Hacettepe Journal of Mathematics and Statistics
Volume 44 (1) (2015), 75 – 81

Gelfand numbers of diagonal matrices

Z.I. Ismailov* , L. Cona† and E. Otkun Çevik‡

Abstract

In this work , a connection between Gelfand numbers of infinite diagonal matrix with linear bounded operator-elements in the direct sum of Banach spaces and its coordinate operators has been investigated.

2000 AMS Classification: 47A05

Keywords: Direct Sum of Banach Spaces and Operators; Gelfand Numbers of Operator.

Received 07/10/2013 : Accepted 13/01/2014 Doi : 10.15672/HJMS.2015449111

1. Introduction

The general theory of so-called singular numbers(singuläre zahlen) for linear compact operators has been explained in the famous book of I.Z.Gohberg and M.G.Krein [1]. But the first results in this area can be found in the papers of E.Schmidt [2] and J.von Neumann, R. Schatten [3] who used this concept in the theory of non-selfadjoint integral equations.

In recent times much attention has been separated to the study of linear bounded operators in Hilbert space and Banach space by means of geometric quantities such as approximation numbers, Gelfand numbers, Weyl numbers and etc. In the last years of 20th century research activity in this area grew considerably. Many of classical problems were solved, interesting new developments started.Deep connections between Banach space geometry and other areas of mathematics were discovered.

The axiomatic theory of Gelfand numbers has been given by A.Pietsch in [4,5]. In generally, in studies concerning to Gelfand numbers have been estimated or found for the special mapping on some functional Banach spaces or Banach spaces of sequences. For

*Karadeniz Technical University, Faculty of Sciences, Department of Mathematics, 61080 Trabzon, Turkey
Email: zameddin.ismailov@gmail.com

†Gumushane University, Department of Mathematical Engineering, 29100 Gumushane, Turkey
Email: lale_cona@hotmail.com

‡Karadeniz Technical University, Faculty of Sciences, Department of Mathematics, 61080 Trabzon, Turkey
Email:e_otkuncevik@hotmail.com

instance, A.Pietsch in [4] proved that for identity operator $id : l_p^n \longrightarrow l_q^n$, $n \in \mathbb{N}$, $1 \leq q \leq p \leq \infty$ the formula is valid that

$$c_k(id : l_p^n \longrightarrow l_q^n) = (n-k+1)^{\frac{1}{q}-\frac{1}{p}}, \ \ , k \geq 1$$

In classical papers mainly identity maps were considered in form

$$id : l_p^n \longrightarrow l_q^n, \ n \in \mathbb{N}, \ 1 \leq q \leq p \leq \infty$$

For example, for the infinite diagonal matrix

$$S(x_n) = (\sigma_n x_n), (x_n) \in l_u, \ 1 \leq u \leq \infty, \quad \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq ... \geq 0$$

has been established such that $c_n(S : l_u \longrightarrow l_u) = \sigma_n, n \geq 1$ and

$$c_n(S : l_u \longrightarrow l_v) = (\sum_{k=n}^{\infty} \sigma_k^r)^{\frac{1}{r}}, 1 \leq v \leq u \leq \infty, \ \frac{1}{r} = \frac{1}{v} - \frac{1}{u} \qquad [4]$$

In special case B.S. Kashin [6] and B.S. Mitiagin [7] proved that very striking result

$$c_n(id : l_1^m \longrightarrow l_2^m) \leq \rho \frac{[\log(m+1)]^{\frac{3}{2}}}{n^{\frac{1}{2}}}, \ n = 1, 2, ..., m$$

In this work these studies will be continued.

