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DEFORMATION RETRACT OF A DIGITAL H-SPACE

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Abstract. In this study we investigate some properties of digital H-spaces and digital H-groups. It is shown that a κ-deformation retract of a digital H-group is itself a digital H-group. Also we prove that there is a functor from the homotopy category of the pointed digital images to the category of groups and homomorphisms.

1. Introduction

Digital topology examines the topological properties of the digital images, such as connectedness, compactness. The first study on this area was carried out by Rosenfeld in [8]. The following years, a lot of concepts of algebraic topology were converted to digital topology. Digital homotopy and digital fundamental group are defined in [2] by Boxer. Digital H-space is defined in [4] by Ege and Karaca. An H-space consists of a pointed topological space P with a continuous multiplication $m : X \times X \longrightarrow X$ and with a constant map $c : X \longrightarrow X$, such that $m \circ (c, 1_X) \simeq 1_X \simeq m \circ (1_X, c)$. A group structure can be established on H-space, called H-group, by homotopy group operations which are similar to group operations.

2. Preliminaries

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}^n$ the set of all lattice points in Euclidean n-dimensional space. A digital image is a finite subset (X, κ) of $\mathbb{Z}^n$, where $X \subset \mathbb{Z}^n$ and κ is a certain adjacency relation for the members of X. There are various adjacency relation in use.

Definition 2.1. [4] Consider the following statements:

(1) Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$.
(2) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ in at most 1 in each coordinate.
(3) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate.
(4) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ in at most 1 in each coordinate.
(5) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ in at most two coordinates.

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(6) Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate.

A digital interval is a set of the form $[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

Definition 2.2. [10] Let (X,κ_1) and (Y,κ_2) be two digital image and $x_1, x_2 \in (X,\kappa_1), y_1, y_2 \in (Y,\kappa_2)$. Then (x_1,y_1) and (x_2,y_2) are adjancent in $X \times Y$ if and only if one of the following is satisfied:

(1) $x_1 = x_2$ and y_1 and y_2 are κ_2-adjacent; or
(2) x_1 and x_2 are κ_1-adjacent and $y_1 = y_2$
(3) x_1 and x_2 are κ_1-adjacent and y_1 and y_2 are κ_2-adjacent.

The adjancency of the cartesian product of digital images (X,κ_1) and (Y,κ_2) is denoted by κ^*.

A $\kappa-$neigbor of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is $\kappa-$adjancent to p. A subset A of a digital image (X,κ) is $\kappa-$connected if and only if for every pair of diffirent points $a, b \in A$, there exists a set $\{y_0, y_1, ..., y_r\}$ of points of A such that $a = y_0, b = y_r$ and y_i and y_{i+1} are $\kappa-$neighbors where $i \in \{0, 1, ..., r-1\}$.

A function $f : (X,\kappa_1) \to (Y,\kappa_2)$ is called $(\kappa_1,\kappa_2)-$continuous if the image under f of every κ_1-connected subset of X is a κ_2-connected subset of (Y,κ_2).

Definition 2.3. [2] Let (X,κ_1) and (Y,κ_2) be two digital image. (κ_1,κ_2)-continuous functions $f, g : (X,\kappa_1) \to (Y,\kappa_2)$ are said to be digitally (κ_1,κ_2)-homotopic in (Y,κ_2) if there exist a positive integer m and a function $H : X \times [0,m]_{\mathbb{Z}} \to Y$, such that,

(1) for all $x \in X$, $H(x,0) = f(x)$ and $H(x,m) = g(x)$
(2) for all $x \in X$, the induced function $H_x : [0,m]_{\mathbb{Z}} \to Y$, $H_x(t) = H(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$, is $(2,\kappa_2)-$continuous,
(3) for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$, $H_t(x) = H(x,t)$ for all $x \in X$, is $(\kappa_1,\kappa_2)-$continuous.

In this case H is called a digital $(\kappa_1,\kappa_2)-$homotopy between f and g, written $f \simeq_{(\kappa_1,\kappa_2)} g$.

The notation $[f]$ is used to denote the digital (κ_1,κ_2)-homotopy class of (κ_1,κ_2)-continuous function $f : X \to Y$, i.e. $[f] = \{g : X \to Y \mid g$ is (κ_1,κ_2)-continuous and $f \simeq_{(\kappa_1,\kappa_2)} g\}$.

