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SOME SPECIAL CURVES IN DUAL 3−SPACE

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Abstract. Rectifying, normal and osculating curves have been worked on many times in different spaces. The aim of this paper is to characterize these curves from another point of view in three dimensional Dual space.

1. Introduction

Curves theory has studied for a long time. It is well-known that to each unit speed curve $x : I \subset \mathbb{R} \to \mathbb{E}^3$ with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $t(s)$, $n(s)$ and $b(s)$, called respectively the tangent, the principal normal and the binormal vector fields. The planes which is spanned by $\{t(s), b(s)\}$, $\{n(s), b(s)\}$ and $\{t(s), n(s)\}$ are known as the rectifying, normal and osculating plane, respectively. The curves $x : I \subset \mathbb{R} \to \mathbb{E}^3$ for which the position vector $x(s)$ always lie in their rectifying plane, are for simplicity called rectifying curves. Similarly, the curves for which the position vector $x(s)$ always lie in their osculating plane, are for simplicity called osculating curves; and finally, the curves for which the position vector always lie in their normal plane, are for simplicity called normal curves [7]. If all normal or osculating planes of a curve in $\mathbb{E}^3$ pass through a particular point, then the curve is spherical or planar, respectively. It is also known that if all rectifying planes of a non-planar curve in $\mathbb{E}^3$ pass through a particular point, then the ratio of its torsion and curvature is a non-constant linear function [2]. Also unit speed curve with nonzero curvatures lies on a sphere if and only if $\dfrac{\tau}{\kappa} = \left(\dfrac{\kappa^{'}}{\tau\kappa^2}\right)^{'}$ [7].

Notions as dual numbers, dual vectors, dual angles, dual orthogonal matrices etc. in general dual elements were defined by W. K. Clifford in 1873 [11]. After him E. Study used dual numbers and dual vectors in his research on geometry of lines and kinematics. Also he used these numbers to explain a mapping from dual unit sphere to three dimensional Euclidean space $\mathbb{E}^3$. This mapping is called Study mapping and this mapping corresponds the dual points of a dual unit sphere to the oriented lines in $\mathbb{E}^3$ [3]. So the set of oriented lines in Euclidean space $\mathbb{E}^3$ is one to one correspondence with the points of dual space in $\mathbb{D}^3$. Recently, dual space curves

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and surfaces have been extensively studied and they are powerful mathematical tools for spherical motion in $\mathbb{D}^3$ [1, 4, 9, 10].

In this study, we study rectifying, normal and osculating curves in Dual 3−space. These curves have already been worked on in different spaces, but the significance of this study is to obtain calculations using different method.

2. Preliminaries

By a dual number $\widehat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathbb{R}$. Let the set $\mathbb{R} \times \mathbb{R}$ be denoted as $\mathbb{D}$. Two inner operations and an equality on $\mathbb{D} = \{(x, x^*) \mid x, x^* \in \mathbb{R}\}$ are defined as follows;

(i) $\oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\widehat{x} = (x, x^*)\,,\ \widehat{y} = (y, y^*)$ defined as

$$\widehat{x} \oplus \widehat{y} = (x, x^*) \oplus (y, y^*) = (x + y, x^* + y^*)$$

is called the addition in $\mathbb{D}$.

(ii) $\odot : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\widehat{x} = (x, x^*)\,,\ \widehat{y} = (y, y^*)$ defined as

$$\widehat{x} \odot \widehat{y} = \widehat{x}\widehat{y} = (x, x^*) \odot (y, y^*) = (xy, xy^* + x^*y)$$

is called the multiplication in $\mathbb{D}$.

(iii) If $x = y$, $x^* = y^*$ for $\widehat{x} = (x, x^*)\,,\ \widehat{y} = (y, y^*) \in \mathbb{D}$, $\widehat{x}$ and $\widehat{y}$ are equal, and it is indicated as $\widehat{x} = \widehat{y}$.

