ is computed with $\tilde{\mathbf{V}}_f^{'}=0$ and given as follows:

$$\begin{cases} q_1^{'}=\dfrac{\sigma_2^{'}}{\tau^{'}}, \\ q_2^{'}=-\dfrac{\sigma_1^{'}}{\tau^{'}}. \end{cases} \quad (26)$$

The pole point $P=(p_1,p_2)$ of the motions $\mathrm{P}_{o}/\mathrm{P}_{o}^{'}$ characterised by $\mathbf{d}_f\mathbf{x}=0$. So, the pole point P of the one-parameter planar motions $\mathrm{P}_{o}/\mathrm{P}_{o}^{'}$ is obtained as follows:

$$\begin{cases} p_1=\dfrac{\sigma_2^{'}-\sigma_2}{\tau^{'}-\tau} \\ p_2=-\dfrac{\sigma_1^{'}-\sigma_1}{\tau^{'}-\tau}, \end{cases} \quad (27)$$

where $\mathbf{BP}=p_1\mathbf{a}_1+p_2\mathbf{a}_2$.

Theorem 4.1.

If three affine CK-planes generate one-parameter planar CK-motions pairwise, there exist three relative rotation poles at every moment t.

Theorem 4.2.

Let P,Q and $Q^{'}$ be the pole points of the affine CK-motions A_{o}/P_{o}, $A_{o}/\mathrm{P}_{o}^{'}$ and $\mathrm{P}_{o}/\mathrm{P}_{o}^{'}$, respectively. Then P,Q and $Q^{'}$ are collinear.

Proof 4.1.

The slopes of $[PQ],[PQ^{'}]$ and $[QQ^{'}]$ are all equal to:

$$\frac{\sigma_1\tau^{'}-\sigma_1^{'}\tau}{\sigma_2^{'}\tau-\sigma_2\tau^{'}}.$$

This completes the proof.

Definition 4.1.

The straight line, indicated the above theorem, is called *the affine CK-pole line* of the one-parameter affine CK-motions A_{o}/P_{o}, $A_{o}/\mathrm{P}_{o}^{'}$ and $\mathrm{P}_{o}/\mathrm{P}_{o}^{'}$.

Corollary 4.1.

Generally, if there are $n-$ affine CK-planes which form one-parameter planar affine CK-motions pairwisely, then we mention about $n-$ member kinematic chain. If the each motions are connected time parameter t (real) , then there exist $\binom{n}{2}$ relative rotation poles at every moment t.

5. Euler-Savary Formula in Affine CK-Planes

In this original section, we will study Euler-Savary formula in affine CK-planes. We choose the relative system $\{B;\mathbf{a}_1,\mathbf{a}_2\}$ satisfying the following conditions:

i) The initial point B of the system is the instantaneous rotation pole P (i.e. $B=P$)
ii) The axis $\{B,\mathbf{a}_1\}$ coincides with the common tangent of the pole curves (P) and $(P^{'})$.

By considering the condition i), we have $p_1=p_2=0$, because of the fact that P is an initial point. Hence, from the equations (27), we obtain the following equalities:

$$\sigma_1' = \sigma_1, \quad \sigma_2' = \sigma_2. \tag{28}$$

So, from the equations (16), we can give the pole tangent as follows:

$$\mathbf{db} = \mathbf{dp} = \sigma_1\mathbf{a_1} + \sigma_2\mathbf{a_2} = \mathbf{d'p} = \mathbf{db'}. \tag{29}$$

The equation (29) means that, the relative and the absolute velocities are equal to each other (i.e. $\mathbf{V}_f = 0$). Thus, the scalar arc elements of the pole curves (P) and (P') can be given as below:

$$ds' = \|\mathbf{V_a}\|_{\grave{o}}\, dt = \|\mathbf{V_r}\|_{\grave{o}}\, dt = ds. \tag{30}$$