For a digital image (X,κ) and its subset (A,κ), (X,A) is called a digital image pair with κ-adjancency. Also, if A is a singleton set $\{p\}$, then (X,p,κ) is called a pointed digital image.

Definition 2.4. [3] Let $f : X \to Y$ be a $(\kappa_1,\kappa_2)-$continuous function and let $g : Y \to X$ be a $(\kappa_2,\kappa_1)-$continuous function, such that $f \circ g \simeq_{(\kappa_2,\kappa_2)} 1_Y$ and $g \circ f \simeq_{(\kappa_1,\kappa_1)} 1_X$. Then (X,κ_1) and (Y,κ_2) are said to be same $(\kappa_1,\kappa_2)-$homotopy type or (κ_1,κ_2)-homotopy equivalent. Also f and g are called (κ_1,κ_2)-equivalences.

Definition 2.5. [4] Let (X,A) be a digital image pair with κ-adjancency. Let $i : A \to X$ be the inclusion map. Then A is called a κ-retract of X if and only if there is a κ-continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Then the function r is called a $\kappa-$retraction of X onto A.

Definition 2.6. [5] Let (X,A) be a digital image pair with $\kappa-$adjancency. If there exists digital pointed κ-homotopy $D : X \times [0,m]_{\mathbb{Z}} \to X$ such that $D(x,0) = x$ and $D(x,m) \subset A, x \in X$, i.e. $i \circ r \simeq_{(\kappa,\kappa)} 1_X$, then A is said to be $\kappa-$deformation retract of X.

3. Digital H-spaces

In this section we investigate some properties of digital H-spaces and digital H-groups.

Definition 3.1. [4] Let (X, p, κ) be a pointed digital image. For a digital continuous multiplication $\mu : X \times X \to X$ and the digital constant map $c : X \to X$, defined by $c(x) = p$, if

$$\mu \circ (c, 1_X) \simeq_{(\kappa,\kappa)} \mu \circ (1_X, c) \simeq_{(\kappa,\kappa)} 1_X,$$

then (X, p, κ) is called a digital H-space.

Definition 3.2. [4] Let (X, p, κ) be a digital H-space.

(1) if $\mu \circ (1_X \times \mu) \simeq_{(\kappa^*,\kappa)} \mu \circ (\mu \times 1_X)$, then μ is called digital homotopy associative.
(2) if $\mu \circ (\varphi, 1_X) \simeq_{(\kappa,\kappa)} \mu \circ (1_X, \varphi) \simeq_{(\kappa,\kappa)} c$, for a map $\varphi : (X, p, \kappa) \to (X, p, \kappa)$, then φ is called a digital homotopy inverse for μ.
(3) if $\mu \circ T \simeq_{(\kappa^*,\kappa)} T$, for a map T, defined by $T(x, y) = T(y, x)$, then μ is called digital homotopy commutative and (X, p, κ) is called abelian digital H-space.

Definition 3.3. A digital H-group is a digital H-space (X, p, κ) with the digital homotopy associative multiplication μ and digital homotopy inverse η.

Theorem 3.4. *[4] A $\kappa-$retract of a digital H-space (X, p, κ) is a digital H-space.*

Theorem 3.5. *A $\kappa-$retract of an abelian digital H-space is itself an abelian digital H-space.*

Proof. Let (X, p, κ) be an abelian digital H-space and $A \subset X$ be a $\kappa-$retract with a $\kappa-$retraction $r : X \to A$. Then (A, q, κ) is a digital H-space.

Since (X, p, κ) is abelian, there exist a function $f(x, y) = (y, x)$ for all $x, y \in X$, such that $\mu \circ f \simeq_{(\kappa^*,\kappa)} \mu$. Define $g = f \mid_{A\times A}$ and $\eta = r \circ \mu \circ (i \times i)$ .Then,

$$\eta = r \circ \mu \circ (i \times i) \simeq_{(\kappa^*,\kappa)} r \circ \mu \circ f \circ (i \times i) = r \circ \mu \circ (i \times i) \circ g = \eta \circ g.$$

So (A, q, κ) is an abelian digital H-space. □

Definition 3.6. Let (X, p, κ) and $(Y, q, \kappa^{'})$ be digital H-spaces. If there exist a map $f : X \to Y$ such that $f \circ \mu \simeq_{(\kappa,\kappa')} \eta \circ (f \times f)$, where μ and η are digital continuous multiplication of X and Y, respectively.