It is known that infinite direct sum of Banach spaces $\mathfrak{X}_m$, $m \geq 1$ in the sense of l_p , $1 \leq p < \infty$ and infinite direct sum of linear densely defined closed operators A_m in $\mathfrak{X}_m$, $m \geq 1$ are defined as

$$\mathfrak{X} = (\oplus_{m=1}^{\infty} \mathfrak{X}_m)_p = \left\{ x = (x_m) : \ x_m \in \mathfrak{X}_m \ , \ m \geq 1, \ \ \|x\|_p = (\sum_{m=1}^{\infty} \|x_m\|_{\mathfrak{X}_m}^p)^{\frac{1}{p}} < \infty \right\}$$

and

$$A = \oplus_{m=1}^{\infty} A_m \ , \ A : D(A) \subset \mathfrak{X} \longrightarrow \mathfrak{X} \ \ ,$$

$$D(A) = \{x = (x_m) \in \mathfrak{X} : \ x_m \in D(A_m), m \geq 1, \ Ax = (A_m x_m) \in \mathfrak{X}\} \ [8]$$

In second section the general facts concerning to the boundedness and compactness properties of direct sum operators in the direct sum of Banach spaces will be given.

In third section, some estimate formulas for the Gelfand numbers of the diagonal matrix with operator elements in form

$$S = \begin{pmatrix} S_1 & & & & & \\ & S_2 & & & 0 & \\ & & S_3 & & & \\ & & & \ddots & & \\ & 0 & & & S_m & \\ & & & & & \ddots \end{pmatrix} \ , \quad S : \mathfrak{X} \longrightarrow \mathfrak{X},$$

where $S_m \in L(\mathfrak{X}_m)$, $m \geq 1$ and $S \in L(\mathfrak{X})$ will be investigated.

Note that many physics problems of today in the modelling of processes of multiparticle quantum mechanics, quantum field theory and in the physics of rigid bodies support to study a theory of direct sum of linear operators in the direct sum of Banach spaces [9].

In this paper, the norms $\|\cdot\|_p$ in $\mathfrak{X}$ and $\|\cdot\|_{\mathfrak{X}_m}$ in $\mathfrak{X}_m$, $m \geq 1$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_m$, $m \geq 1$ respectively. In any Banach space $\mathfrak{B}$ the class of linear bounded and compact operators will be denoted by $L(\mathfrak{B})$ and $C_\infty(\mathfrak{B})$ respectively.

2. Direct sum of bounded and compact operators

In this section continuity and compactness properties of the operator $A = \oplus_{m=1}^{\infty} A_m$ in $\mathfrak{X}$ will be investigated when $A_m \in L(\mathfrak{X}_m)$ and $A_m \in C_\infty(\mathfrak{X}_m)$, $m \geq 1$ respectively.

Using the techniques of the Banach spaces l_p , $1 \leq p < \infty$ and Operator Theory the following two propositions can be proved in general.

2.1. Theorem. Let $A_m \in L(\mathfrak{X}_m)$, $m \geq 1$, $A = \oplus_{m=1}^{\infty} A_m$ in $\mathfrak{X}$. In order to $A \in L(\mathfrak{X})$ the necessary and sufficient condition is $\sup\limits_{m \geq 1} \|A_m\| < \infty$. Morever, in the case when $A \in L(\mathfrak{X})$ it is true that $\|A\| = \sup\limits_{m \geq 1} \|A_m\|$.

Note that from the definition of compactness of operators [10] it is implied that if $A \in C_\infty(\mathfrak{X}_m)$, then for each $m \geq 1$, $A_m \in C_\infty(\mathfrak{X}_m)$.

In general, the following result is true.

2.2. Theorem. Let $A_m \in C_\infty(\mathfrak{X}_m)$ for each $m \geq 1$, $A = \oplus_{m=1}^{\infty} A_m : \mathfrak{X} \longrightarrow \mathfrak{X}$. In this case $A \in C_\infty(\mathfrak{X})$ if and only if $\lim\limits_{m \longrightarrow \infty} \|A_m\| = 0$.

Proof. Assume that $\limsup\limits_{(m)} \|A_m\| > 0$. Then there exists a number $c > 0$ and a sequence $(k_m) \subset \mathbb{N}$ such that

$$\|A_{k_m}\| = \sup \left\{ \frac{\|A_{k_m} x_{k_m}\|_{k_m}}{\|x_{k_m}\|_{k_m}} : \ x_{k_m} \in \mathfrak{X}_{k_m} \diagdown \{0\} \, , m \geq 1 \right\} \geq c > 0$$

In this case there exist a sequence $(x^*_{k_m}) \in \mathfrak{X}_{k_m}$, such that $\frac{\|A_{k_m} x^*_{k_m}\|_{k_m}}{\|x^*_{k_m}\|_{k_m}} \geq c$, $m \geq 1$.

