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# Some remarks for a certain class of holomorphic functions at the boundary of the unit disc

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**Abstract**

We consider a boundary version of the Schwarz Lemma on a certain class which is denoted by $\mathcal{K}(\alpha)$. For the function $f(\lambda) = \lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots$ which is defined in the unit disc $E$ such that the function $f(\lambda)$ belongs to the class $\mathcal{K}(\alpha)$, we estimate from below the modulus of the angular derivative of the function $\frac{\lambda f'(\lambda)}{f(\lambda)}$ at the boundary point $b$ with $\frac{bf'(b)}{f(b)} = \frac{1}{1+\alpha}$. Moreover, we get the Schwarz Lemma for the class $\mathcal{K}(\alpha)$. We also investigate some inequalities obtained in terms of sharpness.

**Keywords:** Schwarz Lemma, Holomorphic function, Jack's Lemma, Angular derivative.

**2010 Mathematics Subject Classication:** 30C80, 32A10.

## 1. INTRODUCTION

One of the main tool of complex functions theory is Schwarz Lemma. Its present statement has been written by Constantin Caratheodory. Schwarz Lemma is an important results which gives the estimates about the values of holomorphic functions defined from the unit disc into itself. This lemma, which is a direct applications of the maximum modulus principle, is commonly used as follows:

Let $E = \{\lambda : |\lambda| < 1\}$ be the unit disc and $T = \{\lambda : |\lambda| = 1\}$ be the boundary of the unit disc. Let

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us consider a holomorphic function $f$ which maps the unit disc to itself and fixes the point zero. Then, under this conditions, $|f(\lambda)| \leq |\lambda|$ for all $\lambda \in E$ and $|f'(0)| \leq 1$. In addition, if the equality $|f(\lambda)| = |\lambda|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then $f$ is a rotation, which means $f(\lambda) = \lambda e^{i\gamma}$, where $\gamma$ is a real. So, unless $f$ is rotation, the function $f$ maps each disc in the unit disc into a strictly smaller one [4].

We use the Jack's lemma which is related to the function $\frac{\lambda f'(\lambda)}{f(\lambda)}$ we shall investigate [5].

Let $\mathcal{A}$ denote the class of holomorphic functions in the unit disc $E$ for which $f(0) = f'(0) - 1 = 0$, that is,

$$f(\lambda) = \lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots$$

Also, let $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{A}$ includes all functions $f(\lambda)$ which satisfying the condition

$$\left| \frac{1 + \frac{\lambda f''(\lambda)}{f'(\lambda)}}{\frac{\lambda f'(\lambda)}{f(\lambda)}} - 1 \right| < \alpha, \ \lambda \in E, \qquad (1.1)$$

where $0 < \alpha \leq 1$. Let $f(\lambda) \in \mathcal{K}(\alpha)$. Let

$$s(\lambda) = \frac{\lambda f'(\lambda)}{f(\lambda)} = \frac{1}{1 + \alpha\varphi(\lambda)} \qquad (1.2)$$

and

$$\varphi(\lambda) = \frac{1}{\alpha}\left(\frac{1}{s(\lambda)} - 1\right).$$

It is clear that $\varphi(\lambda)$ is holomorphic function in $E$ and fixes the point zero. Now, we want to prove that $|\varphi(\lambda)| < 1$ in $E$. By the definition of the class $\mathcal{K}(\alpha)$ and (1.2), we get

$$\left| \frac{1 + \frac{\lambda f''(\lambda)}{f'(\lambda)}}{\frac{\lambda f'(\lambda)}{f(\lambda)}} - 1 \right| = \left| \frac{\lambda s'(\lambda)}{s^2(\lambda)} \right| = \left| \frac{\frac{-\alpha\lambda\varphi'(\lambda)}{(1 + \alpha\varphi(\lambda))^2}}{\left(\frac{1}{1 + \alpha\varphi(\lambda)}\right)^2} \right|$$