Theorem 3.7. *Let (X, p, κ) be a digital H-space and $A \subset X$ be a $\kappa-$ deformation retract of (X, p, κ). Then the inclusion map i and $\kappa-$retraction r are digital H-homomorphism.*

Proof. Let μ be the digital continuous multiplication of (X, p, κ) and $\eta = r \circ \mu \circ (i \times i)$. As $i \circ r \simeq_{(\kappa,\kappa)} 1_X$,

$$i \circ \eta = i \circ r \circ \mu \circ (i \times i) \simeq_{(\kappa^*,\kappa)} 1_X \circ \mu \circ (i \times i) = \mu \circ (i \times i) .$$

So i is a digital H-homomorphism. In addition,

$$\eta \circ (r \times r) = r \circ \mu \circ (i \times i) \circ (r \times r) \simeq_{(\kappa^*,\kappa)} r \circ \mu \circ 1_{X\times X} = r \circ \mu.$$

So r is a digital H-homomorphism. □

Theorem 3.8. *A $\kappa-$deformation retract of a digital H-group is itself a digital H-group.*

Proof. Let (X,p,κ) be a digital H-group with the digital continuous multiplications μ, homotopy inverse θ , constant map c and let $A \subset X$ be a $\kappa-$ deformation retract of (X,p,κ) and $\eta = r \circ \mu \circ (i \times i)$. By Theorem 4 (A,q,κ) is a digital H-space with the digital continuous multiplication η.

$$\begin{aligned} \eta \times 1_A &= r \circ \mu \circ (i \times i) \times 1_A \\ &= r \circ \mu \circ (i \times i) \times r \circ i \\ &= (r \times r) \circ (\mu \times 1_X) \circ (i \times i \times i)\,. \end{aligned}$$

As a similar way $1_A \times \eta = (r \times r) \circ (1_X \times \mu) \circ (i \times i \times i)\,.$

$$\begin{aligned} \eta \circ (\eta \times 1_A) &= r \circ \mu \circ (i \times i) \circ (r \times r) \circ (\mu \times 1_X) \circ (i \times i \times i) \\ &\simeq {}_{(\kappa^*,\kappa)} r \circ \mu \circ 1_{X \times X} \circ (\mu \times 1_X) \circ (i \times i \times i) \\ &= r \circ (\mu \circ (\mu \times 1_X)) \circ (i \times i \times i) \\ &\simeq {}_{(\kappa^*,\kappa)} r \circ (\mu \circ (1_X \times \mu)) \circ (i \times i \times i) \\ &= r \circ \mu \circ 1_{X \times X} \circ (1_X \times \mu) \circ (i \times i \times i) \\ &\simeq {}_{(\kappa^*,\kappa)} r \circ \mu \circ (i \times i) \circ (r \times r) \circ (1_X \times \mu) \circ (i \times i \times i) \\ &= \eta \circ (1_A \times \eta)\,. \end{aligned}$$

Therefore η is homotopy associative.

Let $\theta' = r \circ \theta \circ i$.

$$\begin{aligned} \eta \circ (\theta', 1_Y) &= (r \circ \mu) \circ (i \times i) \circ (\theta', 1_Y) \\ &= (r \circ \mu) \circ (i \times i) \circ (r \circ \theta \circ i, 1_Y) \\ &= (r \circ \mu) \circ (i \circ r \circ \theta \circ i, i \circ 1_Y) \\ &\simeq {}_{(\kappa,\kappa)} (r \circ \mu) \circ (1_X \circ \theta \circ i, i) \\ &= r \circ (\mu \circ (\theta, 1_X)) \circ i \\ &\simeq {}_{(\kappa,\kappa)} r \circ c \circ i \\ &\simeq {}_{(\kappa,\kappa)} c', \end{aligned}$$

where $c' = c\mid_A$. In a similar way $\eta \circ (1_A, \theta') \simeq_{(\kappa,\kappa)} c'$. So (A,q,κ) is a digital H-group with the digital continuous multiplication η, digital homotopy inverse θ' and constant map c'. □

