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On Hamiltonian Properties of Honeycomb Meshes

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Abstract**:** In this paper, we investigated Hamiltonian properties of honeycomb meshes which are created in two different ways. We obtained different Hamilton paths for Honeycomb Meshes for any dimension with using n-bit gray code. Finally, we gave an algorithm which is used to label the nodes of Honeycomb Meshes.

Keywords: *Honeycomb meshes, Interconnection network, Hamilton graph, Gray code, Perfect Matching.*

1. Introduction

Interconnection networks are formed of different components of mechanism and connections between them. A network topology is the pattern for connecting one element to other elements and it may vary depending on formation such as tree, bus, mesh, star, ring, hypercube and tori. In this paper, to construct network topology, we use honeycomb meshes using two different structuring, show labeling with gray codes and then analyze Hamilton properties of honeycomb meshes.

Honeycomb is a pattern that is inspired from nature have been known before by scientist with its structural strengths and studied on them. Long before, people only know honeycomb pattern from bee honeycombs. After technological inventions like microscopes, scientists found honeycomb structure in different natural formations. For example, Robert Hook found a cork has honeycomb like cellular structure (Zhang et al, 2015) and there are many examples have honeycomb like pattern in nature such as insect eyes, marine skeletons, snowflakes, turtle and tortoise shells as in corks. Honeycomb has lightweight, strong and rigid structure that scientist work on from 20th century (Hales, 2001). First application of honeycomb is in 1914, Höfler and Renyi patented the first use of honeycombs as structural element. Then,the usage of honeycomb has been increasead year by year. Honeycomb applications are used in computer graphics (Lester and Sandor J., 1985), satellite component research (Boudjemai et al., 2012), bio-engineering (Engelmary et al., 2008), multiprocessor interconnection networks design (Carle et al., 1999; Manuel et al., 2008), station positioning for cellular phones (Nocetti et al., 2002) and chemical engineering (Rajan et al., 2012).

Interconnection networks are generally represented with a graph $G = (V, E)$ where E is a set of edges and V is a set of nodes. A honeycomb pattern can be used as a graph for constructing interconnection networks. In honeycomb graphs, each node can be labeled with using n-bit gray codes and consecutive labels differ from each other just in a one bit.

In this paper, gray code is used for labeling for honeycomb network. Gray code is a binary code where just one bit differs between adjacent numbers and for n-bit gray code, we can generate with algorithm (Lee and Lee, 1999). 2-bitgray code in the following tables that is used in the rest of the paper;

Table 1. 2-bit Gray Code.

A	B	C	D
00	01	11	10

Hamilton properties of Honeycomb meshes is investigated here. Hamilton cycle is a path in a graph that passing once through every vertex and return to the starting vertex and Hamilton graph is a graph in which a Hamilton cycle exists (Wilson, 1996). Hamilton cycles are important for design of graphs especially graphs with interconnection network topologies. Hamilton properties of graphs have been studied on before such as on random graphs (Janson, 1994), interconnection graphs like star graph (Derakhshan and Hussak, 2013), hypercube graph (Karci and Selçuk, 2014 and Selçuk and Karci, 2017) and honeycomb meshes (Simonraj and George, 2012, Stojmenovic, 1997 and Dong et al., 2015).

The paper is organized as follows: Section 2 presented variants of honeycomb meshes. Hamilton properties of Honeycomb Networks are explained, and algorithms are given in Section 3. These algorithms are the application of labeling the nodes of mesh structure that is given in Section 2. The result of this study is given in section 4 as conclusion.

2. Variants of Honeycomb Meshes

In this paper, honeycomb meshes are created in two following ways;

Case 1: Here after we call the following graph as honeycomb network with dimension one on HC(1) and dimension two on HC(2).