$$= |-\alpha z\varphi'(\lambda)|.$$

Suppose that there exists a point $z_0 \in E$ such that $\max_{|\lambda| \leq |\lambda_0|} |\varphi(\lambda)| = |\varphi(\lambda_0)| = 1$. Hence, $\varphi(\lambda_0) = e^{i\theta}$, where $\theta$ is real. By Jack's Lemma, we get

$$\varphi(\lambda_0) = e^{i\theta} \quad and \quad \frac{\lambda_0\varphi'(\lambda_0)}{\varphi(\lambda_0)} = k.$$

Using the last equality and also by the elementary calculations, we obtain

$$\left| \frac{1 + \frac{\lambda_0 f''(\lambda_0)}{f'(\lambda_0)}}{\frac{\lambda_0 f'(\lambda_0)}{f(\lambda_0)}} - 1 \right| = |-\alpha\lambda_0\varphi'(\lambda_0)|$$

$$= |-\alpha k\varphi(\lambda_0)|$$

$$= |-\alpha k e^{i\theta}| = \alpha k \geq \alpha,$$

which contradicts with (1.1). This means that there is no a point $\lambda_0 \in E$ such that $|\varphi(\lambda_0)| = 1$. Thus, $|\varphi(\lambda)| < 1$ for $|\lambda| < 1$. By the Schwarz Lemma, we take $|c_2| \leq \alpha$. For this last inequality, the extremal function is $f(\lambda) = \frac{\lambda}{1 + \alpha\lambda}$.

So, the following lemma is obtained.

**Lemma 2** *If* $f(\lambda) \in \mathcal{K}(\alpha)$, *then* $|c_2| \leq \alpha$. This inequality is sharp with the extremal function $f(\lambda) = \frac{\lambda}{1 + \alpha\lambda}$.

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies is called the boundary version of Schwarz Lemma. An important result of Schwarz lemma was given by Osserman [9].

In [3], all zeros of the holomorphic function in the unit disc different from $\lambda = 0$ and the holomorphic function which has no zero in the unit disc except $\lambda = 0$ have been considered, respectively. Thus, the stronger inequalities have been obtained.

After these studies, M. Jeong found a necessary and sufficient condition for a holomorphic function with fixed points only at the boundary of the unit disc and had some relations with derivatives of the function at these fixed points (see, [1], [2], [7], [3], [6], [7], [8], [9], [10], [12] and references therein).

For the results obtained, Julia-Wolff Lemma lemma and the result were used. [11].

## 2. MAIN RESULTS

In this section, we consider the function $f(\lambda) = \lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots$ which is defined in the unit disc $E$ and belongs to the class of $\mathcal{K}(\alpha)$. We estimate from below the modulus of the angular derivative of the function $\frac{\lambda f'(\lambda)}{f(\lambda)}$ at the boundary point $b$ with $\frac{bf'(b)}{f(b)} = \frac{1}{1+\alpha}$.

**Theorem 1** *Let* $f(\lambda) \in \mathcal{K}(\alpha)$. Assume that for some $b \in T$, $f'$ has an angular limit $f'(b)$ at $b$, $\frac{bf'(b)}{f(b)} = \frac{1}{1+\alpha}$. Then

$$\left|\left(\frac{\lambda f'(\lambda)}{f(\lambda)}\right)'_{\lambda=b}\right| \geq \frac{\alpha}{(1+\alpha)^2}. \quad (2.1)$$