This means that the moving pole curve (P) and fixed pole curve (P') roll on each other without sliding. By considering the condition ii) yields us that $\sigma_2 = \sigma_2' = 0$. If we take $\sigma_1 = \sigma_1' = \sigma$, then the derivative formulae of the canonical relative system $\{P, \mathbf{a}_1, \mathbf{a}_2\}$ become

$$\begin{cases} d\mathbf{a}_1 = \tau\mathbf{a}_2 \\ d\mathbf{a}_2 = -\grave{o}\tau\mathbf{a}_1 \\ d\mathbf{p} = \sigma\mathbf{a}_1 \end{cases}, \quad \begin{cases} d'\mathbf{a}_1 = \tau'\mathbf{a}_2 \\ d'\mathbf{a}_2 = -\grave{o}\tau'\mathbf{a}_1 \\ d'\mathbf{p} = \sigma\mathbf{a}_1 \end{cases}. \tag{31}$$

The differential forms σ, τ and τ' of the equations (31) have specific meanings: $ds = \sigma$ is the scalar arc element of the pole curves (P) and (P'), τ and τ' are the central cotangent angle, that is, two neighboring tangents angle of (P) and (P'), respectively. Thus, the curvature of the moving pole curve (P) and (P'), at the point P are $\frac{d\varphi}{ds} = \frac{\tau}{\sigma}$ and $\frac{d\varphi'}{ds} = \frac{\tau'}{\sigma}$, respectively. Hence, the curvature radii of the pole curves (P) and (P') can be written as below:

$$r = \frac{\sigma}{\tau} \text{ and } r' = \frac{\sigma'}{\tau'}, \tag{32}$$

respectively. Moving plane $\mathrm{P}_{\grave{o}}$ rotates the infinitesimal instantaneous angle $d\phi = \tau' - \tau$ around the rotation pole (P) within the time scale t with respect to fixed plane $\mathrm{P}_{\grave{o}}'$. Hence, ω is the angular velocity of moving plane $\mathrm{P}_{\grave{o}}$ with respect to the fixed plane $\mathrm{P}_{\grave{o}}'$ is given below:

$$\frac{\tau' - \tau}{dt} = \frac{d\phi}{dt} = \dot{\phi} = \omega. \tag{33}$$

Also, we denote the angular acceleration of moving plane $\mathrm{P}_{\grave{o}}$ with respect to the fixed plane $\mathrm{P}_{\grave{o}}'$ by $\dot{\omega}$, where $\ddot{\phi} = \dot{\omega}$. From the equations (32) and (33), it is seen that

$$\frac{\tau' - \tau}{dt} = \frac{d\phi}{dt} = \frac{1}{r'} - \frac{1}{r}. \tag{34}$$

Let us assume that the direction of unit tangent vector is $\mathbf{a}_1$ and $ds/dt > 0$. Due to the fact that, the curvature center of the moving pole curve (P) stays in the same side of the directed pole curve $(P; \mathbf{a}_1)$, it is written that $r > 0$. Similarly $r' > 0$.

Let us rearrange the equations (17) and (18) with respect to the planes $\mathrm{P}_{\grave{o}}$ and $\mathrm{P}_{\grave{o}}'$ by taking a point $X = (x_1, x_2)$ in the plane $A_{\grave{o}}$, respectively. These differentials can be calculated as below:

$$d\mathbf{x} = (dx_1 - \sigma - \grave{o}\tau x_2)\mathbf{a_1} + (dx_2 + \tau x_1)\mathbf{a_2}$$

and

$$d'\mathbf{x} = (dx_1 - \sigma - \grave{o}\tau' x_2)\mathbf{a_1} + (dx_2 + \tau' x_1)\mathbf{a_2}.$$

On the condition that $d\mathbf{x} = 0$, then the following conditions occur and X is a fixed point $\mathrm{P}_{\grave{o}}$:

$$\begin{cases} dx_1 = \sigma + \grave{o}\tau x_2 \\ dx_2 = -\tau x_1. \end{cases} \tag{35}$$

In a similar way, if $d'\mathbf{x} = 0$, then the following conditions exist and X is a fixed point P_ε':

$$\begin{cases} dx_1 = \sigma + \varepsilon\tau' x_2 \\ dx_2 = -\tau' x_1. \end{cases} \tag{36}$$

Thus, the sliding velocity of the motion can be given as follows:

$$d_f\mathbf{x} = d'\mathbf{x} - d\mathbf{x} = (\tau' - \tau)(\varepsilon x_2\mathbf{a_1} - x_1\mathbf{a_2}).$$