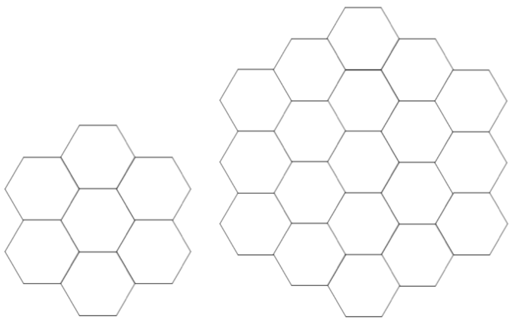


Figure 1. Honeycomb meshes of size one and two (HC)

Case 2: Here after we call the following graph as honeycomb cube network with dimension one on HCC(1) and dimension two on HCC(2).

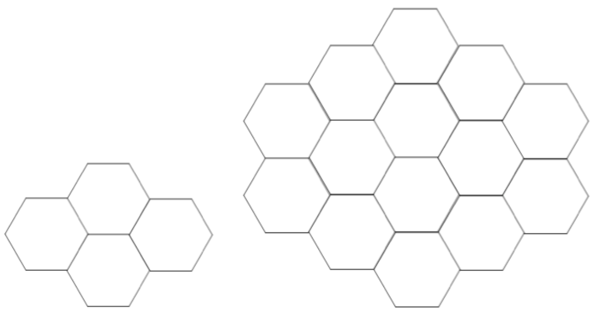


Figure 2. Honeycomb meshes of size one and two (HCC)

The relationship between HC and HCC is shown, as an example in dimension 4, in the figure 3.

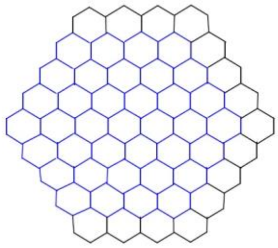


Figure 3. HC(3) (blue color graph) as a sub graph in HCC(4) (whole graph)
(figure 22(a) in (Simonraj and George, 2012)).

The number of vertices and edges of HC(n) are $6(n+1)^2$ and $9(n+1)^2-3(n+1)$, respectively, for $n \geq 0$. The parameter n of HC(n) is called the dimension of HC(n) (Amutha and Mary, 2016). Similarly, the number of vertices and edges of HCC(n) are $2(3n^2+4n+1)$ and $9n^2+9n+1$, respectively, for $n \geq 1$. The parameter n of HCC(n) is called the dimension of HCC(n) (Simonraj and George, 2012).

3. Hamiltonian Properties of Honeycomb Networks

Theorem 1. HC and HCC are bipartite graphs for any dimension using 1-bit gray code. Also, HC and HCC have Hamilton path and HC and HCC are Hamilton graphs via an extra edge.

Proof. We select the starting node from any node with degree 2 in the top level in HC and HCC. As example, with using 1-bit gray code for dimension one and two in HC and HCC, these graphs are bipartite graphs and as well these graphs have Hamilton path. Also, HC and HCC are Hamilton graphs for $n = 1,2$ via an extra edge (blue edge in figure 4 and figure 5). It is seen that HC(n) and HCC(n) are bipartite graphs and Hamilton graphs as intuitively ■

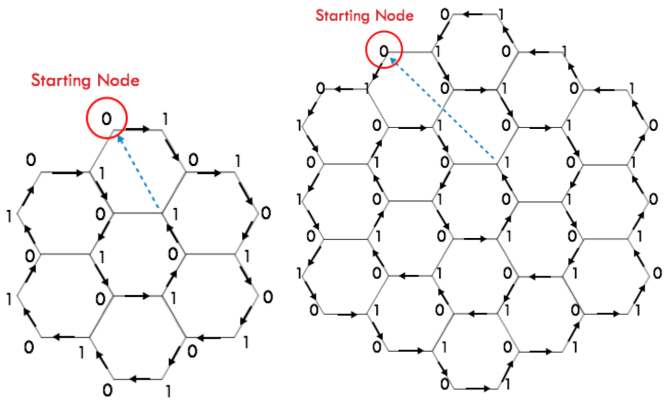


Figure 4. Hamilton path in HC(1) and HC(2).