Corollary 3.9. *A $\kappa-$deformation retract of an abelian digital H-group is itself an abelian digital H-group.*

Theorem 3.10. *[9] For any category C and object X of C, there is a functor Π_X from C to the category of sets and functions which associates to an object Y of C to the set $\Pi_X(Y) = \hom(X,Y)$ and to a morphism $f : Y \to Y'$ the function $\Pi_X(f) = f_* : \hom(X,Y) \to \hom(X,Y')$ defined by $f_*(g) = f \circ g$, for $g : X \to Y'$.*

Definition 3.11. The category whose objects are pointed digital images and the set of morphisms is $\hom((X,p,\kappa_1),(Y,q,\kappa_2)) = [(X,p,\kappa_1),(Y,q,\kappa_2)]$ is called the homotopy category of the pointed digital images.

Theorem 3.12. *Let (X,p,κ_1) be a digital H-space with digital homotopy associative multiplication μ and (Y,q,κ_2) be a pointed digital image. Then $[(Y,q,\kappa_2),(X,p,\kappa_1)]$ is a semigroup with identity.*

Proof. For any $[f], [g] \in [(Y, q, \kappa_2), (X, p, \kappa_1)]$, let define the product

$$[f] \otimes [g] = [\mu \circ (f, g)].$$

Let $[f] = [f']$ and $[g] = [g']$, then $f \simeq_{(\kappa_2,\kappa_1)} f'$ and $g \simeq_{(\kappa_2,\kappa_1)} g'$. Define a digital homotopy $F : Y \times [0, m]_{\mathbb{Z}} \to X$ such that $F = \mu \circ (G, H)$. Then

$$\begin{aligned} F(y, 0) &= \mu \circ (G, H)(y, 0) \\ &= \mu (G(y, 0), H(y, 0)) \\ &= \mu(f(y), g(y)) \\ &= (\mu \circ (f, g))(y) \end{aligned}$$

and similarly

$$F(y, m) = (\mu \circ (f', g'))(y).$$

So $(\mu \circ (f, g)) \simeq_{(\kappa_2,\kappa_1)} (\mu \circ (f', g'))$. Therefore $[\mu \circ (f, g)] = [\mu \circ (f', g')]$, for this reason

$$[f] \otimes [g] = [f'] \otimes [g'].$$

Consequently " $\otimes$ " is well defined.

Let $c : (X, p, \kappa_1) \to (X, p, \kappa_1)$ be the constant map $c(x) = p, \forall x \in X$. Then

$$(\mu \circ (c, f))(x) = \mu (p, f(x)) = (\mu \circ (c, 1_X) \circ f)(x)$$

and since $\mu \circ (c, 1_X) \simeq_{(\kappa_2,\kappa_2)} 1_X$, then $[c] \otimes [f] = [\mu \circ (c, f)] = [f]$. So $[c]$ is the identity element of $[(Y, q, \kappa_2), (X, p, \kappa_1)]$.

Let $[f], [g], [h] \in [(Y, q, \kappa_2), (X, p, \kappa_1)]$.

$$\begin{aligned} [f] \otimes ([g] \otimes [h]) &= [f] \otimes [\mu \circ (g, h)] \\ &= [\mu \circ (f, (\mu \circ (g, h)))] \\ &= [\mu \circ (1_Y \times \mu) \circ (f, g, h)] \\ &= [\mu \circ (\mu \times 1_Y) \circ (f, g, h)] \\ &= [\mu \circ ((\mu \circ (f, g)), h)] \\ &= ([f] \otimes [g]) \otimes [h]. \end{aligned}$$

Therefore " $\otimes$ " is digital homotopy associative. As a result $[(Y, q, \kappa_2), (X, p, \kappa_1)]$ is a semigroup under the product " $\otimes$ ". □

Theorem 3.13. *[4] Let (Y, q, κ') be a digital H-group with the digital multiplication $\mu : Y \times Y \to Y$. For all digital images (X, p, κ), $\left[(X, p, \kappa), (Y, q, \kappa')\right]$ is a group with " $\otimes$ ".*

Theorem 3.14. *Let (Y, q, κ_1) and (Z, r, κ_2) be digital H-spaces and $h : (Y, q, \kappa_1) \to (Z, r, \kappa_2)$ be a digital H-homomorphism. Then there exist a homomorphism from $[(X, p, \kappa), (Y, q, \kappa_1)]$ to $[(X, p, \kappa), (Z, r, \kappa_2)]$ for any pointed digital image (X, p, κ).*

Proof. Let $[g], [g'] \in [(X, p, \kappa), (Y, q, \kappa_1)]$, $[g] \otimes [g'] = [\mu \circ (g, g')]$.

