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A Note On The Series Space $\left|\bar{N}_p^{\theta}\right|(\mu)$

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Abstract

The series space $\left|\bar{N}_p^{\theta}\right|(\mu)$ has recently been introduced and studied by Gökçe and Sarıgöl [1]. The main purpose of this paper is to determine the $\alpha-$, $\beta-$ and $\gamma-$duals of the space $\left|\bar{N}_p^{\theta}\right|(\mu)$ and to show that it is linearly isomorphic to the Maddox's space $l(\mu)$.

Keywords: Absolute summability, Sequence spaces, Weighted mean, Duals, Isomorphism

1. INTRODUCTION

Let ω denote the set of all (real or) complex valued sequences. Any vector subspace of ω is called as a sequence space. Let X, Y be any sequence spaces and $A = (a_{nv})$ be an infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$ if the series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v$$

is convergent for any integer n. If $A(x) \in Y$, whenever $x \in X$, then it is said that A defines a matrix transformation from X into Y, and the class of all infinite matrices A such that $A : X \to Y$ is denoted by (X, Y). Besides, the matrix domain of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x \in \omega : A(x) \in X\}. \quad (1)$$

The set

$$S(X, Y) = \{a \in \omega : \forall x \in X, ax = (a_k x_k) \in Y\}$$

is called the multiplier space of X and Y. With this notation, the $\alpha-$, $\beta-$ and $\gamma-$ duals of the space X are identified as

$$X^{\alpha} = S(X, l), X^{\beta} = S(X, c_s), X^{\gamma} = S(X, b_s).$$

For c, l_{∞}, c_s, b_s and l_p $(1 \le p < \infty)$, we write the space of all convergent, bounded sequences and the space of all convergent, bounded, $p-$absolutely convergent series, respectively.
Also, the Maddox's space defined by

$$l(\mu) = \left\{x = (x_n) : \sum_{n=0}^{\infty} |x_n|^{\mu_n} < \infty\right\}$$

has an important role in summability theory. Note that $l(\mu)$ is an FK space according to its paranorm given by

$$g(x) = \left(\sum_{k=0}^{\infty} |x_k|^{\mu_k}\right)^{1\backslash M}$$

where $M = \max\{1; \sup_k \mu_k\}$ ([2], [3], [4]).

Let $\sum a_v$ be an infinite series with the sequence of partial sum (s_n), $\theta = (\theta_n)$ be any sequence of positive real numbers and $\mu = (\mu_n)$ be any bounded sequence of positive real numbers. The series $\sum a_v$ is said to be summable $|A, \theta_n|(\mu)$ if

$$\sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} |A_n(s) - A_{n-1}(s)|^{\mu_n} < \infty \quad (2)$$

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[1]. It should be noted that the concept of the summability $|A, \theta_n|(\mu)$ includes the some well known summability methods for special cases of the sequences μ, θ and the matrix A (see, for example, [5], [6], [7], [8], [10], [11]). If we take the weighted mean matrix instead of A, the summability $|A, \theta_n|(\mu)$ is reduced to the summability $|\bar{N}, p_n, \theta_n|(\mu)$ and also the space of all series summable by this method is defined as follows ([1]):

$$|\bar{N}_p^\theta|(\mu) = \left\{a : \sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} \left|\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v\right|^{\mu_n} < \infty\right\}.$$

One can give the weighted mean matrix by

$$a_{nv} = \begin{cases} p_v/P_n, & 0 \le v \le n \\ 0, & v > n. \end{cases}$$

According to the notation of the domain given by (2), this space can be redefined by $|\bar{N}_p^\theta|(\mu) = (l(\mu))_{T(\theta,\mu,p)}$, where the matrix $T(\theta, \mu, p)$ is given by

$$t_{nv}(\theta, \mu, p) = \begin{cases} 1, & n = 0, v = 0 \\ \theta_n^{1/\mu_n^*} \frac{p_n P_{v-1}}{P_n P_{n-1}}, & 1 \le v \le n \\ 0, & v > n, \end{cases} \quad (3)$$

whose inverse $S(\theta, \mu, p)$ is

$$s_{nv}(\theta, \mu, p) = \begin{cases} 1, & n = v = 0 \\ -\theta_{n-1}^{-1/\mu_{n-1}^*} \frac{P_{n-2}}{p_{n-1}}, & v = n-1 \\ \theta_n^{-1/\mu_n^*} \frac{P_n}{p_n}, & v = n \\ 0, & v \ne n-1, n \end{cases} \quad (4)$$

where $0 < inf \mu_n \le H < \infty$ and μ_n^* is the conjugate of μ_n, i.e. $1/\mu_n + 1/\mu_n^* = 1$, $\mu_n > 1$, and $1/\mu_n^* = 0$ for $\mu_n = 1$.