If the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ with set of real numbers $\mathbb{R}$ are defined as above, the set $\mathbb{D}$ is called the dual numbers system and the element (x, x^*) of $\mathbb{D}$ is called a dual number. In a dual number $\widehat{x} = (x, x^*) \in \mathbb{D}$, the real number x is called the real part of $\widehat{x}$ and the real number x^* is called the dual part of $\widehat{x}$. The dual number $(1, 0) = 1$ is called unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$. The dual number $(0, 1)$ is to be denoted with ε in short, and the $(0, 1) = \varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2 = 0$. Also, the dual number $\widehat{x} = (x, x^*) \in \mathbb{D}$ can be written as $\widehat{x} = x + \varepsilon x^*$ (see [5, 8]).

The set of $\mathbb{D} = \{\widehat{x} = x + \varepsilon x^* \mid x, x^* \in \mathbb{R}\}$ of dual numbers is a commutative ring according to the operations

i. $(x + \varepsilon x^*) + (y + \varepsilon y^*) = (x + y) + \varepsilon (x^* + y^*)\,,$

ii. $(x + \varepsilon x^*)(y + \varepsilon y^*) = xy + \varepsilon (xy^* + y^*x)\,.$

The dual number $\widehat{x} = x + \varepsilon x^*$ divided by the dual number $\widehat{y} = y + \varepsilon y^*$ provided $y \neq 0$ can be defined as

$$\frac{\widehat{x}}{\widehat{y}} = \frac{x + \varepsilon x^*}{y + \varepsilon y^*} = \frac{x}{y} + \varepsilon \frac{x^*y - xy^*}{y^2}$$

The set of

$$\begin{aligned}
\mathbb{D}^3 &= \mathbb{D} \times \mathbb{D} \times \mathbb{D} \\
&= \left\{ \overrightarrow{\widehat{x}} \,\middle|\, \overrightarrow{\widehat{x}} = (x_1 + \varepsilon x_1^*, x_2 + \varepsilon x_2^*, x_3 + \varepsilon x_3^*) \right\} \\
&= \left\{ \overrightarrow{\widehat{x}} \,\middle|\, \overrightarrow{\widehat{x}} = (x_1, x_2, x_3) + \varepsilon (x_1^*, x_2^*, x_3^*) \right\} \\
&= \left\{ \overrightarrow{\widehat{x}} \,\middle|\, \overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*}, \overrightarrow{x}, \overrightarrow{x^*} \in \mathbb{R}^3 \right\}
\end{aligned}$$

is a module on the ring $\mathbb{D}$. For any $\overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon\overrightarrow{x^*}$, $\overrightarrow{\widehat{y}} = \overrightarrow{y} + \varepsilon\overrightarrow{y^*} \in \mathbb{D}^3$, the scalar or inner product and the vector product of $\overrightarrow{\widehat{x}}$ and $\overrightarrow{\widehat{y}}$ are defined by, respectively,

$$\begin{aligned} \left\langle \overrightarrow{\widehat{x}}, \overrightarrow{\widehat{y}} \right\rangle &= \langle \overrightarrow{x}, \overrightarrow{y} \rangle + \varepsilon \left(\left\langle \overrightarrow{x}, \overrightarrow{y^*} \right\rangle + \left\langle \overrightarrow{x^*}, \overrightarrow{y} \right\rangle \right), \\ \overrightarrow{\widehat{x}} \wedge \overrightarrow{\widehat{y}} &= \left(\widehat{x}_2\widehat{y}_3 - \widehat{x}_3\widehat{y}_2, \widehat{x}_3\widehat{y}_1 - \widehat{x}_1\widehat{y}_3, \widehat{x}_1\widehat{y}_2 - \widehat{x}_2\widehat{y}_1 \right), \end{aligned}$$

where $\widehat{x}_i = x_i + \varepsilon x_i^*, \widehat{y}_i = y_i + \varepsilon y_i^* \in \mathbb{D}$, $1 \leq i \leq 3$. If $x \neq 0$, the norm $\left\| \overrightarrow{\widehat{x}} \right\|$ of $\overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon\overrightarrow{x^*}$ is defined by

$$\left\| \overrightarrow{\widehat{x}} \right\| = \sqrt{\left\langle \overrightarrow{\widehat{x}}, \overrightarrow{\widehat{x}} \right\rangle} = \| \overrightarrow{x} \| + \varepsilon \frac{\left\langle \overrightarrow{x}, \overrightarrow{x^*} \right\rangle}{\| \overrightarrow{x} \|}.$$