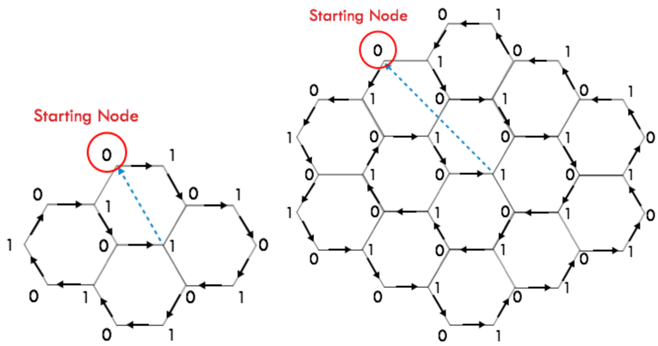


Figure 5. Hamilton path in HCC(1) and HCC(2).

Theorem 2. HC and HCC are Hamilton graphs via an extra edge for any dimension using 2-bit gray code.

Proof. For this, we select the starting node from any node with degree 2 in the top level in HC and HCC. We have two different starting nodes;

- the starting node has two adjacent nodes with degree 3,
- the starting node has one adjacent node with degree 2 and one adjacent node with degree 3.

First strategy: We assume two adjacent of the starting nodes have 3-degree nodes. We walk clockwise in HC when the dimension is odd, and we walk counterclockwise in HC when the dimension is even. (Figure 6-a, Figure 7-a, Figure 8-a and Figure 9-a)

Second strategy: We assume two adjacent of the starting nodes have 2 and 3-degree node. We walk only either clockwise or counterclockwise in HC(n) when the dimension is odd or even. (Figure 6-b, Figure 6-c, Figure 7-b, Figure 8-b and Figure 9-b)

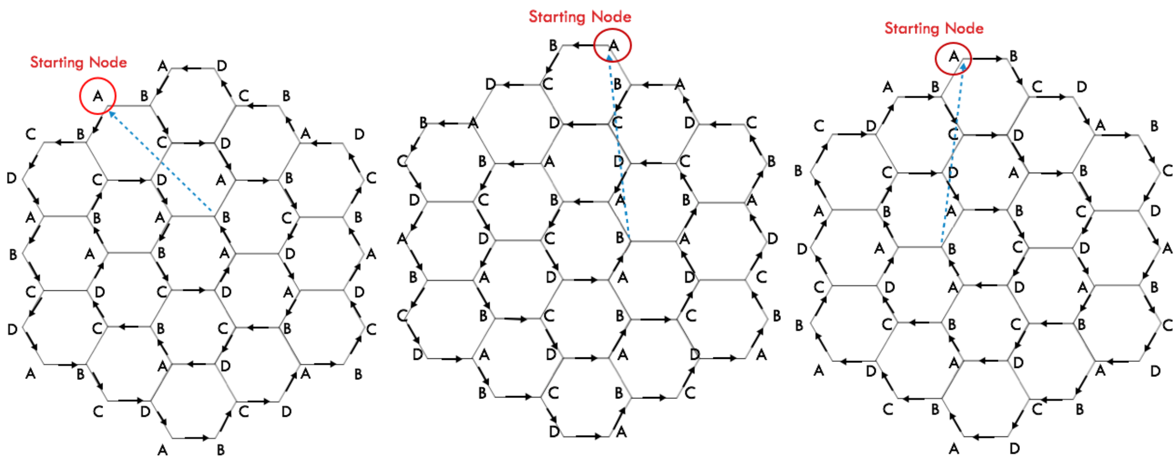
Now, we proved that HCN is a Hamilton graph. Firstly, let $n = 2k + 1, k \in Z^+$. Total number of edges of HC(n) is

$$6(n+1)^2 = 6(2k+2)^2 = 24(k+1)^2 = 4(6(k+1)^2).$$

Using 2-bit gray code, we get perfect matching of labeling of HC(n) and HC(n) has a Hamilton path (Figure 7). That is, the obtained labeling of starting node is A, and labeling of ending node is D. We draw an extra edge which is combined starting node with ending node, we get HC(n) as a Hamilton graph. Let $n = 2k, k \in Z^+$.Total number of edges of HC(n) is