Now, we investigate the curvature centers of trajectory curves which are drawn in the fixed plane by the points of moving plane during the motion $P_\varepsilon / P_\varepsilon'$. Let $M' = (m_1', m_2')$ represents the curvature center of trajectory curves which are drawn in P_ε' by the point $X = (x_1, x_2)$ in P_ε with respect to the canonical relative system at every time t. The points X and M' and the instantaneous rotation pole P lay on an instantaneous trajectory normal related to X at every time t. Therefore, the vectors

$$\mathbf{PX} = x_1\mathbf{a_1} + x_2\mathbf{a_2}$$

and

$$\mathbf{PM}' = m_1'\mathbf{a_1} + m_2'\mathbf{a_2}$$

have the same direction which passes the rotation pole P. Accordingly, we can write

$$x_1 : x_2 = m_1' : m_2'$$

or

$$x_1m_2' - m_1'x_2 = 0. \tag{37}$$

If we differentiate the equation (37), we have

$$dx_1m_2' + x_1dm_2' - dm_1'x_2 - m_1'dx_2 = 0. \tag{38}$$

By using the new form of the equations (35) and (36) for the points X and M' in the equation (38), we obtain the following equation as below:

$$\sigma(m_2' - x_2) - (\tau' - \tau)(x_1m_1' + \varepsilon x_2m_2'). \tag{39}$$

If we substitute the polar coordinates

$$\begin{cases} x_1 = a\cos_\varepsilon\alpha \\ x_2 = a\sin_\varepsilon\alpha \end{cases}, \quad \begin{cases} m_1' = a'\cos_\varepsilon\alpha \\ m_2' = a'\sin_\varepsilon\alpha \end{cases}$$

in the equation (39), we get

$$\sigma\sin_\varepsilon\alpha(a' - a) - (\tau - \tau')aa' = 0,$$

where a and a' are the distance between the points X and M' and rotation pole P, respectively. Besides, α is the angle between the pole ray ($\mathbf{PX}$ and $\mathbf{PM}'$) and the common tangent of pole curves. Finally, by taking into account the equation (33), we obtain the last form of the above equation as follows:

$$\left(\frac{1}{a} - \frac{1}{a'}\right)\sin_\varepsilon\alpha = \frac{1}{r'} - \frac{1}{r} = \frac{d\phi}{ds}. \tag{40}$$

Consequently, the equation (40) is called *Euler-Savary formula for one- parameter motions in affine CK-planes*. Hence, the following theorem can be given:

Theorem 5.1.

Let P_ε and P_ε' be moving and fixed affine CK-planes, respectively. A point X in moving CK-plane P_ε draws a trajectory whose curvature center is at the point M' in fixed plane P_ε' during the one- parameter planar CK-motion $P_\varepsilon / P_\varepsilon'$. In the reverse motion $P_\varepsilon' / P_\varepsilon$ a point M' in P_ε' whose curvature center is at the point X in P_ε'. The relationship between the points X and M' is given by Euler-Savary formula by the equation (40).

6. Discussions and Conclusions

In this paper, the generalization of moving coordinate system and Euler-Savary formula have been successfully applied in affine Cayley-Klein planes (CK-planes) by using one-parameter planar motions [24]. We have considered three affine CK-planes: A_ε, P_ε and P_ε'. The plane P_ε' is a fixed plane relative to two other moving affine CK-planes. We have examined the relationship between the motions $A_\varepsilon / P_\varepsilon$, $A_\varepsilon / P_\varepsilon'$ and $P_\varepsilon / P_\varepsilon'$ by evaluating their derivative formulae, velocity vectors and pole points. We have introduced canonical relative system for one-parameter planar motions in the affine CK-planes by using the notions of moving coordinate system. Furthermore, we have obtained Euler-Savary formula with the aid of canonical relative system by using the H. R. Müller's Method. We have established a simple but effective method by unifying moving coordinate system and Euler-Savary formula in Euclidean, Lorentzian and Galilean planes.

Acknowledgment

The first author was partially supported by TÜBİTAK (2211-Domestic PhD Partially Scholarship)- The Scientific and Technological Research Council of Turkey during her Ph.D. period.