In this paper, we show that the space $|\bar{N}_p^\theta|(\mu)$ is isometrically isomorphic to the space $l(\mu)$ and compute its $\alpha -, \beta -$ and $\gamma -$duals.

Firstly, we consider following conditions:

(a) There exists an integer $M > 1$ such that

$$\sup\left\{\sum_{v=0}^{\infty} \left|\sum_{n \in N} a_{nv} M^{-1}\right|^{\mu_v^*} : N \subset \mathbb{N} \text{ finite}\right\} < \infty.$$

(b) There exists an integer $M > 1$ such that

$$\sup_v \sum_{n=0}^{\infty} \left|a_{nv} M^{-1/\mu_v}\right| < \infty.$$

(c) $\lim_n a_{nv}$ exists for each v.

(d) $\sup_{n,v} |a_{nv}|^{\mu_v} < \infty$.

(e) There exists an integer $M > 1$ such that

$$\sup_n \sum_{v=0}^{\infty} |a_{nv} M^{-1}|^{\mu_v^*} < \infty.$$

Now, we express the lemmas which characterize some well known classes of infinite matrices.

Lemma 1.1 Let $\mu = (\mu_v)$ be arbitrary bounded sequences of strictly positive numbers. For all $v \in \mathbb{N}$,

(i) If $\mu_v > 1$, then $A \in (l(\mu), l)$ if and only if (a) holds.

(ii) If $\mu_v \le 1$, then $A \in (l(\mu), l)$ if and only if (b) holds.

(iii) If $\mu_v \le 1$, then $A \in (l(\mu), c)$ if and only if (c) and (d) hold.

(iv) If $\mu_v \le 1$, then $A \in (l(\mu), l_\infty)$ iff (d) holds.

(v) If $\mu_v > 1$, then $A \in (l(\mu), c)$ if and only if (c) and (e) hold.

(vi) If $\mu_v > 1$, then $A \in (l(\mu), l_\infty)$ iff (e) holds, ([12]).

Lemma 1.2 Let $A = (a_{nv})$ be an infinite matrix with complex numbers, (μ_v) be a bounded sequence of positive numbers. If $U_\mu[A] < \infty$ or $L_\mu[A] < \infty$, then

$$(2K)^{-2} U_\mu[A] \le L_\mu[A] \le U_\mu[A],$$

where $K = \max\{1, 2^{H-1}\}$, $H = \sup_v \mu_v$,

$$U_\mu[A] = \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}|\right)^{\mu_v}$$

and

$$L_\mu[A] = \sup\left\{\sum_{v=0}^{\infty} \left|\sum_{n \in N} a_{nv}\right|^{\mu_v} : N \subset \mathbb{N} \text{ finite}\right\},$$

([9]).

Lemma 1.3 Let $\theta = (\theta_n)$ be a sequence of positive numbers and $\mu = (\mu_n)$ be bounded sequence of positive numbers. Then, the set $|\bar{N}_p^\theta|(\mu)$ is a linear space with the coordinate-wise addition and scalar multiplication. Also, the space $|\bar{N}_p^\theta|(\mu)$ is an FK-space with AK under the paranorm

$$\|x\|_{|\bar{N}_p^\theta|(\mu)} = \left(|x_0| + \sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} \left|\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v\right|^{\mu_n}\right)^{1/M}$$

where $M = \max\{1, \sup_n \mu_n\}$, ([1]).

2. MAIN RESULTS

Theorem 2.1 Assume that $\mu=(\mu_v)$ be a bounded sequence of positive numbers and $\theta=(\theta_n)$ be a sequence of positive numbers. Then, the space $|\bar{N}_p^\theta|(\mu)$ is linear isomorphic to $l(\mu)$; that is, $|\bar{N}_p^\theta|(\mu)\cong l(\mu)$.