The inequality (2.1) is sharp with the extremal function

$$f(\lambda) = \frac{\lambda}{1+\alpha\lambda}.$$

*Proof.* Let us consider the following function

$$\varphi(\lambda) = \frac{1}{\alpha}\left(\frac{1}{s(\lambda)} - 1\right),$$

where $s(\lambda) = \frac{\lambda f'(\lambda)}{f(\lambda)}$. Then $\varphi(\lambda)$ is a holomorphic function in the unit disc $E$, $\varphi(0) = 0$and $|\varphi(\lambda)| < 1$ for $|\lambda| < 1$. Also, we have $|\varphi(b)| = 1$ for $b \in T$. It is clear that

$$\varphi'(\lambda) = \frac{1}{\alpha}\left(\frac{-s'(\lambda)}{s^2(\lambda)}\right)$$

and

$$\varphi'(b) = \frac{1}{\alpha}\left(\frac{-s'(b)}{s^2(b)}\right).$$

Since

$$s(b) = \frac{bf'(b)}{f(b)} = \frac{1}{1+\alpha},$$

from Osseman we get

$$1 \leq |\varphi'(b)| = \frac{1}{\alpha}\left|\frac{s'(b)}{s^2(b)}\right| = \frac{(1+\alpha)^2}{\alpha}|s'(b)|$$

and

$$|s'(b)| \geq \frac{\alpha}{(1+\alpha)^2}.$$

Let's show equality in (2.1). Let

$$f(\lambda) = \frac{\lambda}{1+\alpha}. \quad (2.2)$$

Differentiating (2.2) logarithmically, we obtain

$$ln\, f(\lambda) = ln \frac{\lambda}{(1+\alpha\lambda)} = ln\,\lambda - ln(1+\alpha\lambda),$$

$$\frac{f'(\lambda)}{f(\lambda)} = \frac{1}{\lambda} - \frac{\alpha}{1+\alpha\lambda}$$

and

$$s(\lambda) = \frac{\lambda f'(\lambda)}{f(\lambda)} = 1 - \frac{\alpha\lambda}{1+\alpha\lambda} = \frac{1}{1+\alpha\lambda}.$$

Therefore, we take

$$s'(\lambda) = \frac{-\alpha}{(1+\alpha\lambda)^2}$$

and

$$\left|s'(1)\right| = \frac{\alpha}{(1+\alpha)^2}.$$

**Theorem 2** Assume that conditions of Theorem 1 are satisfied. Then

$$\left|\left(\frac{\lambda f'(\lambda)}{f(\lambda)}\right)'_{\lambda=b}\right| \geq \frac{1}{(1+\alpha)^2}\frac{2\alpha^2}{\alpha+|c_2|}. \tag{2.3}$$

The inequality (2.3) is sharp with the extremal function

$$f(\lambda) = \frac{\lambda}{1+\alpha\lambda}.$$

*Proof.* Let $\varphi(\lambda)$ be as in the above Theorem 1. Therefore, from Osserman,

$$\frac{2}{1+|\varphi'(0)|} \leq \left|\varphi'(b)\right| = \frac{1}{\alpha}\left|\frac{s'(b)}{s^2(b)}\right|$$
$$= \frac{(1+\alpha)^2}{\alpha}\left|s'(b)\right|.$$

Since

$$\varphi'(z) = \frac{1}{\alpha}\left(\frac{-s'(z)}{s^2(z)}\right)$$

and

$$\varphi'(0) = \frac{1}{\alpha}\left(\frac{-s'(0)}{s^2(0)}\right) = \frac{-c_2}{\alpha},$$

it is clear that

$$\left|\varphi'(0)\right| = \frac{|c_2|}{\alpha}.$$

Then

$$\frac{2}{1+\frac{|c_2|}{\alpha}} \leq \left|\varphi'(b)\right| = \frac{(1+\alpha)^2}{\alpha}\left|s'(b)\right|$$

and

$$\left|s'(b)\right| \geq \frac{\alpha}{(1+\alpha)^2}\frac{2\alpha}{\alpha+|c_2|}.$$

The last inequality shows that the inequality intended is obtained.