Now consider the following set in $\mathfrak{X}$ in form

$$M := \left\{ \left\{ 0, 0, ..., 0, \frac{x^*_{k_m}}{\left\|A_{k_m} x^*_{k_m}\right\|_{k_m}}, 0, ... \right\} \in \mathfrak{X} : \ m \geq 1 \right\}$$

It is clear that for $x \in M$, $\|x\| \leq \frac{1}{c} < \infty$, that is, M is a bounded set in $\mathfrak{X}$. On the other hand $AM = \left\{ \left\{ 0, 0, ..., 0, \frac{A_{k_m} x^*_{k_m}}{\left\|A_{k_m} x^*_{k_m}\right\|_{k_m}}, 0, ... \right\} \in \mathfrak{X} : \ m \geq 1 \right\}$.

From this it is easy to see that a set $\overline{AM} \subset \mathfrak{X}$ is not compact. Consequently, $\limsup\limits_{(m)} \|A_m\| = 0$, that is, it is obtained that $\lim\limits_{(m)} \|A_m\| = 0$.

On the contrary, define the following operators $K_n : \mathfrak{X} \longrightarrow \mathfrak{X}$, $n \geq 1$ in form

$$K_n := A_1 \oplus A_2 \oplus ... \oplus A_n \oplus 0 \oplus 0 \oplus ...$$

In this case for $x \in \mathfrak{X}$ we have

$$\begin{aligned} \|(A - K_n)x\|^p &\leq \sum_{m=n+1}^{\infty} \|A_m\|^p \, \|x_m\|_m^p \\ &\leq \sup_{m \geq n+1} \|A_m\|^p \sum_{m=n+1}^{\infty} \|x_m\|_m^p \\ &\leq \left(\sup_{m \geq n+1} \|A_m\|^p \right) \|x\|^p \end{aligned}$$

From this it is obtained that $\|A - K_n\| \leq \sup\limits_{m \geq n+1} \|A_m\|$, $n \geq 1$. Since $\limsup\limits_{(n)} \|A_n\| = 0$, then from last relation it is implied that sequence of operators (K_n) in $L(\mathfrak{X})$ is convergent to the operators A in operator norm. On the other hand $K_n \in C_\infty(\mathfrak{X})$, $n \geq 1$, then by the important theorem of the compact operators theory the operator A belong to the class $C_\infty(\mathfrak{X})$ [10]. $\square$

3. Gelfand numbers of direct sum operators

In this section, the relationship between the Gelfand numbers of the direct sum of operators in the direct sum of Banach spaces and its coordinate operators will be investigated.

Note that firstly the concept of s-number functions (particularly Gelfand number functions) for the operators in Banach spaces was introduced by A.Pietsch in [11].

Now give definitions of these number functions from works [4] and [12].

3.1. Definition. Let $L(E,F)$ be a Banach spaces of linear bounded operators from Banach space E to a Banach space F with operator norm. For the operator $T \in L(E,F)$ the following number

$$c_n(T) := \inf\left\{\|T\|_Z : Z \subset E,\ co\dim Z < n\right\},\ n \geq 1$$

is called the n-th Gelfand number of the operator T.

3.2. Definition. Let E, F, E_0, F_0 be Banach spaces. A map s which to every operators $S \in L(E,F)$ a unique sequence $(s_n(S))$ is called an s-function (or s-number function) if the following conditions are satisfied:

(1) For $S \in L(E,F)$ $\|S\| = s_1(S) \geq s_2(S) \geq ... \geq 0$;
(2) For $S, T \in L(E,F)$ $s_n(S+T) \leq s_n(S) + \|T\|$;
(3) For $T \in L(E_0, E), S \in L(E,F)$ and $R \in L(F, F_0)$ $s_n(RST) \leq \|R\|\, s_n(S)\, \|T\|$;
(4) If $S \in L(E,F)$ and $dim(S) < n$, then $s_n(S) = 0$;
(5) If $id : l_2^n \longrightarrow l_2^n$ is the identity map, then $s_n(id) = 1$.