Proof We should show the existence of a linear bijection map preserving the paranorm between the spaces $|\bar{N}_p^\theta|(\mu)$ and $l(\mu)$. Now, consider the map $T_n(\theta,\mu,p): |\bar{N}_p^\theta|(\mu)\to l(\mu)$ given by

$$T_0(\theta,\mu,p)(x)=x_0,$$
$$T_n(\theta,\mu,p)(x)=\theta_n^{1/\mu_n^*}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}x_v\,,n\geq 1.$$

Since the matrix corresponding to this map given by (3) is a triangle, it is clear that $T(\theta,\mu,p)$ is a linear bijection map. If $x\in|\bar{N}_p^\theta|(\mu)$, then $T(\theta,\mu,p)(x)\in l(\mu)$ and so we get

$$\|x\|_{|\bar{N}_p^\theta|(\mu)}=\left(\sum_{n=0}^{\infty}|T_n(\theta,\mu,p)|^{\mu_n}\right)^{1/M}=\|T(\theta,\mu,p)(x)\|_{l(\mu)}.$$

So, $T(\theta,\mu,p)(x)$ preserves the paranorm which completes the proof.

In the following, we express and prove the theorems determining $\alpha-$, $\beta-$ and $\gamma-$duals of the space $|\bar{N}_p^\theta|(\mu)$. Note that taking $\theta_0=1$ does not disrupt the generality. Hence, in the following proofs, we will accept $\theta_0=1$.

Theorem 2.2 Let $\theta=(\theta_v)$ be a sequence of positive numbers and $\mu=(\mu_v)$ be a bounded sequence of positive numbers. If $\mu_v>1$, for all v, then there exists an integer $M>1$ such that

$$\{|\bar{N}_p^\theta|(\mu)\}^\alpha=\left\{\varepsilon\in\omega:\sum_{v=1}^{\infty}\frac{(Mp_v)^{-1/\mu_v^*}}{\theta_v}(P_v|\varepsilon_v|+P_{v-1}|\varepsilon_{v+1}|)^{\mu_v^*}<\infty\right\},$$

and if $\mu_v\leq 1$, for all v, then there exists an integer $M>1$ such that

$$\{|\bar{N}_p^\theta|(\mu)\}^\alpha=\left\{\varepsilon\in\omega:\sup_v\frac{\theta_v^{-1/\mu_v^*}M^{-1/\mu_v}}{p_v}(P_v|\varepsilon_v|+P_{v-1}|\varepsilon_{v+1}|)<\infty\right\}.$$

Proof Let us define the matrix $C=(c_{nv})$ as follows:

$$c_{nv}=\begin{cases}\theta_n^{-1/\mu_n^*}\dfrac{P_n}{p_n}\varepsilon_n, & v=n\\ -\theta_{n-1}^{-1/\mu_{n-1}^*}\dfrac{P_{n-2}}{p_{n-1}}\varepsilon_n, & v=n-1\\ 0, & v\neq n-1,n.\end{cases}$$

It can be easily seen from the inverse of $T(\theta,\mu,p)$ that

$$\varepsilon_nx_n=\varepsilon_0T_0(\theta,\mu,p)+\varepsilon_n\left(\theta_n^{-1/\mu_n^*}\frac{P_n}{p_n}T_n(\theta,\mu,p)-\theta_{n-1}^{-1/\mu_{n-1}^*}\frac{P_{n-2}}{p_{n-1}}T_{n-1}(\theta,\mu,p)\right)$$
$$=\sum_{v=0}^{n}c_{nv}T_v(\theta,\mu,p)$$

Therefore, we obtain that since $T(\theta,\mu,p)(x)\in l(\mu)$ whenever $x\in|\bar{N}_p^\theta|(\mu)$, $\varepsilon\in\{|\bar{N}_p^\theta|(\mu)\}^\alpha$ if and only if $C\in(l(\mu),l)$. With applying the Lemma 1.1 and Lemma 1.2 to the matrix C, we get the desired results.

Theorem 2.3 Suppose that $\theta=(\theta_v)$ be a sequence of positive numbers and $\mu=(\mu_v)$ be a bounded sequence of positive numbers. Define

$$D_1=\left\{\varepsilon\in\omega:\sup_n\left(\sum_{v=0}^{n-1}\frac{M^{-1/\mu_v^*}}{\theta_v}\left|\frac{P_v}{p_v}\Delta\varepsilon_v+\varepsilon_{v+1}\right|^{\mu_v^*}+\frac{M^{-1/\mu_n^*}}{\theta_n}\left|\frac{P_n}{p_n}\varepsilon_n\right|^{\mu_n^*}\right)<\infty\right\},$$

$$D_2=\left\{\varepsilon\in\omega:\sup_n\left(\left|\theta_n^{-1/\mu_n^*}\left(\frac{P_n}{p_n}\Delta\varepsilon_n+\varepsilon_{n+1}\right)\right|^{\mu_n}+\left|\theta_n^{-1/\mu_n^*}\frac{P_n}{p_n}\varepsilon_n\right|^{\mu_n}\right)<\infty\right\}$$

where $\Delta\varepsilon_v=\varepsilon_v-\varepsilon_{v+1}$ for all $v\geq 0$. Then, for all v,