Now, let us show the case of equality. Let

$$f(\lambda) = \frac{\lambda}{1+\alpha\lambda}.$$

Then, we take by the elementary calculations

$$s(\lambda) = \frac{\lambda f'(\lambda)}{f(\lambda)} = 1 - \frac{\alpha\lambda}{1+\alpha\lambda} = \frac{1}{1+\alpha\lambda}$$

and

$$\left|s'(1)\right| = \frac{\alpha}{(1+\alpha)^2}.$$

Since

$$\lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots = \frac{\lambda}{1+\alpha\lambda},$$

$$1 + c_2\lambda + c_3\lambda^2 + \ldots = \frac{1}{1+\alpha\lambda},$$

$$c_2\lambda + c_3\lambda^2 + \ldots = \frac{1}{1+\alpha\lambda} - 1 = \frac{-\alpha\lambda}{1+\alpha\lambda},$$

$$c_2 + c_3\lambda + \ldots = \frac{-\alpha}{1+\alpha\lambda}$$

and

$$|c_2| = \alpha,$$

we obtain

$$\frac{\alpha}{(1+\alpha)^2}\frac{2\alpha}{\alpha+|c_2|} = \frac{\alpha}{(1+\alpha)^2}\frac{2\alpha}{\alpha+\alpha} = \frac{\alpha}{(1+\alpha)^2}.$$

The last equality shows that the equality intended is obtained.

According to the following theorem, we can strengthen the inequality (2.3) as below by taking into account the third coefficient $c_3$ in the expansion of the function $f(\lambda)$.

**Theorem 3** Let $f(\lambda) \in \mathcal{K}(\alpha)$. Assume that for some $b \in T$, $f'$ has an angular limit $f'(b)$ at $b$, $\frac{bf'(b)}{f(b)} = \frac{1}{1+\alpha}$. Then $\left|\left(\frac{\lambda f'(\lambda)}{f(\lambda)}\right)'_{\lambda=b}\right| \geq \frac{\alpha}{(1+\alpha)^2}\left(1 + \frac{2(\alpha-|c_2|)^2}{\alpha^2-|c_2|^2+2\alpha|c_3-c_2^2|}\right)$. (2.4)

*Proof.* Let $\varphi(\lambda)$ be the same as in the proof of Theorem 1. Let us consider the function

$$k(z) = \frac{\varphi(z)}{B(z)},$$

where $B(\lambda) = \lambda$. The function $k(z)$ is holomorphic in $E$. According to the maximum modulus princible, we have $|k(\lambda)| < 1$ for each $\lambda \in E$. From equality of $k(\lambda)$, we have

$$k(\lambda) = \frac{\varphi(\lambda)}{\lambda} = \frac{\frac{1}{\alpha}\left(\frac{1}{s(\lambda)} - 1\right)}{z}\lambda$$

$$= \frac{\frac{1}{\alpha}\left(\frac{1}{1 + c_2\lambda + (2c_3 - c_2^2)\lambda^2 + \ldots} - 1\right)}{\lambda}$$

$$= \frac{1}{\alpha}\left(\frac{-c_2\lambda - (2c_3 - c_2^2)\lambda^2 - \ldots}{\lambda(1 + c_2\lambda + (2c_3 - c_2^2)\lambda^2 + \ldots)}\right)$$

$$= \frac{1}{\alpha}\left(\frac{-c_2 - (2c_3 - c_2^2)\lambda - \ldots}{1 + c_2\lambda + (2c_3 - c_2^2)\lambda^2 + \ldots}\right).$$

Thus, we get

$$|k(0)| = \frac{|c_2|}{\alpha} \leq 1 \quad (2.5)$$

and

$$|k'(0)| = \frac{1}{\alpha}|2c_3 - 2c_2^2|.$$

Furthermore, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = |\varphi'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