A dual vector $\overrightarrow{\widehat{x}}$ with norm 1 is called a dual unit vector. Let $\overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon\overrightarrow{x^*} \in \mathbb{D}^3$. The set

$$\mathbb{S}^2 = \left\{ \overrightarrow{\widehat{x}} = \overrightarrow{x} + \varepsilon\overrightarrow{x^*} \,\middle|\, \left\| \overrightarrow{\widehat{x}} \right\| = (1,0)\, ; \overrightarrow{x}, \overrightarrow{x^*} \in \mathbb{R}^3 \right\}$$

is called the dual unit sphere with the center $\widehat{O}$ in $\mathbb{D}^3$.

If every $x_i\,(t)$ and $x_i^*\,(t)\,, 1 \leq i \leq 3$ real valued functions, are differentiable, the dual space curve

$$\begin{aligned} \widehat{x} \quad &: \quad I \subset \mathbb{R} \to \mathbb{D}^3 \\ t \quad &\to \quad \overrightarrow{\widehat{x}}\,(t) = \left(x_1\,(t) + \varepsilon x_1^*\,(t)\,, x_2\,(t) + \varepsilon x_2^*\,(t)\,, x_3\,(t) + \varepsilon x_3^*\,(t) \right) \\ &= \quad \overrightarrow{x}\,(t) + \varepsilon\overrightarrow{x^*}\,(t) \end{aligned}$$

in $\mathbb{D}^3$ is differentiable. We call the real part $\overrightarrow{x}\,(t)$ the indicatrix of $\overrightarrow{\widehat{x}}\,(t)$. The dual arc lenght of the curve $\overrightarrow{\widehat{x}}\,(t)$ from t_1 to t is defined as

$$\widehat{s} = \int_{t_1}^{t} \left\| \overrightarrow{\widehat{x}}\,(t)' \right\| dt = \int_{t_1}^{t} \left\| \overrightarrow{x}\,(t)' \right\| dt + \varepsilon \int_{t_1}^{t} \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*} \right)' \right\rangle = s + \varepsilon s^*, \tag{2.1}$$

where $\widehat{t}$ is a unit tangent vector of $\overrightarrow{x}\,(t)$. From now on we will take the arc length s of $\overrightarrow{x}\,(t)$ as the parameter instead of t.

Now we will obtain equations relatively to the derivatives of dual Frenet vectors throughout the curve in $\mathbb{D}^3$. Let

$$\begin{aligned} \widehat{x} \quad &: \quad I \to \mathbb{D}^3 \\ s \quad &\to \quad \overrightarrow{\widehat{x}}\,(s) = \overrightarrow{x}\,(s) + \varepsilon\overrightarrow{x^*}\,(s) \end{aligned}$$

be a dual curve with the arc length parameter s of the indicatrix. Then,

$$\frac{d\overrightarrow{\widehat{x}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{x}}}{ds}\frac{ds}{d\widehat{s}} = \overrightarrow{\widehat{t}}$$

is called the dual unit tangent vector of $\overrightarrow{\widehat{x}}\,(s)$. With the aid of equation (2.1), we have

$$\widehat{s} = s + \varepsilon \int_{s_1}^{s} \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*} \right)' \right\rangle ds$$

and from this $\dfrac{d\widehat{s}}{ds} = 1 + \varepsilon\Delta$, where the prime denotes differentiation with respect to the arc length s of indicatrix and $\Delta = \left\langle \overrightarrow{t}, \left(\overrightarrow{x^*}\right)' \right\rangle$. Since $\overrightarrow{\widehat{t}}$ has constant length 1, its differentiation with respect to $\widehat{s}$, which is given by