On the other hand $s_n(S), n \geq 1$ is called the n-th s-number of the operator S.
The advanced analysis of these numbers has been given in books of A.Pietsch [4,5].Particularly, in Hilbert spaces case for any $n \geq 1$ $c_n(S) = s_n(S) = \lambda_n(|S|) = \lambda_n(|S^*S|^{\frac{1}{2}})$ (for the more informations see [1]). On the other hand for $S \in C_\infty(H)$, where H is a Hilbert space, the significant method for computation of $s_n(S), n \geq 1$ has been given by Dzh.E. Allakhverdiev in [13].

3.3. Theorem. If $S = \oplus_{m=1}^{\infty} S_m$, $S \in L(\mathfrak{X})$ and for any $m \geq 1$, $n_m = \dim \mathfrak{X}_m < \infty$, then for $n > m_1 + m_2 + ... + m_k, k \geq 1$ it is true that $c_n(S) \leq \sup\limits_{m \geq k+1} \|S_m\|$.

Proof. Firstly, for any $k \in \mathbb{N}$ define the operator $P_k : \mathfrak{X} \longrightarrow \mathfrak{X}$ in following form

$$P_k(x_m) := \{x_1, x_2, ..., x_k, 0, ...\},\ \text{for } x = (x_m) \in \mathfrak{X}$$

In this case for $k \in \mathbb{N}$

$$SP_k(x_m) = \left(\oplus_{m=1}^{\infty} S_m\right)(P_k(x_m)) = \{S_1x_1, S_2x_2, ..., S_kx_k, 0, 0, 0, ...\}$$

and $SP_k \in L(\mathfrak{X})$.

Therefore, for any $x = (x_m) \in \mathfrak{X}$ it is clear that

$$\begin{aligned} \|(S - SP_k)(x_m)\| &= \|\{0, ..., 0, S_{k+1}x_{k+1}, ...\}\| = \left(\sum_{m=k+1}^{\infty} \|S_m x_m\|_m^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{m=k+1}^{\infty} \|S_m\|^p \|x_m\|_m^p\right)^{\frac{1}{p}} \leq \sup_{m \geq k+1} \|S_m\| \, \|x\| \end{aligned}$$

Hence $\|S - SP_k\| \leq \sup\limits_{m \geq k+1} \|S_m\|$. From this and definition of Gelfand numbers it is implied that for $n > m_1 + m_2 + ... + m_k$, $k \geq 1$ it is true that $c_n(S) \leq \sup\limits_{m \geq k+1} \|S_m\|$.

$\square$

3.4. Theorem. Let $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$ and the operator $S : \mathfrak{X} \longrightarrow \mathfrak{X}$ is invertible, i.e. there exist S^{-1} and $S^{-1} \in L(\mathfrak{X})$. Then $\inf\limits_{1\leq m\leq n} \frac{1}{\left\|S_m^{-1}\right\|} \leq c_n(S),\ n \geq 1$.

Proof. It is known that in this case $S^{-1} = \oplus_{m=1}^{\infty} S_m^{-1}$ and $\left\|S^{-1}\right\| = \sup\limits_{m\geq 1} \left\|S_m^{-1}\right\|$.

Now define

$$J_n = \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p \longrightarrow \left(\oplus_{m=1}^{\infty} \mathfrak{X}_m\right)_p \quad , \ n \geq 1,$$
$$Q_n = \left(\oplus_{m=1}^{\infty} \mathfrak{X}_m\right)_p \longrightarrow \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p \quad , \ n \geq 1,$$
$$J_n(\{x_1, ..., x_n\}) = \{x_1, x_2, ...x_n, 0, ...\},$$
$$Q_n(\{x_1, ..., x_n, x_{n+1}, ...\}) = \{x_1, x_2, ...x_n\}$$

From these definitions it is obtained that the operator

$$R_n := Q_n S J_n : \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p \longrightarrow \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p$$

is in form $R_n\left(\{x_1, x_2, ..., x_n\}\right) = \{S_1 x_1, S_2 x_2, ..., S_n x_n\}$, $n \geq 1$ for any $\{x_1, x_2, ..., x_n\} \in \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p$.