Let

$$F(\lambda) = \frac{k(\lambda) - k(0)}{1 - \overline{k(0)}k(\lambda)}.$$

This function is holomorphic in $E$, $|\mathrm{F}(\lambda)| \leq 1$ for $|z| < 1$, $\mathrm{F}(0) = 0$, and $|\mathrm{F}(b)| = 1$ for $b \in T$, From Osserman,

$$\frac{2}{1 + |F'(0)|} \leq |F'(b)| = \frac{1 - |k(0)|^2}{\left|1 - \overline{k(0)}k(b)\right|^2}|k'(b)|$$

$$\leq \frac{1 + |k(0)|}{1 - |k(0)|}\left|\frac{\varphi'(b)}{B(b)} - \frac{\varphi(b)B'(b)}{B(b)^2}\right|$$

$$= \frac{1 + |k(0)|}{1 - |k(0)|}\left|\frac{\varphi(b)}{bB(b)}\right|\left|\frac{b\varphi'(b)}{\varphi(b)} - \frac{bB'(b)}{B(b)}\right|$$

$$\leq \frac{1 + |k(0)|}{1 - |k(0)|}\{|\varphi'(b)| - |B'(b)|\}.$$

Since

$$F'(\lambda) = \frac{1 - |k(0)|^2}{\left(1 - \overline{k(0)}k(\lambda)\right)^2}k'(\lambda),$$

$$F'(0) = \frac{k'(0)}{1 - |k(0)|^2},$$

and

$$|F'(0)| = \frac{\frac{1}{\alpha}|2c_3 - 2c_2^2|}{1 - \left(\frac{|c_2|}{\alpha}\right)^2} = \alpha\frac{|2c_3 - 2c_2^2|}{\alpha^2 - |c_2|^2},$$

we obtain

$$\frac{2}{1 + \alpha\frac{|2c_3 - 2c_2^2|}{\alpha^2 - |c_2|^2}} \leq \frac{1 + \frac{|c_2|}{\alpha}}{1 - \frac{|c_2|}{\alpha}}\left\{\frac{(1 + \alpha)^2}{\alpha}|s'(b)| - 1\right\},$$

$$\frac{2(\alpha^2 - |c_2|^2)}{\alpha^2 - |c_2|^2 + \alpha|2c_3 - 2c_2^2|} \leq \frac{\alpha + |c_2|}{\alpha - |c_2|}\left\{\frac{(1 + \alpha)^2}{\alpha}|s'(b)| - 1\right\},$$

$$\frac{2(\alpha-|c_2|)^2}{\alpha^2-|c_2|^2+\alpha|2c_3-2c_2^2|} \leq \frac{(1+\alpha)^2}{\alpha}\left|s^{'}(b)\right|-1$$

and

$$\left|s^{'}(b)\right| \geq \frac{\alpha}{(1+\alpha)^2}\left(1 + \frac{2(\alpha-|c_2|)^2}{\alpha^2-|c_2|^2+\alpha|2c_3-2c_2^2|}\right).$$

In the following theorem, the relation between the Taylor coefficients $c_2$ and $c_3$ is given for the function $f(\lambda)=\lambda+c_2\lambda^2+c_3\lambda^3+\ldots$

**Theorem 4** Let $f(\lambda)\in\mathcal{K}(\alpha)$, $\frac{f(\lambda)}{\lambda f^{'}(\lambda)}-1$ has no zeros in $E$ except $\lambda=0$ and $c_2>0$. Suppose that for some $b\in T$, $f^{'}$ has an angular limit $f^{'}(b)$ at $b$, $\frac{bf^{'}(b)}{f(b)}=\frac{1}{1+\alpha}$. Then we take the inequality

$$|c_3-c_2^2| \leq \left|c_2\, ln\left(\frac{c_2}{\alpha}\right)\right|. \tag{2.6}$$