$$\frac{d\overrightarrow{\widehat{t}}}{d\widehat{s}} = \frac{d\overrightarrow{\widehat{t}}}{ds}\frac{ds}{d\widehat{s}} = \frac{d^2\overrightarrow{\widehat{x}}}{d\widehat{s}^2} = \widehat{\kappa}\overrightarrow{\widehat{n}},$$

measures the way the curve is turning in $\mathbb{D}^3$. The norm of the vector $\dfrac{d\overrightarrow{\widehat{t}}}{d\widehat{s}}$ is called curvature function of $\overrightarrow{\widehat{x}}(s)$. We impose the restriction that the function $\widehat{\kappa} : I \to \mathbb{D}$ is never pure dual. Then, the dual unit vector $\overrightarrow{\widehat{n}} = \dfrac{1}{\widehat{\kappa}}\dfrac{d\overrightarrow{\widehat{t}}}{d\widehat{s}}$ is called the principal normal of $\overrightarrow{\widehat{x}}(s)$. The dual vector $\overrightarrow{\widehat{b}}$ is called the binormal of $\overrightarrow{\widehat{x}}(s)$. The dual vectors $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ are called the dual Frenet trihedron of $\overrightarrow{\widehat{x}}(s)$ at the point $\widehat{x}(s)$. The equalities relative to derivatives of dual Frenet vectors $\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}$ throughout the dual space curve are written in the matrix form

$$\frac{ds}{d\widehat{s}}\begin{bmatrix} \overrightarrow{\widehat{t}} \\ \overrightarrow{\widehat{n}} \\ \overrightarrow{\widehat{b}} \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa} & 0 \\ -\widehat{\kappa} & 0 & \widehat{\tau} \\ 0 & -\widehat{\tau} & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{\widehat{t}} \\ \overrightarrow{\widehat{n}} \\ \overrightarrow{\widehat{b}} \end{bmatrix}, \tag{2.2}$$

where $\widehat{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\widehat{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion. The formulae (2.2) are called the Frenet formulae of dual curve in $\mathbb{D}^3$ (see [8]). If a moving point moves along a dual unit speed curve $\overrightarrow{\widehat{x}}(s)$, then the moving dual frame $\left\{\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}\right\}$ moves as above. The planes spanned by $\left\{\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{b}}\right\}$, $\left\{\overrightarrow{\widehat{t}}, \overrightarrow{\widehat{n}}\right\}$ and $\left\{\overrightarrow{\widehat{n}}, \overrightarrow{\widehat{b}}\right\}$ at each point of the dual space curve are called the dual rectifying plane, the dual osculating plane, and the dual normal plane, respectively [1].

3. Some Special Curves in $\mathbb{D}^3$

In this section we give some new corollaries related to special curves in three dimensional dual space. Assume that $\widehat{x}(s)$ is a unit speed curve with nonzero curvature in dual space $\mathbb{D}^3$. Since the rectifying plane of $\widehat{x}(s)$ is the perpendicular plane to $\widehat{n}(s)$, we have

$$\langle \widehat{x}(s) - \widehat{x_0}, \widehat{n}(s)\rangle = 0.$$

If we take the derivative of this expression,

$$\left\langle \widehat{t}(s), \widehat{n}(s)\right\rangle + \langle \widehat{x}(s) - \widehat{x_0}, \widehat{n}'(s)\rangle = 0$$

then by substituting from the Frenet-Serret formula from the equation (2.2),

$$\begin{aligned} \left\langle \widehat{x}(s) - \widehat{x_0}, -\widehat{\kappa}(s)\widehat{t}(s) + \widehat{\tau}(s)\,\widehat{b}(s)\right\rangle &= 0, \\ -\widehat{\kappa}(s)\left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{t}(s)\right\rangle + \widehat{\tau}(s)\left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}(s)\right\rangle &= 0. \end{aligned} \tag{3.1}$$

So we can easily see that,

$$
\begin{aligned}
\widehat{\kappa}(s)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle &= \widehat{\tau}\,(s)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{b}(s)\right\rangle \\
\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{b}(s)\right\rangle &= \left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle . \qquad (3.2)
\end{aligned}
$$