Let $h_* : [(X, p, \kappa), (Y, q, \kappa_1)] \to [(X, p, \kappa), (Z, r, \kappa_2)]$ be a map such that $h_*([g]) = [h \circ g]$. Then

$$h_* ([g] \otimes [g']) = h_* ([\mu \circ (g, g')]) = [h \circ \mu \circ (g, g')].$$

Since $\eta \circ (h \times h) \simeq_{(\kappa^*, \kappa_2)} h \circ \mu$,

$$\begin{aligned} [h \circ \mu \circ (g, g')] &= [\eta \circ (h \times h) \circ (g, g')] \\ &= [h \circ g] \otimes [h \circ g'] \\ &= h_* ([g]) \otimes h_* ([g']) . \end{aligned}$$

Therefore h_* is a homomorphism. □

Corollary 3.15. *Let (A, q, κ) be a $\kappa-$deformation retract of a digital H-space (X, p, κ). It is clear that, since $\kappa-$retraction r is a digital H-homomorphism, there exist an homomorphism from $[(Y, q, \kappa_1), (X, p, \kappa)]$ to $[(Y, q, \kappa_1), (A, q, \kappa)]$ for any pointed digital image (Y, q, κ_1). Also since inclusion map i is a digital H-homomorphism, there exist an homomorphism from $[(Y, q, \kappa_1), (A, q, \kappa)]$ to $[(Y, q, \kappa_1), (X, p, \kappa)]$.*

Theorem 3.16. *Let (X, p, κ) be a digital H-group. Then Π_X is a functor from the homotopy category of digital H-groups and digital H-homomorphisms $(\mathcal{D})$ to the category of groups and homomorphisms $(\mathcal{G})$.*

Proof. Let (Y, q, κ_1), (Z, r, κ_2) be objects of $\mathcal{D}$ and $[h] \in [(Y, q, \kappa_1), (Z, r, \kappa_2)]$ be a morphism of $\mathcal{D}$. $\Pi_X((Y, q, \kappa_1)) = \hom((X, p, \kappa), (Y, q, \kappa_1)) = [(X, p, \kappa), (Y, q, \kappa_1)]$. By theorem 10, $[(X, p, \kappa), (Y, q, \kappa_1)]$ is a group and by theorem 9 $\Pi_X([h]) = h_*$ is an homomorphism. Let $[1_Y] \in [(Y, q, \kappa_1), (Y, q, \kappa_1)]$ be the unit morphism. Then

$$\Pi_X([1_Y]) = 1_{Y*} : [(X, p, \kappa), (Y, q, \kappa_1)] \to [(X, p, \kappa), (Y, q, \kappa_1)]$$

such that $1_{Y*}([h]) = [1_Y \circ h] = [h]$. Therefore $\Pi_X([1_Y])$ is the unit. Let $[g] \in [(Z, r, \kappa_2), (W, t, \kappa_3)]$. Then,

$$\Pi_X([f])([h]) = [f \circ h]$$
$$\Pi_X([g])([f \circ h]) = [(g \circ f \circ h)] = \Pi_X([g \circ f])([h]).$$

So,

$\Pi_X([g \circ f])([h]) = \Pi_X([g])([f \circ h]) = \Pi_X([g])(\Pi_X([f])([h])) = (\Pi_X([g]) \circ \Pi_X([f]))([h]) . \Rightarrow \Pi_X([g \circ f]) = \Pi$

Consequently Π_X is a functor. □

Corollary 3.17. *Let (X, p, κ) be an abelian digital H-group, then Π_X is a functor from the digital homotopy category of the pointed digital images to the category of abelian groups and homomorphisms.*

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