*Proof.* Let $c_2>0$ and let us consider the function $k(\lambda)$ as in Theorem 3. Taking account of the equality (2.5), we denote by $ln\, k\,(\lambda)$ the holomorphic branch of the logarithm which is normed by the following condition

$$ln\, k\,(0)=ln\left(\frac{c_2}{\alpha}\right)<0.$$

Consider the following composite function

$$p(\lambda)=\frac{ln\, k\,(\lambda)-ln\, k\,(0)}{ln\, k\,(\lambda)+ln\, k\,(0)}.$$

It is obvious that $p(z)$ is a holomorphic function in $E$, $p(0)=0$ and $|p(\lambda)|<1$ for $\lambda\in E$. Thus, the function $p(z)$ satisfies assumptions of the Schwarz Lemma. Since

$$p^{'}(\lambda)=\frac{2\, ln\, k\,(0)}{(ln\, k\,(\lambda)+ln\, k\,(0))^2}\frac{k^{'}(\lambda)}{k(\lambda)}$$

and

$$p^{'}(0)=\frac{1}{2\, ln\, k\,(0)}\frac{k^{'}(0)}{k(0)},$$

we obtain

$$1\geq\left|p^{'}(0)\right|=\left|\frac{1}{2\, ln\, k\,(0)}\frac{k^{'}(0)}{k(0)}\right| = \frac{-1}{2\, ln\left(\frac{c_2}{\alpha}\right)}\frac{\frac{1}{\alpha}|2c_3-2c_2^2|}{\frac{|c_2|}{\alpha}} = \frac{-1}{ln\left(\frac{c_2}{\alpha}\right)}\frac{|c_3-c_2^2|}{|c_2|}$$

and

$$|c_3-c_2^2| \leq \left|c_2\, ln\left(\frac{c_2}{\alpha}\right)\right|.$$

## 3. CONCLUSIONS

In this study, the boundary behaviour of the bounded holomorphic function in the unit disc has been examined and the different versions of boundary Schwarz lemma have been discussed. In a class of analytic functions on the circle, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

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2018 (ICMS 2018) Maltepe University, Istanbul, Turkey, and the statements of some results in this paper will be appeared in AIP Conference Proceeding of 2nd International Conference Mathematical Sciences, (ICMS 2018) Maltepe University, Istanbul, Turkey [12].

## REFERENCES

[1] H. P. Boas, Julius and Julia: Mastering the Art of the Schwarz lemma, *Amer. Math. Monthly*, 117, 770-785, 2010.

[2] D. Chelst, A generalized Schwarz lemma at the boundary, *Proc. Amer. Math. Soc.*, 129, 3275-3278, 2001.

[3] V. N. Dubinin, The Schwarz inequality on the boundary for functions regular in the disc, *J. Math. Sci.*, 122, 3623-3629, 2004.

[4] G. M. Golusin, Geometric Theory of Functions of Complex Variable [in Russian], 2nd edn., Moscow 1966.

[5] I. S. Jack, Functions starlike and convex of order $\alpha$, *J. London Math. Soc.*, 3, 469-474, 1971.

[6] M. Jeong, The Schwarz lemma and its applications at a boundary point, *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* 21, 275-284, 2014.

[7] D. M. Burns and S. G. Krantz, Rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary, *J. Amer. Math. Soc.*, 7, 661-676, 1994.

[8] M. Mateljević, Rigidity of holomorphic mappings & Schwarz and Jack lemma, DOI: 10.13140/RG.2.2.34140.90249, In press.

[9] R. Osserman, A sharp Schwarz inequality on the boundary, *Proc. Amer. Math. Soc.*, 128, 3513-3517, 2000.

[10] T. Aliyev Azeroğlu and B. N. Örnek, A refined Schwarz inequality on the boundary, *Complex Variables and Elliptic Equations*, 58, 571-577, 2013.

[11] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag, Berlin, 1992.

[12] B. N. Örnek and T. Akyel, Some results a certain class of holomorphic functions at the boundary of the unit disc, 2nd International Conference of Mathematical Sciences (ICMS 2018) 31 July-06 August 2018, Istanbul, Turkey.