If we take the derivative of equation (3.1), we obtain following equalities

$$
\begin{aligned}
&\left(\left\langle \widehat{t}(s),-\widehat{\kappa}(s)\widehat{t}(s)\right\rangle+\left\langle \widehat{x}(s)-\widehat{x_0},-\widehat{\kappa}'(s)\widehat{t}(s)-\widehat{\kappa}^2\,(s)\,\widehat{n}\,(s)\right\rangle\right) \\
&+\left(\left\langle \widehat{t}(s),\widehat{\tau}(s)\widehat{b}(s)\right\rangle+\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{\tau}'(s)\widehat{b}(s)-\widehat{\tau}^2\,(s)\,\widehat{n}\,(s)\right\rangle\right)=0 \\
&-\widehat{\kappa}(s)-\widehat{\kappa}'(s)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle+\widehat{\tau}'(s)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{b}(s)\right\rangle=0
\end{aligned}
$$

If equation (3.2) is written in last equation, we get

$$
\begin{aligned}
\widehat{\kappa}(s) &= -\widehat{\kappa}'(s)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle+\widehat{\tau}'(s)\left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right)\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle \\
\widehat{\kappa}(s) &= \left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle\left(-\widehat{\kappa}'(s)+\widehat{\tau}'(s)\left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right)\right)
\end{aligned}
$$

$$
\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle=\frac{\widehat{\kappa}(s)}{-\widehat{\kappa}'(s)+\widehat{\tau}'(s)\left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right)} \tag{3.3}
$$

$$
\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{b}(s)\right\rangle=\left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right)\frac{\widehat{\kappa}(s)}{-\widehat{\kappa}'(s)+\widehat{\tau}'(s)\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}} \tag{3.4}
$$

Then working with the denominator of equation (3.3),

$$
\begin{aligned}
-\widehat{\kappa}'(s)+\widehat{\tau}'(s)\left(\frac{\widehat{\kappa}(s)}{\widehat{\tau}(s)}\right) &= \frac{\widehat{\tau}'(s)\widehat{\kappa}(s)-\widehat{\kappa}'(s)\widehat{\tau}(s)}{\widehat{\tau}(s)} \\
&= \frac{\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'\widehat{\kappa}^2(s)}{\widehat{\tau}(s)}.
\end{aligned}
$$

If we using this equation in equation (3.3), we can see that

$$
\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle=\frac{\widehat{\kappa}(s)}{\frac{\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'\widehat{\kappa}^2(s)}{\widehat{\tau}(s)}}
$$

$$
\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle=\frac{\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)}{\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'} \tag{3.5}
$$

So, we obtain that

$$\left\langle \widehat{x}(s)-\widehat{x_0},\widehat{b}(s)\right\rangle=\frac{1}{\left(\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'} \tag{3.6}$$

From the equations (3.5) and (3.6), the equation of the curve $\widehat{x}(s)$ is as follows

$$\widehat{x}(s)-\widehat{x_0}=\frac{\left(\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)}{\left(\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'}\widehat{t}(s)+\frac{1}{\left(\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)'}\widehat{b}(s) \tag{3.7}$$

If we say $\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}=\widehat{H}(s)$ which is the harmonic curvature function, we have

$$\widehat{x}(s)-\widehat{x_0}=\frac{\widehat{H}(s)}{\widehat{H}'(s)}\widehat{t}(s)+\frac{1}{\widehat{H}'(s)}\widehat{b}(s)$$