References

[1] Blaschke, W., Müller, H. R. 1956. Ebene Kinematik, R. Oldenbourg, München, 269p.

[2] Klein F. 1985. Ueber die sogenannte Nicht-Euklidische Geometrie. In: Gauß und die Anfänge der nicht-euklidischen Geometrie. Teubner-Archiv zur Mathematik, Springer, Vienna, Vol 4., pp. 224-238

[3] Klein, F., 1967. Vorlesungen über nicht-Euklidische Geometrie, Springer, Berlin, 330p.

[4] Yaglom, I. M. 1979. A simple non-Euclidean geometry and its Physical Basis, Springer-Verlag, New York, 307p.

[5] Röschel, O. 1992. Zur Krümmungsverwandtschaft von zwanglaufen in affinen CK-Ebenen I, Journal of Geometry, Vol. 44, No. 1-2, pp. 160-170.

[6] Röschel, O. 1993. Zur Krümmungsverwandtschaft von zwanglaufen in affinen CK-Ebenen II, Journal of Geometry, Vol. 47, No. 1-2, pp. 131-140.

[7] Es, H. 2003. Motions and Nine Different Geometry, Ankara University, Graduate School of Natural and Applied Sciences, Ph.D. Thesis, 130p, Ankara, Turkey.

[8] Helzer, G. 2000. Special Relativity with Acceleration, The American Mathematical Monthly, Vol. 107, No. 3, pp. 219-237.

[9] Herranz, F. J., Santader, M. 1997. Homogeneous Phase Spaces: The Cayley-Klein framework. http://arxiv.org/pdf/physics/9702030v1.pdf (Access Date: 26.03.2013).

[10] Salgado, R. 2006. Space-Time Trigonometry. In: AAPT Topical Conference: Teaching General Relativity to Undergraduates,
AAPT Summer Meeting, July 20-21, Syrauce University, NY, 22-26.

[11] Sanjuan, M. A. F. 1984. Group Contraction and Nine Cayley-Klein Geometries, International Journal of Theoretical Physics, Vol. 23, No.1, pp 1-14.

[12] Spirova, M. 2009. Propellers in Affine Cayley-Klein Planes, Journal of Geometry, Vol. 93, pp. 164-167.

[13] Urban, H. 1994. Über drei zwangslaufig gegeneinander bewegte Cayley/Klein-Ebenen, Geometriae Dedicata, Vol. 53, pp. 187-199.

[14] McRae, A.S. 2009. Clifford Fibrations and Possible Kinematics, Symmetry, Integrability and Geometry: Methods and Applications, Vol. 5, 072, 18p.

[15] Kisil, V. V. 2012. Geometry of Möbius Transformations: Eliptic, Parabolic and Hyperbolic Actions of $SL_2(\square)$,Imperial College Press, London, 208p.

[16] Martini, H., Spirova, M. 2008. Circle Geometry in Affine Cayley-Klein Planes, Periodica Math. Hungar. Vol. 57, No. 2, pp. 197–206.

[17] Spirova, M. 2006. On the Napoleon-Torricelli Configuration in Affine Cayley-Klein Planes, Abh. Math. Sem. Univ. Hamburg, Vol. 76, pp. 131–142.

[18] Urban, H., 1993. Verallgemeinerung des Wendekreises der Ebenen Euklidischen Kinematik auf Cayley/Klein-Zwanglaufe 1. Art, Journal of Geometry, Vol. 47, pp. 163–180.

[19] Gunn, C. 2011. Geometry, Kinematics, and Rigid Body Mechanics in Cayley-Klein Geometries, Technische Universitat Berlin, Ph.D. thesis, Berlin, Germany.

[20] Dijskman, E. A. 1976. Motion Geometry of Mechanism, Cambridge University Press, Cambridge, 280p.

[21] Hall, A. S. Jr. 1986. Kinematics and Linkage Design, Waveland Press, Inc., Prospect Heights, Illinois (Originally published by Prentice-Hall,Inc.,),162p.

[22] Ergin, A. A. 1991. On the one-parameter Lorentzian motion, Communications, Faculty of Science, University of Ankara, Series A, Vol. 40, pp. 59-66.