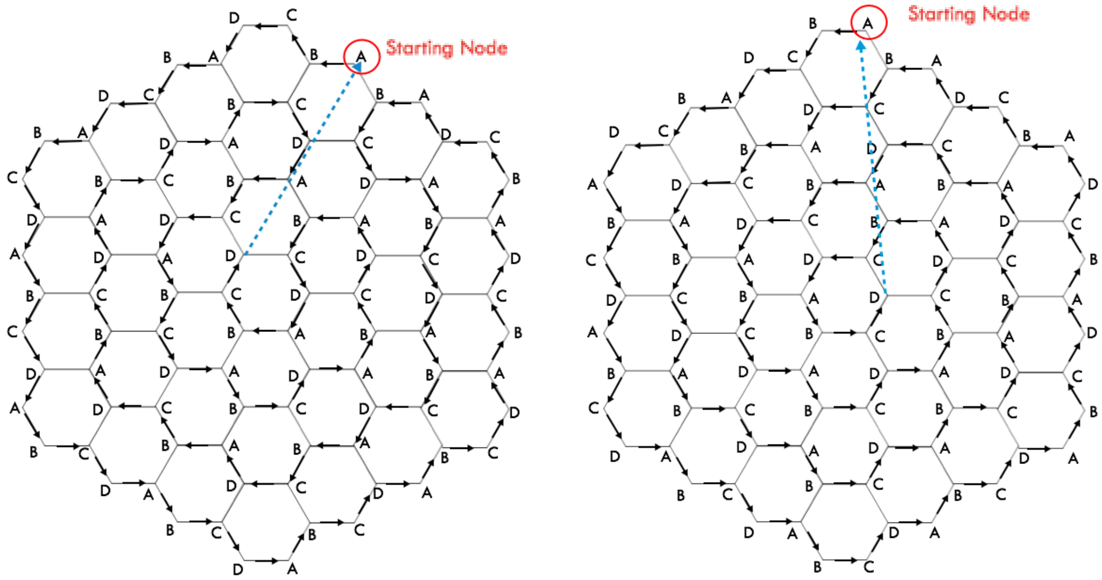
$$6(n+1)^2 = 6(2k+1)^2 = 24k^2 + 24k + 6 = 4(6k^2 + 6k + 1) + 2.$$

Using 2-bit gray code, we get perfect matching of labeling of HC(n) and HC(n) has a Hamilton path (Figure 6). That is, the obtained labeling of starting node is A, and labeling of ending node is B. We draw an extra edge which is combined starting node with ending node, we get HC(n) as a Hamilton graph.



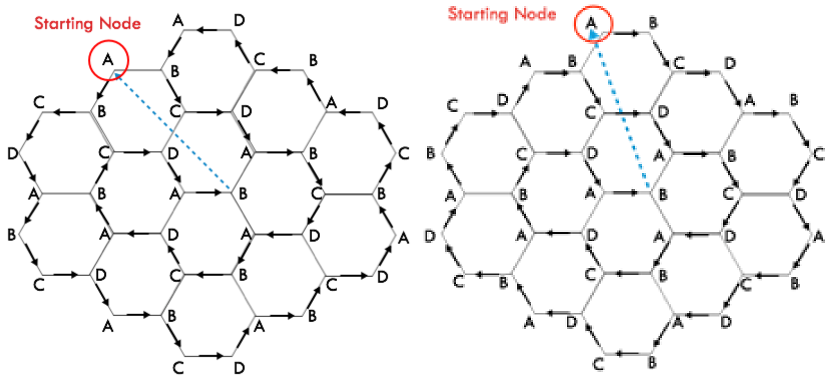
(a) **(b)** **(c)**

Figure 6. Different Hamilton paths in HC(2).