Therefore, there exist R_n^{-1} ,the inverse of the operator R_n, $n \geq 1$ and

$$R_n^{-1} = \oplus_{m=1}^{n} S_m^{-1}, \quad R_n^{-1} : \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p \longrightarrow \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p, \quad \left\|R_n^{-1}\right\| = \sup_{1\leq m\leq n} \left\|S_m^{-1}\right\|, \ n \geq 1$$

Since the mapping $c : S \longrightarrow (c_n(S))$, $S \in L(\mathfrak{X})$ is a s-number function, then from the property of s-number function it is clear that

$$\begin{aligned} 1 \quad &= \quad c_n(id : \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p \longrightarrow \left(\oplus_{m=1}^{n} \mathfrak{X}_m\right)_p) = c_n(R_n R_n^{-1}) \leq c_n(Q_n S J_n)\left\|R_n^{-1}\right\| \\ &\leq \|Q_n\|\, c_n(S)\, \|J_n\| \left\|R_n^{-1}\right\| \leq c_n(S)\left\|R_n^{-1}\right\|, n \geq 1 \end{aligned}$$

Hence $\frac{1}{\left\|R_n^{-1}\right\|} \leq c_n(S)$, i.e. $\frac{1}{\sup\limits_{1\leq m\leq n}\left\|S_m^{-1}\right\|} \leq c_n(S)$, $n \geq 1$. In other words, for each $n \geq 1$ it is true that $\inf\limits_{1\leq m\leq n} \frac{1}{\left\|S_m^{-1}\right\|} \leq c_n(S)$.

$\square$

3.5. Corollary. If $S = \oplus_{m=1}^{\infty} S_m$, $S_m = \alpha_m id$, $\alpha_m \in \mathbb{C}$, $id : \mathfrak{X}_m \longrightarrow \mathfrak{X}_m$ for each $m \geq 1$, then $\inf\limits_{1\leq m\leq n} |\alpha_m| \leq c_n(S) \leq \sup\limits_{m\geq n} |\alpha_m|, \ n \geq 1$.

3.6. Remark. In case when $\alpha_m \in \mathbb{R}$, $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m \geq ... \geq 0$ and $\dim \mathfrak{X}_m = 1, m \geq 1$, then from Corollary 3.5 it is obtained that for every $n \geq 1$ $c_n(S) = \alpha_n, \ n \geq 1$.

This result has been obtained in [4].

Now prove the following results which explained some relation between Gelfand numbers of direct sum operator and its coordinate operators.

3.7. Theorem. Let us $S = \oplus_{m=1}^{\infty} S_m$, $S : \mathfrak{X} \longrightarrow \mathfrak{X}$. In this case for every $n \geq 1$

$$\sup_{n\geq 1} c_n^{(m)}(S_m) \leq c_n(S),$$

where $c_n^{(m)}(S_m)$ is denoted by the n-th Gelfand number of the operator $S_m \in L(\mathfrak{X}_m)$, $m \geq 1$.

Proof. If the following operators

$$D_m : \mathfrak{X}_m \longrightarrow \mathfrak{X}\ , \ T_m : \mathfrak{X} \longrightarrow \mathfrak{X}_m, \ m \geq 1$$

define in forms

$$D_m x_m = \{0, 0, ..., 0, x_m, 0, ...\}, x_m \in \mathfrak{X}_m,$$
$$T_m(x_m) = x_m, (x_m) \in \mathfrak{X}, x_m \in \mathfrak{X}_m, \ m \geq 1,$$

then D_m, T_m are linear bounded operators and $\|D_m\| \leq 1, \|T_m\| \leq 1, m \geq 1$. Moreover, it is clear that $S_m = T_m S D_m,\ m \geq 1$.

Hence from third condition in definition of s-functions for any $n \geq 1$ and $m \geq 1$ it is established that $c_n^{(m)}(S_m) = c_n^{(m)}(T_m S D_m) \leq \|T_m\| \, c_n(S) \, \|D_m\| \leq c_n(S)$.

From last relation the validity of claim is evident.