If we take the derivative of this equation,

$$\widehat{t}(s)=\left(\frac{\widehat{H}(s)}{\widehat{H}'(s)}\right)'\widehat{t}(s)+\left(\frac{\widehat{H}(s)}{\widehat{H}'(s)}\right)\widehat{\kappa}(s)\widehat{n}(s)+\left(\frac{1}{\widehat{H}'(s)}\right)'\widehat{b}(s)-\left(\frac{1}{\widehat{H}'(s)}\right)\widehat{\tau}(s)\widehat{n}(s)$$

$$\left(\left(\frac{\widehat{H}(s)}{\widehat{H}'(s)}\right)'-1\right)\widehat{t}(s)+\left(\frac{\widehat{H}(s)\widehat{\kappa}(s)-\widehat{\tau}(s)}{\widehat{H}'(s)}\right)\widehat{n}(s)+\left(\frac{1}{\widehat{H}'(s)}\right)'\widehat{b}(s)=0$$

and since $\widehat{t}(s),\widehat{n}(s)$ and $\widehat{b}(s)$ are linearly independent, we obtain that

$$\left(\frac{\widehat{H}(s)}{\widehat{H}'(s)}\right)'-1=0 \tag{3.8}$$

$$\frac{\widehat{H}(s)\widehat{\kappa}(s)-\widehat{\tau}(s)}{\widehat{H}'(s)}=0 \tag{3.9}$$

and

$$\left(\frac{1}{\widehat{H}'(s)}\right)'=0 \tag{3.10}$$

Corollary 3.1. *Let $\widehat{x}(s):I\subset\mathbb{R}\longrightarrow\mathbb{D}^3$ be a unit speed curve nonzero curvature in Dual 3−space. If every rectifying plane contains the point $\widehat{x_0}$, i.e, if $\widehat{x}(s)$ is a rectifying curve, then $\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}$ is a linear function.*

Proof. Using the equations (3.8) or (3.10), we can easily see that $\widehat{H}''(s)=0$ and $\widehat{H}(s)=\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}=\widehat{c}\widehat{s}+\widehat{d}$ for some constants $\widehat{c}\neq 0,\widehat{d}$ and arc length $\widehat{s}$. □

With similar thought, assume that $\widehat{x}(s)$ is a unit speed curve with nonzero curvature in Dual 3−space. Since the normal plane of $\widehat{x}(s)$ is orthogonal to $\widehat{t}(s)$, we have

$$\left\langle\widehat{x}(s)-\widehat{x_0},\widehat{t}(s)\right\rangle=0.$$

If we take the derivative of this expression, we get

$$\left\langle \widehat{t}(s), \widehat{t}(s) \right\rangle + \left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{t}'(s) \right\rangle = 0$$

then by substituting from the Frenet-Serret formula, we have

$$1 + \langle \widehat{x}(s) - \widehat{x_0}, \widehat{\kappa}(s)\, \widehat{n}(s) \rangle = 0 \tag{3.11}$$

$$\langle \widehat{x}(s) - \widehat{x_0}, \widehat{n}(s) \rangle = \frac{-1}{\widehat{\kappa}(s)} \tag{3.12}$$

If we take the derivative of equation (3.11),

$$\langle \widehat{x}(s) - \widehat{x_0}, \widehat{\kappa}'(s)\, \widehat{n}(s) \rangle - \widehat{\kappa}^2(s) \left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{t}(s) \right\rangle + \widehat{\kappa}(s)\, \widehat{\tau}(s) \left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}(s) \right\rangle = 0$$

So if we write equation (3.12) in the last equation, we obtain

$$\frac{-\widehat{\kappa}'(s)}{\widehat{\kappa}(s)} + \widehat{\kappa}(s)\, \widehat{\tau}(s) \left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}(s) \right\rangle = 0$$

$$\left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}(s) \right\rangle = -\frac{\left(\dfrac{1}{\widehat{\kappa}(s)}\right)'}{\widehat{\tau}(s)} \tag{3.13}$$

Using the equations (3.12) and (3.13), we can say that

$$\widehat{x}(s) - \widehat{x_0} = \frac{-1}{\widehat{\kappa}(s)} \widehat{n}(s) - \frac{\left(\dfrac{1}{\widehat{\kappa}(s)}\right)'}{\widehat{\tau}(s)} \widehat{b}(s) \tag{3.14}$$