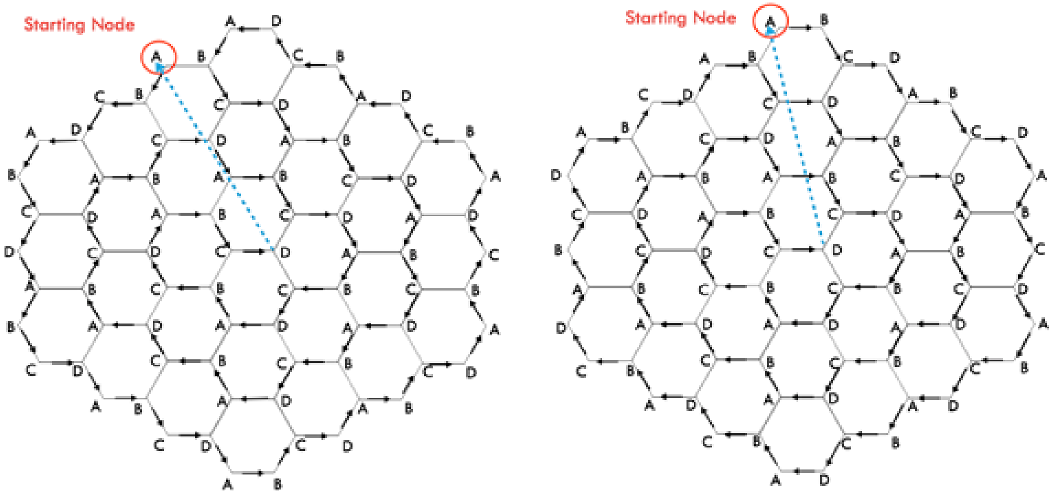
(a) **(b)**

Figure 7. Different Hamilton paths in HC(3).



(a) **(b)**

Figure 8. Different Hamilton paths in HCC(2).



(a) **(b)**

Figure 9. Different Hamilton paths in HCC(3).

We prove that HCCN is a Hamilton graph. Firstly, let $n = 2k + 1, k \in Z^+$. Total number of edges of HCC(n) is

$$2(3n^2 + 4n + 1) = 2(3(2k + 1)^2 + 4(2k + 1) + 1) = 4(6k^2 + 10k + 4).$$

Using 2-bit gray code, we get perfect matching of labeling of HCC(n) and HCC(n) has a Hamilton path (Figure 9). That is, the obtained labeling of starting node is A, and labeling of ending node is D. We draw an extra edge which is combined starting node with ending node, we get HCC(n) is a Hamilton graph. Let $n = 2k, k \in Z^+$. Total number of edges of HCC(n) is

$$2(3n^2 + 4n + 1) = 2(3(2k)^2 + 4(2k) + 1) = 4(6k^2 + 4k) + 2.$$

Using 2-bit gray code, we get perfect matching of labeling of HCC(n) and HCC(n) has a Hamilton path (Figure 8). That is, the obtained labeling of starting node is A, and labeling of ending node is B. We draw an extra edge which is combined starting node with ending node, we get HCC(n) is a Hamilton graph. The theorem is proved. ■

Remark1. As shown in the following figure, we can start to labelling in order as $\{A \to B \to \mathrm{C} \to \mathrm{D}\}$ in each dimension of HC(n) or HCC(n).

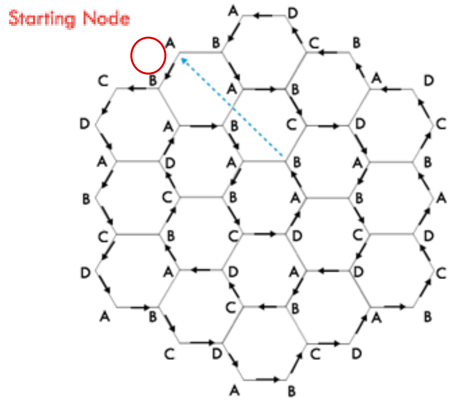


Figure 10. Alternative Hamilton path in HC(2). (Maintaining A-B-C-D order)

As shown Theorem 2, we can use labeling of nodes of HC(n) and HCC(n) using m −bit Gray Code for $m = 1,2, \ldots$. Now, we find upper bound for m via generalized this problem.

Theorem 3.HC and HCC are Hamilton graphs via an extra edge for $2^k - 1$ dimension using maximum$(2k + 1)$-bit and $(k + 1)$-bit gray code, respectively.