On the other hand the following assertion is true. □

3.8. Theorem. If $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$, then for any $n, m \geq 1$ it is valid that

$$c_n(S) \leq c_n(S_m) + \sup_{n \neq m} \|S_n\|$$

Proof. Indeed, from second condition in definition of s-functions it is established that

$$\begin{aligned} c_n(S) &= c_n(0 \oplus 0 \oplus ... \oplus 0 \oplus S_m \oplus 0 \oplus ... + S_1 \oplus S_2 \oplus ... \oplus S_{m-1} \oplus 0 \oplus S_{m+1} \oplus ...) \\ &\leq c_n(0 \oplus 0 \oplus ... \oplus 0 \oplus S_m \oplus 0 \oplus ...) + \|S_1 \oplus S_2 \oplus ... \oplus S_{m-1} \oplus 0 \oplus S_{m+1} \oplus ...\| \\ &= c_n(S_m) + \sup_{n \neq m} \|S_n\|,\ n \geq 1,\ m \geq 1 \end{aligned}$$

□

3.9. Theorem. If $S = \oplus_{m=1}^{\infty} S_m \in L(\mathfrak{X})$ and $S^{(p)} := S_1 \oplus S_2 \oplus ... \oplus S_p \oplus 0 \oplus ...$, $S^{(p)} : \mathfrak{X} \longrightarrow \mathfrak{X}$, $p \geq 1$, then $\left| c_n(S) - c_n(S^{(p)}) \right| \leq \sup_{m \geq p+1} \|S_m\|$, $n \geq 1$.

In particular, if $S \in C_\infty(\mathfrak{X})$, then $\lim_{p \longrightarrow \infty} c_n(S^{(p)}) = c_n(S)$, $n \geq 1$.

Proof. Since Gelfand number function is a s-number function, then in this case the validity of assertion is clear from inequality $\left\| c_n(S) - c_n(S^{(p)}) \right\| \leq \left\| S - S^{(p)} \right\|$, $p \geq 1$ and 2.2. Theorem. □

3.10. Remark. In Hilbert spaces case the analogous results have been obtained in [14].

Acknowledgement The authors would like to thank to Prof. C. Orhan (Ankara University, Turkey) for his interesting and encouragement discussions.

References

[1] Gohberg, I.Z. and Krein, M.G. Introduction to the theory of linear non-selfadjoint operators in Hilbert space (Moskow, 1965; Providence, 1969; Paris, 1971).

[2] Schmidt, E. Zur Theorie der linearen und nichtlinearen integralgleichungen, Math. Ann. 63, 433-476,1907; 64, 161-174(1907).

[3] von Neumann, J. and Schatten,R. The cross-space of linear transformations, Ann. Math. 47, 608-630,1946; 49, 557-582,1948.

[4] Pietsch, A. Operators ideals (North-Holland Publishing Company, 1980).

[5] Pietsch,A. Eigenvalues and s-numbers (Cambridge Universty Press, 1987).

[6] Kashin, B.S. Diameter of some finite dimensional sets and of same classes of smooth functions , Izv. Akad. Nauk SSSR, Ser. Mat., 41, 334-351,1977(in Russian).

[7] Mitjagin, B.S. Random matrices and supspaces , Collection of papers on "Geometry of linear spaces and operator theory", 175-202 (Jaroslavl, 1977,in Russian).

[8] Lindenstrauss, J. and Tzafriri, L. Classical Banach spaces, I(Springer-Verlag, Berlin 1977).

[9] Zettl, A. Sturm-Liouville Theory, V.121,Amer. Math. Soc.(Math. Survey and Monographs USA, 2005).

[10] Dunford, N. and Schwartz, J.T. Linear operators, I (Interscience, New-York,1958).

[11] Pietsch, A. s-numbers of operators in Banach spaces, Studia Math. 51, 201-223,1974.

[12] König, H. Eigenvalues of operators and applications, Mathematisches Seminar, 18 March 2008, Elsevier Preprint, p. 1-40.

[13] Allakhverdiev, Dzh. E. On the rate of appoximation to completely continuous operator with finitely dimension operators, Sci. notes of Azerbaijan State University, 2,27-35 ,1957(in Russian).
[14] Ismailov,Z.I., Otkun Çevik, E. and Unluyol, E. Compact inverses of multipoint normal differential operators for first order, Electronic J. Differential Equations, 89, 1-11, 2011.