If we say $\dfrac{1}{\widehat{\kappa}(s)} = \widehat{m}(s)$, equation (3.14) takes the form the following equation

$$\widehat{x}(s) - \widehat{x_0} = -\widehat{m}(s)\widehat{n}(s) - \frac{\widehat{m}'(s)}{\widehat{\tau}(s)} \widehat{b}(s)$$

So, if we take the derivative of this equation

$$\begin{aligned} \widehat{t}(s) &= -\widehat{m}'(s)\widehat{n}(s) - \widehat{m}(s)(-\widehat{\kappa}(s)\, \widehat{t}(s) + \widehat{\tau}(s)\, \widehat{b}(s)) \\ &\quad - \left(\frac{\widehat{m}'(s)}{\widehat{\tau}(s)}\right)' \widehat{b}(s) + \frac{\widehat{m}'(s)}{\widehat{\tau}(s)} \widehat{\tau}(s)\, \widehat{n}(s) \end{aligned}$$

and

$$(-1 + \widehat{\kappa}(s)\, \widehat{m}(s))\widehat{t}(s) - \left(\widehat{\tau}(s)\, \widehat{m}(s) + \left(\frac{\widehat{m}'(s)}{\widehat{\tau}(s)}\right)'\right) \widehat{b}(s) = 0.$$

We know that $\widehat{t}(s)$ and $\widehat{b}(s)$ are linearly independent. Thus,

$$-1 + \widehat{\kappa}(s)\, \widehat{m}(s) = 0 \tag{3.15}$$

$$\widehat{\tau}(s)\, \widehat{m}(s) + \left(\frac{\widehat{m}'(s)}{\widehat{\tau}(s)}\right)' = 0 \tag{3.16}$$

Corollary 3.2. *Let $\widehat{x}(s) : I \subset \mathbb{R} \longrightarrow \mathbb{D}^3$ be a unit speed curve nonzero curvature in Dual 3−space. If every normal plane contains the point $\widehat{x_0}$ in $\mathbb{D}^3$, i.e, if $\widehat{x}(s)$ is a normal curve, then the curve is a spherical curve in $\mathbb{D}^3$, i.e,* $\dfrac{\widehat{\tau}(s)}{\widehat{\kappa}(s)} = \left(\dfrac{\widehat{\kappa}'(s)}{\widehat{\kappa}^2(s)\, \widehat{\tau}(s)}\right)'$.

Proof. From the equation (3.16), we can say that

$$\begin{aligned} \widehat{\tau}\,(s)\,\widehat{m}(s) &= -\left(\frac{\widehat{m}'(s)}{\widehat{\tau}\,(s)}\right)' \\ \frac{\widehat{\tau}\,(s)}{\widehat{\kappa}\,(s)} &= -\left(\frac{-\widehat{\kappa}'\,(s)}{\widehat{\kappa}^2\,(s)\,\widehat{\tau}\,(s)}\right)' \end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $\widehat{x}(s) : I \subset \mathbb{R} \longrightarrow \mathbb{D}^3$ be a unit speed curve nonzero curvature in Dual 3−space. If every osculating plane contains the point $\widehat{x_0}$ in $\mathbb{D}^3$, i.e, if $\widehat{x}(s)$ is a osculating curve, then the curve is a planar curve in three dimensional dual space.*

Proof. Since the osculating plane of $\widehat{x}(s)$ is the perpendicular plane to $\widehat{b}(s)$, we have $\left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}(s) \right\rangle = 0$. If we take the derivative of this expression,

$$\left\langle \widehat{t}(s), \widehat{b}(s) \right\rangle + \left\langle \widehat{x}(s) - \widehat{x_0}, \widehat{b}'(s) \right\rangle = 0$$

then by substituting from the Frenet-Serret formula we have

$$\begin{aligned} \langle \widehat{x}(s) - \widehat{x_0}, -\widehat{\tau}\,(s)\,\widehat{n}\,(s) \rangle &= 0 \\ -\widehat{\tau}\,(s)\,\langle \widehat{x}(s) - \widehat{x_0}, \widehat{n}\,(s) \rangle &= 0 \end{aligned}$$

$\widehat{\tau} = 0$ is obtained from the last equation. □

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