Proof. Let $n = 2^k - 1$ in HC(n). From, the number of vertices of HC(n) is $6(n + 1)^2$. Putting $n = 2^k - 1$, we get

$$6(n + 1)^2 = 6(2^k - 1 + 1)^2 = 6.2^{2k} = 3 \times 2^{2k+1}.$$

Therefore, $3 \times 2^{2k+1}$ total nodes in HC(n) can be traveled using $(2k + 1)$-bit gray code 3 times. These nodes can be labeled with up to $(2k + 1)$-bits.

Similarly, the number of vertices of HCC(n) is $2(n + 1)(3n + 1)$. Firstly, let $n = 2^k - 1$. Putting $n = 2^k - 1$, we get

$$2(n + 1)(3n + 1) = 2(2^k - 1 + 1)\big(3(2^k - 1) + 1\big) = 2^{k+1}(3 \times 2^k - 2)$$

Therefore, $2^{k+1}(3 \times 2^k - 2)$ total nodes in HCC(n) can be traveled using $(k + 1)$-bit gray code $3 \times 2^k - 2$ times.

Lastly, let $n = (2^k - 1)/3 \in Z$ (n is an integer number). Putting $n = (2^k - 1)/3$, we get

$$2(n + 1)(3n + 1) = 2\big((2^k - 1)/3 + 1\big)\big(3(2^k - 1)/3 + 1\big) = 2^{k+1}\big((2^k + 2)/3\big).$$

Therefore, $2^{k+1}\left((2^k+2)/3\right)$ total nodes in HCC(n) can be traveled using $(k+1)$-bit gray code $(2^k+2)/3$ times. These nodes can be labeled with up to $(k+1)$-bits.■

Algorithm 1. (Labeling of HC(n)) The following algorithm is used for labelling to the nodes of mesh structure, HC(n) in Theorem 2, using 2-bit gray code. HC(n) has two same strategies to get Hamilton path and HC(n) has$12n+6$node for $n=1,\dots,k$ in each dimension. The time complexity of the algorithm is $O(n^2)$.

```
Input: n
Output: Labeled L = HC(n)                               // Theorem 2's process
if initial node is degree two with degree 3 neighbor edges
   for i=n to 0
      ccw=n%2            // direction: ccw=1 counter clockwise, ccw=0 clockwise
      for j=1 to 12*i+6
         if ccw==1
           p=(j+2)%4
         else
           p=j%4
         end
         if p==0
               L(i,j)= A
         elseif p==1
               L(i,j)= B
         elseif p==2
               L(i,j)= C
         elseif p==3
               L(i,j)= D
         end
      end
   end
else     // select direction (ccw) to the side of the adjacent node with degrees 2
 next=1          // initial node is degree two with degree 2 and 3 neighbor edges
   for i=n to 0
      for j=1 to 12*i+6
         if next==1
           p=(j+2)%4
         else
           p=j%4
         end
         if p==0
               L(i,j)= A
         elseif p==1
               L(i,j)= B
         elseif p==2
               L(i,j)= C
         elseif p==3
               L(i,j)= D
         end
      end
   end
   next=(next+1)%2
end
```

Remark 2. (a) As HC(n)'s strategies, HCC(n) has two same strategies to get Hamilton path and HCC(n) also has $12n+14$ nodes for $n=1,\dots,k$ in each dimension. For this, if we change $12i+14$ ending parameter of second for loop in the above algorithm, then we have obtained a new algorithm which is used for labelling the nodes of mesh structure, HCC(n) in Theorem 2. The time complexity of the algorithm is $O(n^2)$.

(b) If we assume $n = 2^k - 1$ as in Theorem 3 and modify if controls, then we have obtained a new algorithm which is used labelling the nodes of HC(n) via $(2k + 1)$-bit Gray code.

(c) If we assume $n = 2^k - 1$ or$n = (2^k - 1)/3 \in Z$ in Theorem 3 and modify if controls, then we have obtained a new algorithm which is used labelling the nodes of HCC(n) via $(k + 1)$-bit gray code.

4. Conclusion

We get labeling of nodes in Honeycomb meshes (HC and HCC) for any dimensions, using gray code. Also, we obtained perfect matching in honeycomb network for any dimensions, using 1-bit and 2-bit gray code. Also, we find the upper bound for gray code bit number to use it is used labelling of HC and HCC.

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