[23] Akar, M., Yüce, S., Kuruoğlu, N. 2013. One-Parameter Planar Motion in the Galilean Plane, International Electronic Journal of Geometry (IEJG), Vol. 6, No. 1, pp. 79-88.

[24] (Bayrak) Gürses, N., Yüce, S. 2014. One-Parameter Planar Motions in Affine Cayley-Klein Planes, European Journal of Pure and Applied Mathematics, Vol. 7, No. 3, pp. 335-342.

[25] Tutar, A., Kuruoğlu, N., Düldül, M. 2001. On the Moving Coordinate System and Pole Points on the Lorentzian Plane, International Journal of Applied Mathematics, Vol. 7, No. 4, pp. 439-445.

[26] Akbıyık, M., Yüce, S. 2015. The Moving Coordinate System and Euler-Savary's Formula for the One-Parameter Motions On Galilean (Isotropic) Plane, International Journal of Mathematical Combinatorics, Vol. 2, pp. 88-105.

[27] Ergüt, M., Aydın, A. P., Bildik, N. 1988. The Geometry of the Canonical Relative System and the One-Parameter Motions in 2-dimensional Lorentzian Space, The Journal of Fırat University, Vol. 3, No. 1, pp. 113-122.

[28] Akbıyık, M.2012. Moving coordinate system and Euler Savary formula on Galilean plane, Yıldız Technical University, Graduate School of Natural and Applied Sciences, Master Thesis, 90p, Istanbul, Turkey.

[29] Ergin, A. A. 1992. Three Lorentzian Planes Moving With Respect to One Another And Pole Points, Communications, Faculty of Science, University of Ankara, Series A, Vol. 14, pp. 79-84.

[30] Aytun, I. 2002. Euler-Savary formula for one-parameter Lorentzian plane motion and its Lorentzian geometrical interpretation, Celal Bayar University, Graduate School of Natural and Applied Sciences, Master Thesis, 54p, Manisa, Turkey.

[31] Ikawa, T. 2003. Euler-Savary's Formula on Minkowski Geometry, Balkan Journal of Geometry and Its Applications, Vol. 8, No. 2, pp. 31-36.
[32] Dooner, D. B., Griffis, M. W. 2007. On the Spatial Euler-Savary Equations for Envelopes, Journal of Mechanical Design, Vol. 129, No. 8, pp. 865-875.
[33] Buckley, R., Whitfield, E. V. 1949. The Euler-Savary Formula, The Mathematical Gazette, Vol. 33, No. 306, p. 297-299.
[34] Garnier, R. 1951. Cours de cinématique, géométrie et cinématique cayleyennes, Gauthier-Villars, Paris.
[35] Sandor, G. N., Xu, Y., Weng, T-C.1990. A Graphical Method for Solving the Euler-Savary Equation, Mechanism and Machine Theory, Vol. 25, No. 2, pp. 141-147.
[36] Sandor, G. N., Arthur, G.E., Raghavacharyulu, E. 1985. Double Valued Solutions of the Euler-Savary Equation and Its Counterpart in Bobillier's Construction, Mechanism and Machine Theory, Vol. 20, No. 2, pp. 145-178.
[37] Sá Pereira, N.T., Ersoy, S. 2016. Elliptical Harmonic Motion and Euler–Savary Formula, Adv. Appl. Clifford Algebras, Vol. 26, pp. 731–755.
[38] Ergüt, M., Aydın, A.P., Bildik, N. 1989. The Curvature of the Trajectory curves of the pole curves (P), (P') and the point correspondence $X \leftrightarrow X'$ in 2-dimensional Lorentzian Plane L^2, The Journal of Fırat University, Vol. 4, No. 1, pp. 2-33.
[39] Ito, N., Takahashi, K. 1999. Extension of the Euler-Savary Equation to Hypoid Gears, JSME Int. Journal. Ser C. Mech Systems, Vol. 42, No. 1, pp. 218-224.
[40] Röschel, O. 1983. Zur Kinematik der isotropen Ebene, Journal of Geometry, Vol. 21, pp. 146-156.
[41] Röschel, O. 1985. Zur Kinematik der isotropen Ebene II., Journal of Geometry, Vol. 24, pp. 112-122.