(i) if $\mu_v>1$, then $\{|\bar{N}_p^\theta|(\mu)\}^\beta=D_1$,

(ii) if $\mu_v\leq 1$, then $\{|\bar{N}_p^\theta|(\mu)\}^\beta=D_2$.

Proof Recall that $\varepsilon\in\{|\bar{N}_p^\theta|(\mu)\}^\beta$ if and only if $\varepsilon x\in c_s$ whenever $x\in|\bar{N}_p^\theta|(\mu)$. It can be written by (4) that

$$\sum_{k=0}^{n}\varepsilon_kx_k=T_0(\theta,\mu,p)\varepsilon_0+\sum_{k=1}^{n}\varepsilon_k\left(\theta_k^{\frac{-1}{\mu_k^*}}\frac{P_k}{p_k}T_k(\theta,\mu,p)-\theta_{k-1}^{-1/\mu_{k-1}^*}\frac{P_{k-2}}{p_{k-1}}T_{k-1}(\theta,\mu,p)\right)$$
$$=T_0(\theta,\mu,p)\varepsilon_0+\sum_{k=1}^{n}\theta_k^{\frac{-1}{\mu_k^*}}\frac{P_k}{p_k}T_k(\theta,\mu,p)\varepsilon_k-\sum_{k=1}^{n}\theta_{k-1}^{\frac{-1}{\mu_{k-1}^*}}\frac{P_{k-2}}{p_{k-1}}T_{k-1}(\theta,\mu,p)\varepsilon_k$$
$$=T_0(\theta,\mu,p)\varepsilon_0+\theta_n^{-1/\mu_n^*}\frac{P_n}{p_n}T_n(\theta,\mu,p)\varepsilon_n$$

$$+\sum_{k=1}^{n-1}\frac{\theta_k^{\frac{-1}{\mu_k^*}}}{p_k}(\varepsilon_k P_k-\varepsilon_{k+1}P_{k-1})T_k(\theta,\mu,p)$$
$$=\theta_n^{-1/\mu_n^*}\frac{P_n}{p_n}T_n(\theta,\mu,p)\varepsilon_n$$
$$+\sum_{k=0}^{n-1}\frac{\theta_k^{-1/\mu_k^*}}{p_k}(P_k\Delta\varepsilon_k+\varepsilon_{k+1}p_k)T_k(\theta,\mu,p)$$
$$=\sum_{k=0}^{n}b_{nk}T_k(\theta,\mu,p)$$

where $B=(b_{nk})$ is given by

$$b_{nk}=\begin{cases}\theta_n^{-1/\mu_n^*}\frac{P_n}{p_n}\varepsilon_n, & k=n\\ \frac{\theta_k^{-1/\mu_k^*}}{p_k}(P_k\Delta\varepsilon_k+\varepsilon_{k+1}p_k), & 0\le k\le n-1\\ 0, & k>n.\end{cases}$$

Since $T(\theta,\mu,p)(x)\in l(\mu)$ whenever $x\in\left|\bar{N}_p^\theta\right|(\mu)$, it is clear that $\varepsilon\in\left\{\left|\bar{N}_p^\theta\right|(\mu)\right\}^\beta$ if and only if $B\in(l(\mu),c)$. So, it follows from Lemma 1.1 that if $\mu_v>1$ for all v, then $\varepsilon\in D_1$; otherwise $\varepsilon\in D_2$.

Theorem 2.4 Let $\theta=(\theta_v)$ be a sequence of positive numbers and $\mu=(\mu_v)$ be a bounded sequence of positive numbers. Then, for all v,

(i) $\left\{\left|\bar{N}_p^\theta\right|(\mu)\right\}^\gamma=D_1$ where $\mu_v>1$,

(ii) $\left\{\left|\bar{N}_p^\theta\right|(\mu)\right\}^\gamma=D_2$ where $\mu_v\le 1$.

Since the proof is simple and similar, we omitted it.

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